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Uniqueness and Singularities of Cylindrically Symmertric Surfaces Moving by Mean Curvature

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This note is an announcement of our recent work about the singularities and the uniqueness of cylindrically symmetric hypersurfaces in $\mathbb{R}^3$ which move by mean curvature. The details and proofs will appear in the authors' forthcoming paper [SoS 1991].

We begin with a very brief description of the notion of hypersurfaces moving by mean curvature. Let Γ_t denote a hypersurface expressed as the boundary of a bounded open set D_t in $\mathbb{R}^N$ at time t. If n is the exterior unit normal vector field to $\Gamma_t = \partial D_t$, we say that Γ_t moves by mean curvature if the speed $V(x,t)$ of Γ_t at x is given by

$$V = -\text{div } n. \tag{1}$$

An alternative approach is to think of Γ_t as the level set (for definiteness the zero-level set) of the solution $u : \mathbb{R}^N x(0,\infty) \to \mathbb{R}$ of the mean curvature equation

$$u_t = \text{ trace}[(I - \frac{Du \otimes Du}{|Du|^2})D^2 u]. \tag{2}$$

A fundamental analytic question related to (1) and (2) is to construct a global in time unique solution $\{\Gamma_t\}_{t\geq 0}$ for a given initial data Γ_0 (allowing that Γ_t becomes empty in a finite time). This question has been studied extensively over the last few years. In particular, Evans and Spruck [ES 1990] and Chen, Giga and Goto [CGG 1989] have used the theory of viscosity solutions (cf. [CIL 1990]) to construct global solutions to (2). Soner [So 1990] studied (1) directly; he gave an intrinsic characterization for the motion of Γ_t in terms of its distance function. For all the details we refer to the above mentioned references.

The question of uniqueness of the front Γ_t is a very fundamental one from both the analytic point of view as well as for applications, since fronts moving by mean curvature arrive quite naturally in the theory of phase transitions (cf. Evans, Soner and Souganidis [ESoS 1991]).

The following theorem gives a characterization of uniqueness in terms of the regularity of the level sets of the solution of (2).

Theorem 1 ([ES 1989], [So 1990]): *The front Γ_t is determined uniquely by its initial value Γ_0 if, for each $t > 0, \partial[u(\cdot,t)\geq 0] = \partial[u(\cdot,t) > 0]$, where u is the solution of (2), with initial data u_0 such that $\Gamma_0 = [u_0 = 0]$.* □

So far the most general sufficient condition for uniqueness is given by Barles, Soner and Souganidis [BSoS 1991].

Theorem 2 ([BSoS 1991]) *Let Γ_0 be smooth and denote by d the signed distance from Γ_0. If there exist $x_0 \in I\!\!R^N, K \in I\!\!R$ and a skew symmetric matrix Q such that*

$$(I+Q)(x-x_0)\cdot Dd + K\Delta d \neq 0 \text{ on } \Gamma_0, \tag{3}$$

then the surface Γ_t is uniquely determined by Γ_0. □

Condition (3) is by no means necessary. It has, however, a very natural geometric interpretation (cf [BSoS 1991]).

We now turn our attention to the flow by mean curvature of cylindrically symmetric hypersurfaces in $I\!\!R^N (N \geq 3)$. Such surfaces, as long as they are smooth, can be parametrized by

$$r = h(z,t), \tag{4}$$

where h solves the equation

$$h_t = \frac{h_{zz}}{1+(h_z)^2} - \frac{N-2}{h} \tag{5}$$

in some time dependent domain. At this point we need to remark that the parametrization (4) involves two different functions h, which satisfy (5) and match appropriately on the boundary. It is immediate that as long as $h \neq 0$, (5) has classical solutions, therefore Γ_t is unique. We are interested on what happens when h becomes 0. For definiteness let us assume that h becomes 0 for the first time at $(0, T^*)$ and that (5) holds in $\Omega = (-2A, 2A) \times (0, T^*)$, for some $A > 0$. Finally, let us denote by $\tilde{\Omega}$ the set $[-2A, 2A] \times [0, T^*)$.

Our main result is:

Theorem 3 ([SoS 1991]): *Assume that*

$$h > 0, zh_z \geq 0 \text{ and } h(t,0) \leq h(t,z) \text{ in } \tilde{\Omega}.$$

Then

$$\lim_{t \uparrow T^*} (T^* - t)^{-\frac{1}{2}} h(y(T^* - t)^{\frac{1}{2}}, t) = \sqrt{2(N-2)}, \tag{6}$$

with the limit uniform for y bounded. □

A similar type of result was obtained by Huisken [H 1990], under the condition that at $t = 0$ the surface was positive mean curvature. No such assumption is made here. In the sequel we sketch the main steps of the proof of the theorem. The proof goes along the general scheme developed in Giga and Kohn [GK 1985] for the study of the blow-up of solutions of semilinear heat equations.

Sketch of the proof of Theorem 3: In the sequel for simplicity we let $N = 3$. The main step in the proof is to establish the rate at which h vanishes and the fact that h_t remains strictly negative in $[-\delta, , \delta] \times [T^* - \delta, T^*)$ for some sufficiently small $\delta > 0$. The rate follows after the latter is established. To this end observe that if

$$\psi = \frac{hh_{zz}}{1 + (h_z)^2},$$

then $h_t < 0$ if $\psi < 1$.
We have:

Lemma: Under the assumption of Theorem 3 there exists a constant $c < 1$ and $\delta > 0$ such that

$$\psi < c \text{ in } [-\delta, \delta] \times [T^* - \delta, T^*). \qquad \square \tag{7}$$

The proof of this lemma is rather technical. It is based upon constructing appropriate barriers which touch h at any given point and time. The heuristic idea behind (7) is that near focusing the curvature is controlled by its angular component. We refer to [SoS 1991] for the details.

We next establish the rate at which h becomes 0. Since $z = 0$ is a local minimum of h for each t, it is immediate from (5) that

$$h(0,t) \leq (2(T^* - t))^{\frac{1}{2}}.$$

On the other hand, (7) yields

$$h(0,t) \geq (2(1-c)(T^* - t))^{\frac{1}{2}} \text{ in } [T^* - \delta, T^*). \tag{8}$$

Once (7) and (8) have been established we define

$$w(y,s) = (T^* - t)^{-\frac{1}{2}} h(y(T^* - t)^{\frac{1}{2}}, t) \qquad (T^* - t = e^{-s}),$$

which solves the equation

$$w_s = \frac{w_{yy}}{1 + w_y^2} - \frac{1}{2} y w_y + \frac{1}{2} w - \frac{1}{w}.$$

Estimates (7) and (8) yield that there exist a, b such that

$$|w| \leq a(1 + e^{b|y|}), \ |w_s| \leq a(1 + |y|)e^{b|y|} \text{ and } |w_y| \leq a. \tag{9}$$

Therefore, along subsequences, w converges to w^∞ as $s \to \infty$, which, however, may dependent on the choice of the subsequence and s.

The s dependence is eliminated by a monotonicity identity, which we state here under the simplifying assumption that w is defined for all $y \in \mathbb{R}$. We have:

$$\frac{d}{ds} \int \rho w (1 + (w_y)^2)^{\frac{1}{2}} = - \int \rho w (1 + (w_y)^2)^{-\frac{1}{2}} w_s^2 \tag{10}$$

where $\rho(y,s) = \exp(-\frac{1}{4}(w^2(y,s)+y^2))$.

An immediate consequence of (10) is that all limits w^∞ of w are solutions of the stationary equation.

$$\frac{w_{yy}^\infty}{1+(w_y^\infty)^2} - \frac{1}{2}yw_y^\infty + \frac{1}{2}w^\infty - \frac{1}{w^\infty} = 0 \text{ in } \mathbb{R},$$

with $|w^\infty| \leq a(1+e^{|y|b})$ and w_y^∞ locally bounded.

Looking at the equation satisfied by $\left(\frac{w^\infty w_{yy}^\infty}{1+(w_y^\infty)^2}\right)^2$ and utilizing the maximum principle we conclude that $w^\infty = \sqrt{2}$. □

We conclude this short note with some results concerning the uniqueness of the surface Γ_t. In particular, we consider the continuation of Γ_t past T^*, which is given by [ES 1990], [CGG 1989] and [So 1990] and ask whether it is unique.

An immediate consequence of the proof of Theorem 3 is that the singularity at $(0,T^*)$ does not create an "interior". This follows from:

Proposition 4 ([SoS 1991]): *Let u be the solution of (2) with $[u(\cdot,t=0] = [r = h(z,t)]$ for $t < T^*$. Then $(0,0,0) \notin$ int $[u(\cdot,T^*) = 0]$.* □

Next we state a global uniqueness result for torus-like surfaces. A similar result holds for general dumbbell shapes. In [SoS 1991] we also consider uniqueness questions in more general situations.

To state the result, we parametrize the surface by

$$z = g(r,t), \qquad (g > 0)$$

where g solves the equation

$$g_t = \frac{g_{rr}}{1+g_r^2} + \frac{N-2}{r}g_r.$$

Theorem 5 ([SoS 1991]). If $rg_r - g - 2T^*g_t$ has at most 2 zeroes at $t = 0$, then the surface $t \to \Gamma_t$ is unique.

Proof. As long as $h \neq 0$, Γ_t is classical and therefore unique. On the other hand, the assumption at $t = 0$ yields that $h - zh_z \leq 0$ at $t = T^*$ with equality only at $z = 0$. In view then of the fact that $h_t(0,t) \uparrow -\infty$ as $t \uparrow T^*$, we can show that (3) holds for some K for t near T^*. We then conclude by Theorem 2. □

In the case of the torus the surface is given by

$$(r-1)^2 + z^2 = R^2 \quad \text{with } 0 < R < 1.$$

The following is an immediate consequence of Theorem 5.

Proposition 6 [(SoS 1991]): *There exists $R_0 \in (0,1)$ such that for any $R < R_0$ the torus shrinks to a circle and then becomes extinct. For $R > R_0$ the torus "focuses" at 0 at some T_R, but then it opens up ; it becomes "topologically" a sphere, shrinks to a point and then becomes extinct. The surface Γ_t is unique throughout this motion. Finally, for $R = R_0$ the torus focuses at exactly the same time it shrinks to a circle.* □

The above result is related to a conjecture by DeGiorgi [DeG 1990].

Remark: After this note was completed the authors learned that X.-Y. Chen [C 1991] has shown that, in general, bodies of rotation give rise to point singularities.

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We consider the evolution of rotationally symmetric surfaces under mean curvature flow. We obtain local information around the geometric singularities and then prove uniqueness of weak solutions for a large class of initial data. The torus and dumbell are included in this class.

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