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Jacobians and Hardy Spaces

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FRANCE

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Center for Nonlinear Analysis

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Introduction

We wish to illustrate here the links between various nonlinear quantities identified by the compensated–compactness method (L. Tartar [9],[10]), F. Murat [6],[7]) and the Hardy spaces. A systematic study of this matter can be found in R. Coifman, P. L. Lions, Y. Meyer and S. Semmes [3], and we want to illustrate it on the particular example of the Jacobian.

More precisely, we consider $J(u) = \det(\nabla u)$ when $u \in W^{1,p}(\mathbb{R}^N)^N$ for some $p \in [1,\infty[$ where $N \geq 2$. This quantity clearly makes sense in $L^1(\mathbb{R}^N)$ if $p=N$; however, in that case, there are various reasons to guess that $J(u)$ might be slightly better than L^1. First of all, it is a classical fact that J is weakly sequentially continuous on $W^{1,N}(\mathbb{R}^N)^N$ – this fact is one of the key ingredients in J. Ball's theory of polyconvex functionals in Nonlinear Elasticity [1]. Next, several results by H. Wente [12], L. Tartar [11] indicate that L^1 is not optimal – see also H. Brézis and J. M. Coron [2]. Finally, the last piece of evidence is a striking recent result due to S. Müller [4] showing that if $u \in W^{1,N}_{loc}(\mathbb{R}^N)^N$ and $J(u) \geq 0$ a.e., then $J(u)\log(1+J(u)) \in L^1_{loc}(\mathbb{R}^N)$.

We will show below in Section 2 how all these results can be recovered from the following statement: $J(u) \in \mathcal{H}^1(\mathbb{R}^N)$ if $u \in W^{1,N}(\mathbb{R}^N)^N$. Here and everywhere below, we denote by $\mathcal{H}^p(\mathbb{R}^N)$ the following Hardy spaces:

(1) $$\mathscr{H}^p(\mathbb{R}^N) = \{f \in S'(\mathbb{R}^N) \,/\, \sup_{t>0} |h_t * f| \in L^p(\mathbb{R}^N)\} \text{ , } p > 0,$$

where h_t is a regularization kernel satisfying, for example

(2) $$h_t = \frac{1}{t^N} h\left(\frac{\cdot}{t}\right),\ h \in \mathscr{D}(\mathbb{R}^N),\ 0 \le h \text{ on } \mathbb{R}^N,\ \text{Supp}\, h \subset B_1,$$

where we denote by B_λ the open ball of radius λ and by $B_\lambda(x) = B(x,\lambda)$ the open ball centered at x of radius λ ($B_\lambda = B_\lambda(0) = B(0, \lambda)$). We shall see in fact that $J(u) \in \mathscr{H}^p(\mathbb{R}^N)$ if $u \in W^{1,pN}(\mathbb{R}^N)^N$, $\frac{N}{N+1} < p \le 1$ and that $J(\nabla\phi) = \det(D^2\phi) \in \mathscr{H}^p(\mathbb{R}^N)$ if $\phi \in W^{2,pN}(\mathbb{R}^N)^N$, $\frac{N}{N+2} < p \le 1$. Of course, one has to interpret $J(u)$ or $J(\nabla\phi)$ in a distribution sense explained below when $p < 1$.

We present in Section 3 a proof of these facts. To simplify the presentation and the notations we restrict our attention here to the case when $N = 2$.

2. Main results and consequences.

Of course, if $u \in W^{1,2}(\mathbb{R}^2)^2$, $J(u) = \det(\nabla u) \in L^2(\mathbb{R}^2)$. But, one easily checks that

(3) $$J(u) = \partial_1 (u_1 \partial_2 u_2) - \partial_2 (u_2 \partial_1 u_2) \text{ in } \mathcal{D}'(\mathbb{R}^2),$$

and this last expression is well–defined (in the sense of distributions) whenever $u \in W^{1,\frac{4}{3}}(\mathbb{R}^2)^2$: indeed, one then deduces from Sobolev embeddings that $u \in L^4(\mathbb{R}^2)^2$ and thus $|u|\ |\nabla u|$ is integrable.

Next, if $u = \nabla\phi$, further cancellations of $J(u)$ take place and we may write

(4) $$J(\nabla\phi) = -\frac{1}{2}\partial_1^2((\partial_2\phi)^2) - \frac{1}{2}\partial_2^2((\partial_1\phi)^2) + \partial_{12}^2(\partial_1\phi\,\partial_2\phi) \text{ in } \mathcal{D}'(\mathbb{R}^2)$$

or

(5) $$J(\nabla\phi) = \frac{1}{2}\partial_1^2(\phi\,\partial_2^2\phi) + \frac{1}{2}\partial_2^2(\phi\,\partial_1^2\phi) - \partial_{12}^2(\phi\,\partial_{12}^2\phi) \text{ in } \mathcal{D}'(\mathbb{R}^2).$$

Note that (4) makes sense as soon as $\phi \in W^{1,2}(\mathbb{R}^2)$ while (5) makes sense if $\phi \in W^{2,1}(\mathbb{R}^2)$ (since it implies $\phi \in C_0(\mathbb{R}^2)$). In fact, the two expressions are easily shown to be equal if $\phi \in W^{2,1}(\mathbb{R}^2)$ since $W^{2,1}(\mathbb{R}^2)$ embeds into $W^{1,2}(\mathbb{R}^2)$ and $\partial_j(\phi\,\partial_j\phi) = \phi\,\partial_j^2\phi + (\partial_j\phi)^2$ $(\forall_j)$, $\partial_1\phi\,\partial_2\phi + \phi\,\partial_{12}^2\phi = \frac{1}{2}\partial_1(\phi\,\partial_2\phi) + \frac{1}{2}\partial_2(\phi\,\partial_1\phi)$. Notice, finally, that by Sobolev embeddings, (4) makes sense if $\phi \in \bar{W}^{2,1}(\mathbb{R}^2) = \{\phi \in W^{1,1}(\mathbb{R}^2) /\ \partial_{i,j}^2\phi$ is a bounded measure on $\mathbb{R}^2(\forall_{i,j})\}$.

We may now state our main results:

Theorem 1: Let $p \in (\frac{4}{3}, 2]$ and let $u \in W^{1,p}(\mathbb{R}^2)^2$, then $J(u) \in \mathscr{H}^{p/2}(\mathbb{R}^2)$.

Theorem 2: Let $p \in (1,2]$ and let $\phi \in W^{2,p}(\mathbb{R}^2)$, then $J(\nabla\phi) \in \mathscr{H}^{p/2}(\mathbb{R}^2)$.

Remarks: 1) These results also hold locally.

2) The borderline cases $p = \frac{4}{3}$ or $p = 1$ can also be studied – see also section 3 below.

3) One recovers immediately S. Müller's result [4] from Theorem 1 since if $f \in L^1_{loc}$, $f \geq 0$ a.e. then $f \in \mathscr{H}^1_{loc}(\mathbb{R}^N)$ if and only if $f \log(1+f) \in L^1_{loc}$ (see E. Stein [8]).

4) Observing that $\mathscr{H}^1(\mathbb{R}^2) \in W^{-1,2}(\mathbb{R}^2)$, one deduces that $W = (-\Delta)^{-1} J(u) \in W^{1,2}(\mathbb{R}^2)$ and $J(u) \in W^{-1,2}(\mathbb{R}^2)$ if $u \in W^{1,2}(\mathbb{R}^2)^2$. Furthermore, in that case, one sees that $\partial_{i,j} W \in \mathscr{H}^1(\mathbb{R}^2)$ $(\forall_{i,j})$ and this yields: $W \in \mathscr{F}L^1(\mathbb{R}^2)$. We recover in this way the results mentioned in the Introduction.

5) One can define a linear continuous map P from $\mathscr{H}^q(\mathbb{R}^N)$ into $L^q(\mathbb{R}^N)$ for $0 < q < 1$ which consists in taking the "a. e. part" of a distribution f in $\mathscr{H}^q(\mathbb{R}^N)$: more precisely, $Pf = \lim_{t\to 0} \text{a.e. } h_t * f$. Of course, $Pf = f$ if $f \in \mathscr{H}^1$ or L^1 and Pf is the regular part of f if f is a bounded measure.

Next, one remarks that when $p < 2$, one can also define a.e. $\det(\nabla u)$ or $\det(D^2\phi)$ obtaining thus a measurable function which lies obviously in $L^{p/2}$. We denote by $\mathrm{Det}(\nabla u)$ or $\mathrm{Det}(D^2\phi)$ these functions. Then, the above results yield easily: $\mathrm{Det}(\nabla u) = P(J(u))$,

$\mathrm{Det}(D^2\phi) = P(J(\nabla\phi))$.

These relations yield and extend another recent result of S. Müller [5].

3 Proofs

Theorems 1 – 2 follow immediately from the following lemma and the classical maximal theorem:

Lemma 3: 1) Let $u \in W^{1,4/3}(\mathbb{R}^2)^2$; then we have for all $t > 0$, $x \in \mathbb{R}^2$

(6) $$| h_t * J(u) | \leq C_0 \left(⨍_{B(x,t)} | Du |^{4/3} dx \right)^{3/2}.$$

(2) Let $\phi \in W^{2,1}(\mathbb{R}^2)$; then we have for all $t > 0$, $x \in \mathbb{R}^2$

(7) $$| h_t * J(\nabla\phi) | \leq C_0 \left(⨍_{B(x,t)} | D^2\phi | dx \right)^2.$$

Remarks:

1) Here and everywhere below, C_0 denotes various constants independent of t, x, u, ϕ.

2) The estimate (7) is still true for $\phi \in \bar{W}^{2,1}$ provided we define $J(\nabla\phi)$ by (4) and the right hand side is replaced by the total mass of the measure $\sum_{i,j} \left| \frac{\partial^2 \phi}{\partial x_j \partial x_j} \right|$ on the ball $B(x,t)$.

3) Those estimates allow, in fact, to investigate the borderline cases $p = \frac{4}{3}$ or $p = 1$.

Proof of Lemma 3: We begin with Part 1). Using (3) and integrating by parts, we find

$$(8)\qquad h_t * J(u) = \int u_1(y)\,[\partial_2 u_2(y)\frac{1}{t^3}\,\partial_1 h(\tfrac{x-y}{t}) - \partial_1 u_2(y)\frac{1}{t^3}\,\partial_2 h(\tfrac{x-y}{t})]dy.$$

But, clearly, these expressions are left invariant if we subtract constants from u_1 and u_2. Therefore, in particular, we have, denoting by $\displaystyle ⨍_{B(x,t)} u = \frac{1}{|B(x,t)|}\int_{B(x,t)} u(y)dy$,

$$(9)\qquad h_t * J(u) = \int \tfrac{1}{t}(u_1 - ⨍_{B(x,t)} u_1)\,[\partial_2 u_2(y)\frac{1}{t^2}\,\partial_1 h(\tfrac{x-y}{t}) - \partial_1 u_2(y)\frac{1}{t^2}\,\partial_2 h(\tfrac{x-y}{t})]dy.$$

We now apply Hölder's inequality to find

$$(10)\qquad |h_t * J(u)| \le C_0 (⨍_{B(x,t)} |\tfrac{1}{t}(u_1 - ⨍_{B(x,t)} u_1)|^4\,dy)^{1/4} (⨍_{B(x,t)} |Du|^{4/3}dy)^{3/4}.$$

And we obtain (6) by recalling the Sobolev–Poincaré's inequality

$$(11)\qquad (⨍_{B(x,t)} \tfrac{1}{t}(u_1 - ⨍_{B(x,t)} u_1)|^4 dy)^{1/4} \le C_0 (⨍_{B(x,t)} |Du|^{4/3}\,dy)^{3/4}.$$

We now turn to part 2), which is proven in a similar way using either (4) or (5): we use for instance (4) and obtain as above

$$(12)\qquad h_t * J(\nabla\phi) = -\tfrac{1}{2}\int (\partial_2\phi(y)^2\,\frac{1}{t^4}\,\partial_1^2\, h(\tfrac{x-y}{t})dy +$$

$$-\tfrac{1}{2}\int (\partial_1\phi(y)^2\,\frac{1}{t^4}\,\partial_2^2\, h(\tfrac{x-y}{t})dy + \int \partial_1\phi(y)\,\partial_2\phi(y)\,\frac{1}{t^4}\,\partial_{12}^2\, h(\tfrac{x-y}{t})dy.$$

Then we observe that this quantity is left invariant if we add to ϕ an arbitrary affine function so that, in other words, we may subtract respectively from $\partial_1\phi, \partial_2\phi$ the following quantities

$$\int_{B(x,t)} \partial_1\phi\,, \quad \int_{B(x,t)} \partial_2\phi.$$

We then find

$$|h_t * J(\nabla\phi)| \le C_0 \fint_{B(x,t)} \left| \frac{1}{t}\left\{\nabla\phi - \fint_{B(x,t)} \nabla\phi\right\}\right|^2 dy, \tag{13}$$

and we conclude again by Sobolev–Poincaré's inequality:

$$\fint_{B(x,t)} \left|\frac{1}{t}\left\{f - \fint_{B(x,t)} f\right\}\right|^2 dy \le C_0 \left(\fint_{B(x,t)} |Df|\, dy\right)^2, \text{ for all } f \in W^{1,1}(\mathbb{R}^2).$$

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13. ABSTRACT *(Maximum 200 words)*

Motivated by questions in nonlinear elasticity, Stefan Müller has recently proved that if $u \in (W^{1,N}(R^N))^N$ satisfies $J(u) = \det \nabla u \geq 0$ almost everywhere, then one has $J(u)\log(1+J(u))$ $L^1loc(R^N)$. This kind of result has been generalized in the context of Hardy spaces by R. Coifman and P.L. Lions and Y. Meyer and S. Semmes. The present report gives some elementary proofs, explained in the simple solution of R^2, of the embedding results of Jacobian determinants into Hardy spaces.

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