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ON THE FORMULATION OF MECHANICAL BALANCE LAWS FOR STRUCTURED CONTINUA

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1. INTRODUCTION

Ericksen [1961,1991], in his studies of liquid crystals, and Toupin [1964], in his theory of oriented hyperelastic materials,[1] introduce the **macroforce** and **microforce balances**

$$\int_{\partial P} \mathbf{T}\mathbf{n} + \int_{P} \mathbf{b} = \mathbf{0}, \tag{1.1}$$

$$\int_{\partial P} \mathbb{T}\mathbf{n} + \int_{P} (\mathbb{b} + \mathbb{k}) = \mathbf{0}, \tag{1.2}$$

in conjunction with the **macromoment balance**

$$\int_{\partial P} \mathbf{p}\times\mathbf{T}\mathbf{n} + \int_{P} \mathbf{p}\times\mathbf{b} + \int_{\partial P} \mathbb{d}\times\mathbb{T}\mathbf{n} + \int_{P} \mathbb{d}\times\mathbb{b} = \mathbf{0}. \tag{1.3}$$

Here (and throughout the paper) the time is fixed, P is a control volume, $\mathbf{n}$ is the outward unit normal to ∂P, and $\mathbf{p}(\mathbf{x})$ is the position vector from the origin:

[1] Cf. Truesdell and Noll [1965], Part (β) of Sect. 98.

$$p(x) = x - o. \tag{1.4}$$

Further, $\mathbf{v}$ is the material velocity, while $\mathbb{d}$, the **director**, is a vector field that represents the fine structure; and the indicated forces and stresses have a physical interpretation suggested by the following nomenclature:

$\mathbf{T}$	**macrostress**
$\mathbf{b}$	**body macroforce**
$\mathbb{T}$	**microstress**
$\mathbb{b}$	**body microforce**
$\mathbb{k}$	**interactive microforce**[2]

with *inertial forces included in the body forces.*

The derivations of (1.1)-(1.3) given by Ericksen and Toupin are essentially variational in nature, and therefore the resulting balance laws are appropriate only to nondissipative systems, although they are often extended by fiat to more general continua.

Here we take a different viewpoint: we derive the basic balance laws as consequences of invariance of power under changes in observer.[3] In a purely mechanical theory the relevant power is that expended to induce changes in energy with time; in the theories described by Ericksen and Toupin this **expended power** has the form

$$\pi(P) = \int_{\partial P} \mathbf{T}\mathbf{n}\cdot\mathbf{v} + \int_{P} \mathbf{b}\cdot\mathbf{v} + \int_{\partial P} \mathbb{T}\mathbf{n}\cdot\dot{\mathbb{d}} + \int_{P} \mathbb{b}\cdot\dot{\mathbb{d}}. \tag{1.5}$$

The standard invariance of (1.5) yields the macroforce balance (1.1) and the macromoment balance (1.2), but it cannot yield the microforce balance (1.3), since the interactive force $\mathbb{k}$ is not present in (1.5), and since the director $\mathbb{d}$ is invariant under Galilean changes.

[2] i.e., the interactive force between the gross and fine structures, to be described later.

[3] Cf. Noll [1963], Green and Rivlin [1964a,1964b].

The novelty of our approach consists in our use of two hypotheses: the availability of a "microscope", mathematized by the introduction of a scale parameter ε; the existence of total and fine power expenses and the invariance of the corresponding power functionals in the limit $\varepsilon \to 0+$. These hypotheses yield, as consequences, the balance laws (1.1)-(1.3) together with a **micromoment balance**

$$\int_{\partial P} p \times \mathbb{T}n + \int_{P} p \times (\mathbb{b} + \mathbb{k}) + \int_{P} \mathbb{m} = 0, \qquad (1.6)$$

where $\mathbb{m}$ is an **interactive microcouple**. The relation (1.6) is not included among the balances derived by Ericksen and Toupin. We view the microcouple $\mathbb{m}$ as indeterminate, since it does not enter our expression (3.6) for the expended power; granted that interpretation, the balance law (1.6) adds nothing essential to the theory.

Finally, for completeness, we write a (mechanical) expression for the second law in terms of a global dissipation inequality, and use its localization, in the standard manner, to deduce constitutive relations for oriented hyperelastic materials and nondissipative liquid crystals.

We do not discuss the form of the inertial terms.[4]

2. GROSS AND FINE CONSTITUENTS. FORCE SYSTEMS. EXPENDED POWER.

We view each material point of the body as a superposition of a gross and a fine structure.[5] We consider the time as fixed in the ensuing discussion, and we assume that the body occupies a region B at this time. We write $v(x)$ for the **material velocity** at the point $x \in B$,

$$\mathbf{v}(x) = \tfrac{1}{2}\operatorname{curl} v(x) \qquad (2.1)$$

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[4]Cf. Ericksen [1961,1991], Toupin [1964].

[5]The terms "gross" and "fine" refer to particular structures, while "macro" and "micro" refer to quantities that enter balance laws such as (1.1) and (1.2).

for the corresponding **spin**, and $f^{\cdot}$ for the material time-derivative of a function f.

Our theory differs from the classical theories of continuum mechanics in the presence of a *scale* for the fine structure: the fine structure "appears" when the system is observed through a microscope of **magnification** $1+\varepsilon$, $\varepsilon>0$. To within terms of $o(\varepsilon)$ this fine structure manifests itself, at each $\mathbf{x}\in B$, through a **director** $\mathbb{d}(\mathbf{x})$ such that $\mathbf{x}+\varepsilon\mathbb{d}(\mathbf{x})$ marks the force center of the fine structure at $\mathbf{x}$. The material point of the fine structure that occupies $\mathbf{x}$ is therefore seen — under the microscope — at $\mathbf{x}+\varepsilon\mathbb{d}(\mathbf{x})$, and hence the effective material velocity of the fine constituent is the **fine velocity**

$$\mathbb{v}_\varepsilon = \mathbf{v} + \varepsilon\mathbb{d}^{\cdot} + o(\varepsilon), \tag{2.2}$$

while the effective spin of the fine constituent is presumed given by

$$\mathbb{W}_\varepsilon = \mathbf{W} + o(1). \tag{2.3}$$

We now introduce force systems for the gross and fine structures; each of these systems should be considered as accounting, in principle, for the self interactions within each structure as well as for the complex mutual interactions between structures. Since the kinetic contribution of the fine structure is $O(\varepsilon)$, corresponding forces will contribute to the power (in the limit $\varepsilon\to0+$) only if these forces are $O(\varepsilon^{-1})$. For that reason we allow the asymptotic expansions of each of the forces to begin with a term of that order. Precisely, there are two distinct force systems, and, to within terms that vanish with ε, they are specified as follows:

gross force system

$$\mathbf{T}_\varepsilon = \varepsilon^{-1}\mathbf{T}_{-1} + \mathbf{T}_0 \qquad \text{stress}$$
$$\mathbf{b}_\varepsilon = \varepsilon^{-1}\mathbf{b}_{-1} + \mathbf{b}_0 \qquad \text{body force}$$

fine force system

$$\mathbb{T}_\varepsilon = \varepsilon^{-1}\mathbb{T}_{-1} + \mathbb{T}_0 \qquad \text{stress}$$
$$\mathbb{b}_\varepsilon = \varepsilon^{-1}\mathbb{b}_{-1} + \mathbb{b}_0 \qquad \text{body force}$$
$$\mathbb{k}_\varepsilon = \varepsilon^{-1}\mathbb{k}_{-1} + \mathbb{k}_0 \qquad \text{interactive force}$$
$$\mathbb{m}_\varepsilon = \varepsilon^{-1}\mathbb{m}_{-1} + \mathbb{m}_0 \qquad \text{interactive couple.}$$

We will refer to the terms subscripted by "-1" and "0" (which are $O(1)$ in ε) as the **singular** and **regular** parts, respectively, of the corresponding fields.

We characterize forces and couples through the manner in which they expend power. The gross stress and body force at a point $x \in B$ expend power over the material velocity $\mathbf{v}(\mathbf{x})$, the fine stress, body force, and interactive force expend power over the fine velocity $\mathbb{v}_\varepsilon(\mathbf{x})$, and the fine couple $\mathbb{m}_\varepsilon(\mathbf{x})$ expends power over the fine spin $\mathbb{w}_\varepsilon(\mathbf{x})$. In the list of gross forces we could also have included an interactive force $\mathbf{k}_\varepsilon$ and an interactive couple $\mathbf{m}_\varepsilon$; since we never study the gross structure by itself, these interactions are irrelevant to our presentation.

In view of this discussion, the **total power expended** on a control volume P is given by

$$\pi_{tot}(\varepsilon, P, \mathbf{v}, \mathbb{d}^\cdot) = \int_{\partial P} (\mathbf{T}_\varepsilon \mathbf{n}\cdot\mathbf{v} + \mathbb{T}_\varepsilon \mathbf{n}\cdot\mathbb{v}_\varepsilon) + \int_P (\mathbf{b}_\varepsilon\cdot\mathbf{v} + \mathbb{b}_\varepsilon\cdot\mathbb{v}_\varepsilon) + o(1); \tag{2.4}$$

interactive forces are not included, since they are presumed *internal* to the combined system of gross and fine structures. Further, the **power expended by the fine structure** has the form

$$\pi_{fine}(\varepsilon,P,v,\mathbb{v},\mathbb{d}') = \int_{\partial P} \mathbb{T}_\varepsilon n\cdot\mathbb{v}_\varepsilon + \int_{P}(\mathbb{b}_\varepsilon\cdot\mathbb{v}_\varepsilon + \mathbb{k}_\varepsilon\cdot\mathbb{v}_\varepsilon + \mathbb{m}_\varepsilon\cdot\mathbb{w}_\varepsilon) + o(1). \qquad (2.5)$$

The **actual power** is the total power expended as observed without a microscope

$$\pi(P,v,\mathbb{d}') = \lim_{\varepsilon\to 0+} \pi_{tot}(\varepsilon,P,v,\mathbb{d}'), \qquad (2.6)$$

and we assume that:

(A1) *the total power* $\pi(P,v,\mathbb{d}')$ *is finite.*

This assumption has two consequences. Firstly, (2.6) and (A1) yield the relation

$$\int_{\partial P}(T_{-1}n + \mathbb{T}_{-1}n)\cdot v + \int_{P}(b_{-1} + \mathbb{b}_{-1})\cdot v = 0, \qquad (2.7)$$

showing that the singular terms in the expansions for the gross and fine stresses and body forces measure mutual interactions of the two structures. (It is tempting to assume that $T_{-1} = -\mathbb{T}_{-1}$, $b_{-1} = -\mathbb{b}_{-1}$, but we do not find this necessary.) Secondly, (2.4) and (2.7) yield

$$\pi(P,v,\mathbb{d}') = \int_{\partial P}(T_0 n + \mathbb{T}_0 n)\cdot v + \int_{P}(b_0 + \mathbb{b}_0)\cdot v + \int_{\partial P}\mathbb{T}_{-1}n\cdot\mathbb{d}' + \int_{P}\mathbb{b}_{-1}\cdot\mathbb{d}'. \qquad (2.8)$$

We therefore arrive at the desired expression (1.5) for the power provided we define

$$\begin{aligned} \mathbb{T} &= \mathbb{T}_{-1}, & \mathbb{b} &= \mathbb{b}_{-1}, \\ T &= T_0 + \mathbb{T}_0, & b &= b_0 + \mathbb{b}_0. \end{aligned} \qquad (2.9)$$

Thus the *microstress is the singular part of the fine stress, while the* *macrostress is the sum of the regular parts of the gross and fine stresses;* analogous assertions apply to the body force.

The power expended by the fine structure is infinite in the limit $\varepsilon \to 0+$, but scaled by ε it is not: we refer to

$$\pi_{s\text{-fine}}(P,\mathbf{v},\mathbb{w},\mathbb{d}^{\cdot}) = \lim_{\varepsilon\to 0+}\{\varepsilon\,\pi_{\text{fine}}(\varepsilon,P,\mathbf{v},\mathbb{w},\mathbb{d}^{\cdot})\} \tag{2.10}$$

as the **scaled fine power**. By (2.5) and (2.9),

$$\pi_{s\text{-fine}}(P,\mathbf{v},\mathbb{w},\mathbb{d}^{\cdot}) = \int_{\partial P}\mathbb{T}\mathbf{n}\cdot\mathbf{v} + \int_{P}(\mathbb{b}\cdot\mathbf{v} + \mathbb{k}\cdot\mathbf{v} + \mathbb{m}\cdot\mathbb{w}), \tag{2.11}$$

with

$$\mathbb{k} = \mathbb{k}_{-1}, \qquad \mathbb{m} = \mathbb{m}_{-1}. \tag{2.12}$$

3. INVARIANCE UNDER CHANGES IN OBSERVER

We consider only observer changes that leave the body invariant at the particular time in question. A change of this type is specified by vectors $\mathbf{c}$ and $\mathbf{q}$ in conjunction with the following transformation rules for the material velocity $\mathbf{v}$, the spin $\mathbb{w}$, and the director velocity $\mathbb{d}^{\cdot}$:

$$\begin{aligned} \mathbf{v}(\mathbf{x}) &\longrightarrow \mathbf{v}^*(\mathbf{x}) = \mathbf{v}(\mathbf{x}) + \mathbf{c} + \mathbf{q}\times\mathbf{p}(\mathbf{x}),\\ \mathbb{w}(\mathbf{x}) &\longrightarrow \mathbb{w}^*(\mathbf{x}) = \mathbb{w}(\mathbf{x}) + \mathbf{q},\\ \mathbb{d}^{\cdot}(\mathbf{x}) &\longrightarrow (\mathbb{d}^{\cdot})^*(\mathbf{x}) = \mathbb{d}^{\cdot}(\mathbf{x}) + \mathbf{q}\times\mathbb{d}(\mathbf{x}), \end{aligned} \tag{3.1}$$

with $\mathbf{c}$ the velocity, $\mathbf{q}$ the spin about the origin, and $\mathbf{p}(\mathbf{x})$, defined by (1.4), the position vector from the origin. The stresses, forces, and couples of the theory are assumed invariant under an observer change of this type.

A basic assumption of our theory is:

(A2) *the actual power as well as the scaled fine power are invariant:*

$$\pi(P,\mathbf{v},\mathbb{d}') = \pi(P,\mathbf{v}^*,(\mathbb{d}')^*),$$

$$\pi_{s\text{-fine}}(P,\mathbf{v},\mathbb{v},\mathbb{d}') = \pi_{s\text{-fine}}(P,\mathbf{v}^*,\mathbb{v}^*,(\mathbb{d}')^*)$$

for all changes in observer.

Because of the linearity (in the kinematic variables) of the expressions for the power, these assumptions of invariance are equivalent to the simpler requirements:

$$\begin{aligned} &\pi(P,\mathbf{c},0) = 0, && \pi(P,\mathbf{q}\times\mathbf{p},\mathbf{q}\times\mathbb{d}) = 0, \\ &\pi_{s\text{-fine}}(P,\mathbf{c},0,0) = 0, && \pi_{s\text{-fine}}(P,0,\mathbf{q}\times\mathbf{p},\mathbf{q}\times\mathbb{d}) = 0 \end{aligned} \tag{3.2}$$

for all vectors $\mathbf{c}$ and $\mathbf{q}$. The relations involving $\mathbf{c}$ yield force balances, while the relations involving $\mathbf{q}$ yield moment balances. Precisely, we have the following:

Result. *If the actual power expense is well defined and invariant under observer changes, and if the fine power expense scaled by* ε *is so invariant in the limit* $\varepsilon \to 0+$, *then:*

(i) *the macro and micro stresses, body forces, and couple defined by* (2.9) *and* (2.12) *obey the force and moment balances* (1.1)-(1.3) *and* (1.6);

(ii) *the expended power has the form* (1.5).

The balances are required to hold for every control volume P and therefore imply the **local force balances**

$$\mathrm{div}\mathbf{T} + \mathbf{b} = 0, \qquad \mathrm{div}\mathbb{T} + \mathbb{b} + \mathbb{k} = 0 \tag{3.3}$$

and the **local moment balances**

$$\begin{aligned} &\mathrm{skw}(T + \mathbb{T}\mathbb{G}^{T} + \mathbb{d}\otimes\mathbb{k}) = 0, \\ &\mathbb{m} = \text{the axial vector of } -2\,\mathrm{skw}\,\mathbb{T}, \end{aligned} \tag{3.4}$$

where $\mathbb{G}$ is the **director gradient**

$$\mathbb{G} = \mathrm{grad}\,\mathbb{d}. \tag{3.5}$$

Further, using the force balances we can express the expended power (1.5) in the form

$$\pi(P) = \int_{P} \{(T + \mathbb{G}^{T}\mathbb{T})\cdot \mathrm{grad}\,v + \mathbb{T}\cdot\mathbb{G}^{\cdot} - \mathbb{k}\cdot\mathbb{d}^{\cdot}\}. \tag{3.6}$$

All of the above relations are with respect to the current configuration of the body. If, instead, we use a fixed reference configuration, and refer the basic quantities to this configuration using "Piola-Kirchhoff" descriptions[6]

$$\begin{aligned} &T_R = JTF^{-T}, \qquad \mathbb{T}_R = J\mathbb{T}F^{-T}, \\ &b_R = Jb, \qquad \mathbb{k}_R = J\mathbb{k}, \qquad \mathbb{b}_R = J\mathbb{b}, \end{aligned} \tag{3.7}$$

where F is the deformation gradient of the (gross) motion relative to this reference configuration and $J = \det F$, then the local balance laws (3.3) take the form

$$\mathrm{Div}\,T_R + b_R = 0, \qquad \mathrm{Div}\,\mathbb{T}_R + \mathbb{b}_R + \mathbb{k}_R = 0, \tag{3.8}$$

with Div the divergence relative to position in the reference configuration.

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[6] Cf., e.g., Gurtin [1981], p. 178.

4. ENERGETICS. DISSIPATION INEQUALITY

Within a mechanical theory of the type considered here, the second law is the assertion that the rate of energy increase cannot be greater than the expended power; precisely, given an arbitrary control volume P,

$$\Big\{\int_P \rho\psi\Big\}^{\cdot} \leq \pi(P), \tag{4.1}$$

where ρ is the **macromass density** (consistent with the usual local mass balance), ψ is the **macroenergy** per unit mass, and the superscript dot here indicates the derivative with respect to time following the material subbody that occupies P at the time in question. Since P is arbitrary, (3.6) and (4.1) yield the **local dissipation inequality**

$$\rho\psi^{\cdot} \leq (T + \mathbb{G}^{T}\mathbb{T})\cdot \mathrm{grad}\, v + \mathbb{T}\cdot\mathbb{G}^{\cdot} - \mathbb{k}\cdot\mathbb{d}^{\cdot}. \tag{4.2}$$

An equivalent relation holds using the referential quantities described in (3.7) in conjunction with the macroenergy (per unit referential volume),

$$\psi_R = J\rho\psi; \tag{4.3}$$

the resulting relation is

$$\psi_R^{\cdot} \leq T_R\cdot F^{\cdot} + \mathbb{T}_R\cdot\mathbb{G}_R^{\cdot} - \mathbb{k}_R\cdot\mathbb{d}^{\cdot}, \tag{4.4}$$

with

$$\mathbb{G}_R = \nabla\mathbb{d}, \tag{4.5}$$

∇ denoting the referential gradient.

5. ORIENTED HYPERELASTIC MATERIALS

We now specialize the theory to oriented hyperelastic materials defined by constitutive equations of the form

$$\psi_R, \quad T_R, \quad \mathbb{T}_R, \quad \mathbb{k}_R \quad \text{functions of} \quad (F, \mathbb{G}_R, \mathbb{d}). \tag{5.1}$$

The standard proceedure[7] of requiring that all "processes" related through the constitutive relations (5.1) be consistent with the dissipation inequality (4.2) then yields the result that the constitutive equation

$$\psi_R = \Psi(F, \mathbb{G}_R, \mathbb{d}) \tag{5.2}$$

for the macroenergy determines the other constitutive quantities through the relations

$$\begin{aligned} T_R &= \partial_F \Psi(F, \mathbb{G}_R, \mathbb{d}), \\ \mathbb{T}_R &= \partial_{\mathbb{G}_R} \Psi(F, \mathbb{G}_R, \mathbb{d}), \\ \mathbb{k}_R &= -\partial_{\mathbb{d}} \Psi(F, \mathbb{G}_R, \mathbb{d}). \end{aligned} \tag{5.3}$$

6. LIQUID CRYSTALS

As is standard with theories of fluid behavior, we refer all quantities to the current configuration. We assume that the body is incompressible and homogeneous in the sense that

$$\operatorname{div} v = 0, \qquad \rho = \text{constant}; \tag{6.1}$$

we write the stress $T + \mathbb{G}^T \mathbb{T}$ in the form

[7] Cf. Coleman and Noll [1963].

$$T + \mathbb{G}^T\mathbb{T} = S + p1,$$

with S, traceless, the "extra stress" and p the (indeterminate) pressure; and we lay down constitutive equations of the form

$$\psi,\quad S,\quad \mathbb{T},\quad \mathbb{k} \quad \text{functions of} \quad (\mathbb{G},\mathbb{d}). \tag{6.2}$$

Compatibility with the dissipation inequality (4.2) then yields the reduced equations

$$\begin{aligned} T &= -p1 - \mathbb{G}^T\mathbb{T}, \\ \mathbb{T} &= \partial_{\mathbb{G}}\Phi(\mathbb{G},\mathbb{d}), \\ \mathbb{k} &= -\partial_{\mathbb{d}}\Phi(\mathbb{G},\mathbb{d}), \end{aligned} \tag{6.3}$$

with Φ the response function for the macroenergy:

$$\psi = \Phi(\mathbb{G},\mathbb{d}). \tag{6.4}$$

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