


# A Regularized Equation for Anisotropic Motion-by-Curvature

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## 1. Introduction.

There are situations of interest in which the motion of a phase interface is essentially independent of the behavior of the corresponding bulk phases. One of the first models of such phenomena was proposed by Mullins [Mu] to study the *planar* motion of grain boundaries; this model is based on the relation[1]

$$V = K, \tag{1.1}$$

with $V$ the *normal velocity* and $K$ the *curvature* of the interfacial curve. The relation (1.1) yields a *parabolic* partial differential equation for the evolution of the interface; when the interface is represented (locally) as a graph $y=h(x,t)$, this equation has the form[2]

$$h_t = (\sin^2\theta)h_{xx}\,, \tag{1.2}$$

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[1] Allen and Cahn [AC] and Rubinstein, Sternberg, and Keller [RSK] deduce the equation $V=K$ as a formal approximation to the Landau-Ginzburg equation.

[2] Subscripts denote partial derivatives with respect to the corresponding variable; a prime will denote differentiation with respect to $\theta$.

where $\theta$ is the angle to the interface normal, chosen so that $\theta\in(0,\pi)$ and $h_x \tan\theta = -1$. The relation (1.1) has a large mathematical literature;[3] its major consequence[4] is that all interfaces, no matter their initial shape, shrink to a point in finite time, with asymptotic shape a circle.

Mullins's theory was generalized in [G1] (cf. [AG]) to include anisotropy and the possibility of a difference in bulk energies between phases. The resulting equation is[5]

$$a(\theta)V = [f(\theta) + f''(\theta)]K - F, \tag{1.3}$$

with $f(\theta)>0$ the interfacial energy, $a(\theta)>0$ a kinetic coefficient, and $F$ the (constant) difference in bulk energies. When

$$g(\theta) := f(\theta) + f''(\theta) \tag{1.4}$$

is strictly positive, the underlying partial differential equation is parabolic and the interface is fairly well-behaved;[6] linearization of (1.3) about a flat interface yields the "heat equation"

$$u_t = \alpha u_{xx} \tag{1.5}$$

for the slope

$$u := h_x. \tag{1.6}$$

Here $\alpha = g(\pi/2)/a(\pi/2)$ when the angle of the flat interface is $\theta = \pi/2$, an assumption that involves no loss in generality.

Material scientists use energies for which $g(\theta)<0$ for certain intervals of $\theta$ [CH,Gj]. Unfortunately, this renders the underlying

[3] Cf. Brakke [BR], Sethian [Se], Abresch and Langer [AL], Gage and Hamilton [GH], Grayson [Gr], Osher and Sethian [OS], Evans and Spruck [ES], and the references therein.

[4] [GH], [Gr].

[5] The special case $V=-\Psi(\theta)$ was introduced by Frank [Fr].

[6] Cf. Angenent [An]; Chen, Giga, and Goto [CGG]; Soner [So].

differential equation *backward parabolic* and inherently unstable; in fact, (1.5) is then the backward heat-equation, since $\alpha < 0$. One can exclude the unstable angles by inserting corners in the crystal [AG]; this leads to facets and wrinklings, and to a free-boundary problem since the positions of the corners vary with time.

To analyze behavior within the unstable range of angles a regularization of the underlying equations is needed. Such a regularization, proposed in [AG], has the form

$$a(\theta)V = g(\theta)K - F - \varepsilon(K_{ss} + \tfrac{1}{2}K^3) \qquad (s = \text{arc length}) \qquad (1.7)$$

with $\varepsilon > 0$ small, and leads to a fourth-order *parabolic* equation for the evolution of the interface.

The angle-intervals in which $g(\theta) < 0$ are very much like the spinodals encountered in the Korteweg-Cahn-Hilliard theory of phase transitions (cf., e.g., [Ca]), and we refer to such intervals as *interfacial spinodals.* When linearized about a flat interface at angle $\theta = \pi/2$, (1.7) has the form

$$u_t = \alpha u_{xx} - \delta u_{xxxx} \qquad (1.8)$$

with $\alpha = g(\pi/2)/a(\pi/2)$, $\delta = \varepsilon/a(\pi/2)$; and $\alpha < 0$ when $\theta = \pi/2$ lies in a spinodal. The relation (1.8) is the linear equation used by Cahn [Ca] in his treatment of spinodal decomposition.

The main purpose of this paper is to present a derivation of (1.7). The chief ingredient is a *constitutive* dependence of interfacial energy on curvature, an idea that traces back to Gibbs [Gi] and Herring [He]. A curvature-dependent energy requires interfacial couples and a corresponding balance law for torques, a possibility not encompassed within the framework of [G1,G2,AG].

We begin with a discussion of force and torque balances for evolving interfacial curves. We next introduce an inequality that represents a mechanical version of the second law, and use this inequality to deduce suitable constitutive relations for the interface. The resulting constitutive equations and the force and torque balances together yield the evolution equation (1.7). Finally, we discuss the

similarities between the theory based on (1.7) and Cahn's theory of spinodal decomposition.

## 2. Grains.

We consider a planar theory in which a *grain* $\Omega$ is a region in $\mathbb{R}^2$ with boundary $\partial\Omega$ the *interface* between the grain and its *ancillary phase*. We assume that $\partial\Omega$ is a smooth curve, we write $\mathbf{n}(\mathbf{x})$ for the outward unit normal to $\partial\Omega$ at $\mathbf{x}$, and we define a unit tangent $\boldsymbol{\ell}(\mathbf{x})$, the direction of increasing *arc length* $s$, so that $\{\boldsymbol{\ell}(\mathbf{x}),\mathbf{n}(\mathbf{x})\}$ is a positively oriented basis of $\mathbb{R}^2$. We then have the Frenet formulae

$$\mathbf{n}_s = -K\boldsymbol{\ell}, \qquad \boldsymbol{\ell}_s = K\mathbf{n}, \tag{2.1}$$

with $K(\mathbf{x})$ the *curvature* of $\partial\Omega$. We define the *angle* $\theta(\mathbf{x})$, as a smooth function of $\mathbf{x}$, through $\mathbf{n} = (\cos\theta,\sin\theta)$, $\boldsymbol{\ell} = (\sin\theta,-\cos\theta)$; then

$$K = \theta_s. \tag{2.2}$$

We consider grains $\Omega(t)$ that evolve with time $t$, under the assumption that $\partial\Omega(t)$ is a smoothly evolving curve (in the sense of [AG]). We write $V(\mathbf{x},t)$ for the *normal velocity* of $\partial\Omega(t)$ in the direction $\mathbf{n}(\mathbf{x},t)$. Let $\mathbf{V}(\mathbf{x},t) = V(\mathbf{x},t)\mathbf{n}(\mathbf{x},t)$. Fix $t$ and $\mathbf{x}\in\partial\Omega(t)$ and (for $\tau$ sufficiently close to $t$) let $\mathbf{y}(\tau)$ denote the curve that passes through $\mathbf{x}$ at time $t$ and has

$$d\mathbf{y}(\tau)/d\tau = \mathbf{V}(\mathbf{y}(\tau),\tau). \tag{2.3}$$

Then the *normal time-derivative* $\Phi^\circ(\mathbf{x},t)$ (following $\partial\Omega(t)$) of a scalar or vector function $\Phi(\mathbf{x},t)$ is defined by

$$\Phi^\circ(\mathbf{x},t) := (d/d\tau)\Phi(\mathbf{y}(\tau),\tau)\big|_{\tau=t}, \tag{2.4}$$

and we have the identities[7]

[7] Cf. (2.17) and (2.16) of [AG].

$$\theta^\circ = V_s, \qquad K^\circ = V_{ss} + K^2 V,$$
$$(\Phi^\circ)_s = (\Phi_s)^\circ - KV\Phi_s. \tag{2.5}$$

An interface is *convex* if $K$ nowhere vanishes. In this case we may use $(\theta,t)$ as independent variables when describing interfacial quantities: writing $K=K(\theta,t)$ and $V=V(\theta,t)$, we have the identity[8]

$$K_t = K^2(V_{\theta\theta} + V). \tag{2.6}$$

By an *interfacial subcurve* we mean a smoothly evolving curve $\mathcal{n}(t)$ with $\mathcal{n}(t)\subset\partial\Omega(t)$ at each time $t$. Let $\mathcal{n}(t)$ be an interfacial subcurve; let $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ denote the initial and terminal points of $\mathcal{n}(t)$; let $\mathbf{v}_1(t)$ and $\mathbf{v}_2(t)$ denote the corresponding *endpoint velocities*

$$\mathbf{v}_1(t) := d\mathbf{x}_1(t)/dt, \qquad \mathbf{v}_2(t) := d\mathbf{x}_2(t)/dt. \tag{2.7}$$

Further, for any smooth function $f(\mathbf{x},t)$, let

$$f_i(t) := f(\mathbf{x}_i(t),t), \qquad f_i^{\cdot}(t) := (d/dt)f(\mathbf{x}_i(t),t), \tag{2.8}$$

so that the superscript dot here signifies time-differentiation following an endpoint. We then have the identities

$$\mathbf{n}_i\cdot\mathbf{v}_i = V_i, \qquad f_i^{\cdot} = (f^\circ)_i + (f_s)_i\,\boldsymbol{\ell}_i\cdot\mathbf{v}_i. \tag{2.9}$$

The identity $(2.9)_1$ follows from $(2.20)_1$ of [AG]; to establish $(2.9)_2$ we take $f=f(s,t)$ in (2.8) and use (2.16) and $(2.20)_1$ of [AG].

Given a smooth function $\Phi(\mathbf{x},t)$ and a smooth vector function $\mathbf{C}(\mathbf{x},t)$, we write[9]

---

[8] Cf. $(2.23)_1$ of [AG].

[9] Here and henceforth we omit the argument $t$ in the integration-set.

$$\begin{aligned}
&\int_{\partial\mathcal{R}}\Phi \;:=\; \Phi_2 - \Phi_1,\\
&\int_{\partial\mathcal{R}}\mathbf{C}\cdot\mathbf{v}_{\partial\mathcal{R}} \;:=\; \mathbf{C}_2\cdot\mathbf{v}_2 - \mathbf{C}_1\cdot\mathbf{v}_1, \qquad (2.10)\\
&\int_{\partial\mathcal{R}}\Phi(f_{\partial\mathcal{R}})^{\cdot} \;:=\; \Phi_2 f_2^{\cdot} - \Phi_1 f_1^{\cdot}.
\end{aligned}$$

We then have the following integral identities; in the third of these $\sigma$ and $\xi$ are the tangential and normal components of $\mathbf{C}$ (see (3.1) below).

$$\begin{aligned}
\int_{\partial\mathcal{R}}\Phi &= \int_{\mathcal{R}}\Phi_s\,ds,\\
(d/dt)\int_{\mathcal{R}}\Phi\,ds &= \int_{\mathcal{R}}(\Phi^{\circ} - \Phi KV)\,ds + \int_{\partial\mathcal{R}}\Phi\boldsymbol{\ell}\cdot\mathbf{v}_{\partial\mathcal{R}}, \qquad (2.11)\\
\int_{\partial\mathcal{R}}\mathbf{C}\cdot\mathbf{v}_{\partial\mathcal{R}} &= \int_{\mathcal{R}}(\xi_s V + \xi\theta^{\circ})\,ds + \int_{\partial\mathcal{R}}\sigma\boldsymbol{\ell}\cdot\mathbf{v}_{\partial\mathcal{R}},\\
\int_{\partial\mathcal{R}}\Phi(f_{\partial\mathcal{R}})^{\cdot} &= \int_{\mathcal{R}}[\Phi_s f^{\circ} + \Phi(f_s)^{\circ} - KV\Phi f_s]\,ds + \int_{\partial\mathcal{R}}\Phi f_s\boldsymbol{\ell}\cdot\mathbf{v}_{\partial\mathcal{R}}.
\end{aligned}$$

Equation $(2.11)_1$ is a direct consequence of $(2.10)_1$; $(2.11)_2$ follows from $(2.20)_1$, (2.32), and (2.34) of [AG]; $(2.11)_3$ follows from $(2.5)_1$, (2.9), and $(2.11)_1$; $(2.11)_4$ is a consequence of $(2.5)_3$ and $(2.9)_2$.

## 3. Mechanics. Balance of forces and torques.

We consider a theory in which the mechanics of an evolving grain $\Omega(t)$ is described by four functions of $\mathbf{x}\in\partial\Omega(t)$ and time $t$:

| | |
|---|---|
| $\mathbf{C}(\mathbf{x},t)$ | *interfacial stress,* |
| $\mathbf{M}(\mathbf{x},t)$ | *interfacial couple-stress,* |
| $\mathbf{b}(\mathbf{x},t)$ | *body force,* |
| $\mathbf{m}(\mathbf{x},t)$ | *body couple.* |

$\mathbf{C}(\mathbf{x},t)$ and $M(\mathbf{x},t)$ represent the force and couple *within the interface* exerted across $\mathbf{x}$ at time t, while $\mathbf{b}(\mathbf{x},t)$ and $m(\mathbf{x},t)$ represent the force and couple exerted on the interface by the bulk material of the grain and its ancillary phase.

We decompose the interfacial stress into normal and tangential components,

$$\mathbf{C} = \sigma\boldsymbol{\ell} + \xi\mathbf{n}, \tag{3.1}$$

with $\sigma(\mathbf{x},t)$ the *surface tension* and $\xi(\mathbf{x},t)$ the *surface shear,* and we refer to

$$\pi := \mathbf{b}\cdot\mathbf{n}, \tag{3.2}$$

as the *normal interaction.*

The balance laws of our theory are *balance of forces and torques:*

$$\begin{aligned} \int_{\partial\mathcal{n}}\mathbf{C} + \int_{\mathcal{n}}\mathbf{b}\,ds &= \mathbf{0}, \\ \int_{\partial\mathcal{n}}(\mathbf{r}\times\mathbf{C} + M) + \int_{\mathcal{n}}(\mathbf{r}\times\mathbf{b} + m)ds &= 0, \end{aligned} \tag{3.3}$$

with $\mathbf{r}(\mathbf{x}) = \mathbf{x} - \mathbf{x}_0$ the position vector from an arbitrary point $\mathbf{x}_0$, and with "$\times$" denoting the scalar-valued vector product in $\mathbb{R}^2$. These balance laws must hold for all interfacial subcurves $\mathcal{n}(t)$, and are hence equivalent to the local balance laws

$$\mathbf{C}_s + \mathbf{b} = \mathbf{0}, \qquad M_s + m + \xi = 0, \tag{3.4}$$

where we have used the identities $\boldsymbol{\ell} = \mathbf{r}_s$ and $\xi = \boldsymbol{\ell}\times\mathbf{C}$. Note that, by (2.1), the normal component of $(3.4)_1$ is

$$\xi_s + \sigma K + \pi = 0. \tag{3.5}$$

We characterize forces and couples by the manner in which they

expend power. Consider an arbitrary interfacial subcurve $\mathcal{R}(t)$. Since $\mathbf{C}$ and $M$ act on the endpoints of $\mathcal{R}(t)$, we assume that $\mathbf{C}$ expends power over the endpoint velocities, while $M$ expends power over the rotation rate of the interface following the endpoints. On the other hand, since $\mathbf{b}$ and $m$ act along the length of $\mathcal{R}(t)$, we assume that $\mathbf{b}$ expends power over the normal velocity $V$ of the interface, while $m$ expends power over the rotation rate $\theta^\circ$ following the interface. The *power expended* on $\mathcal{R}(t)$ is therefore given by[10]

$$\int_{\partial\mathcal{R}} [\mathbf{C}\cdot\mathbf{v}_{\partial\mathcal{R}} + M(\theta_{\partial\mathcal{R}})^\cdot] + \int_{\mathcal{R}} (\mathbf{b}\cdot V\mathbf{n} + m\theta^\circ)\,ds \tag{3.6}$$

and, in view of $(2.11)_{3,4}$, $(3.4)_2$, and (3.5), has the alternative form

$$\int_{\mathcal{R}} [MK^\circ - (\sigma + MK)KV]\,ds + \int_{\partial\mathcal{R}} (\sigma + MK)\boldsymbol{\ell}\cdot\mathbf{v}_{\partial\mathcal{R}}. \tag{3.7}$$

The term $MK^\circ$ represents power expended in bending the interface, while $-(\sigma + MK)KV$ represents power expended in creating new interface. Here, interestingly, *both the surface tension and the interfacial couple-stress work to create new surface.* The final term $(\sigma + MK)\boldsymbol{\ell}\cdot\mathbf{v}_{\partial\mathcal{R}}$ compensates for the tangential motion of the interface.

Combining $(3.4)_2$ and (3.5), we arrive at the basic balance law of our theory:

$$M_{ss} + m_s - \sigma K - \pi = 0. \tag{3.8}$$

## 4. Energetics. The dissipation inequality.

We associate with each interfacial motion two energies:

$f(\mathbf{x},t)$ *interfacial energy,*
$F$ *bulk energy.*

---

[10] Cf. [G1-G3] for the terms involving $\mathbf{C}$ and $\mathbf{b}$.

$f(x,t)$, defined for $x \in \partial\Omega(t)$, represents the energy of the interface per unit length; $F$, measured per unit area and assumed constant, is the bulk energy of the grain minus that of the ancillary phase.

Let $R$ be a fixed region of space and let

$$\mathcal{R}(t) := \Omega(t) \cap R, \qquad \mathcal{r}(t) := \partial\Omega(t) \cap R.$$

Then

$$\int_{\mathcal{r}} f\,ds + \int_{\mathcal{R}} F\,da$$

represents the total energy of $R$, while

$$\int_{\partial\mathcal{r}} [\mathbf{C}\cdot\mathbf{v}_{\partial\mathcal{r}} + M(\theta_{\partial\mathcal{r}})^{\cdot}]$$

is the power expended on $R$ by the interfacial force and moment. The body force $\mathbf{b}$ and the body couple $m$ represent interactions *within* $R$ (cf. (3.6)); consequently they do not expend power on $R$.

For a purely mechanical theory the *second law* is the assertion that the rate of energy increase cannot be greater than the power expended. A precise statement of this *global dissipation inequality* is that

$$(d/dt)\left\{\int_{\mathcal{r}} f\,ds + \int_{\mathcal{R}} F\,da\right\} \leq \int_{\partial\mathcal{r}} [\mathbf{C}\cdot\mathbf{v}_{\partial\mathcal{r}} + M(\theta_{\partial\mathcal{r}})^{\cdot}] \tag{4.1}$$

for any region $R$.

The only portion of $\mathcal{R}(t)$ not stationary is $\mathcal{r}(t)$. Thus

$$(d/dt)\,\text{area}(\mathcal{R}) = \int_{\mathcal{r}} V\,ds, \tag{4.2}$$

and, since $F$ is constant, we may use $(2.11)_2$, (3.1), and the identity of

(3.6) and (3.7) to write (4.1) in the form

$$\int_{\mathcal{R}} \{ f^\circ - MK^\circ + m\theta^\circ + (\sigma + MK - f)KV + (\pi + F)V \} ds + \int_{\partial \mathcal{R}} (f - \sigma - MK) \boldsymbol{\ell} \cdot \mathbf{v}_{\partial \mathcal{R}} \leq 0. \tag{4.3}$$

This inequality must hold for all regions $R$, or equivalently, for all interfacial subcurves $\mathcal{R}(t)$. By arbitrarily varying the endpoints of $\mathcal{R}(t)$ we can arbitrarly vary the term $\boldsymbol{\ell} \cdot \mathbf{v}_{\partial \mathcal{R}}$ without affecting the other terms in (4.3); thus[11]

$$f = \sigma + MK, \tag{4.4}$$

and interfacial energy is generally not equal to surface tension alone: there is an additional term to account for interfacial bending.

Finally, since $\mathcal{R}(t)$ is arbitrary, (4.3) and (4.4) yield the *local dissipation inequality*

$$f^\circ - MK^\circ + m\theta^\circ + (\pi + F)V \leq 0. \tag{4.5}$$

## 5. Constitutive equations.

As constitutive equations we allow the interfacial energy, the interfacial couple-stress, the body couple, and the normal interaction to depend on the orientation of the interface through a dependence on the angle $\theta$, on the "bending" of the interface through a dependence on the curvature $K$, and on the kinetics of the interface through a dependence on the normal velocity $V$. Thus

$$z := (\theta, K, V)$$

is the list of constitutive variables, and the constitutive assumptions have the form

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[11] Cf. [G2,G3].

$$\begin{aligned} f &= \hat{f}(z), \qquad & M &= \hat{M}(z), \\ m &= \hat{m}(z), \qquad & \pi &= \hat{\pi}(z). \end{aligned} \tag{5.1}$$

Because of (4.4) we do not specify a separate constitutive equation for the surface tension $\sigma$. The surface shear $\xi$ and the tangential component $\mathbf{b}\cdot\boldsymbol{\ell}$ of $\mathbf{b}$ do not enter the dissipation inequality (4.5), and for that reason are taken to be indeterminate (not specified by constitutive equations).[12] In principle, once the balance equation (3.8) has been solved for the interfacial motion, then $(3.4)_2$ determines $\xi$ and, with this, the tangential component of $(3.4)_1$ determines $\mathbf{b}\cdot\boldsymbol{\ell}$.

Given an evolving grain $\Omega(t)$, the constitutive equations (5.1) may be used to compute a corresponding *constitutive process*. A basic hypothesis[13] of our theory is that *constitutive processes be consistent with the local dissipation inequality* (4.5). This assumption has important consequences.[14]

Substituting (5.1) into (4.5), we find that

$$\hat{f}_V(z)V^\circ + [\hat{f}_K(z) - \hat{M}(z)]K^\circ + [\hat{f}_\theta(z) + \hat{m}(z)]\theta^\circ + [\hat{\pi}(z) + F]V \le 0.$$

Given a value $z_0$ of the constitutive list, one can find an evolving grain $\Omega(t)$ such that, at some point and time, $(\theta,K,V) = z_0$, but $(\theta^\circ,K^\circ,V^\circ)$ is arbitrary. Thus

$$\hat{f}_V(z) = 0, \quad \hat{m}(z) = -\hat{f}_\theta(z), \quad \hat{M}(z) = \hat{f}_K(z), \quad [\hat{\pi}(z) + F]V \le 0,$$

and we are led to the following constitutive restrictions:

---

[12] Cf. classical beam-theory, in which the shear force is regarded as a reaction to the constraint that the cross section remain normal to the center line, and is hence indeterminate.

[13] We assume the availability of external body forces and body couples to ensure balance of forces and moments (cf. [G1], Footnote 13).

[14] We sketch the derivation of these consequences; they are minor modifications of results given in [G1,AG] (Compatibity Theorem).

(i) f, m, *and* M *are independent of* V, *and* $\hat{f}(\theta,K)$ *generates* m *and* M *through*

$$m = -\hat{f}_{\theta}(\theta,K), \qquad M = \hat{f}_{K}(\theta,K); \tag{5.2}$$

(ii) *the normal interaction is given by a relation of the form*

$$\pi = -F - aV, \qquad a = \hat{a}(\theta,K,V) \geq 0. \tag{5.3}$$

Conditions (i) and (ii) are also sufficient that (4.5) hold in all constitutive processes.

The relations (5.2) imply the Gibbs relation

$$f^{\circ} = MK^{\circ} - m\theta^{\circ}. \tag{5.4}$$

In view of (5.3) and (5.4), the left side of (4.5) is $-aV^2$, which identifies this quantity as the *sole* rate of energy dissipation; indeed, tracing backwards the argument leading to (4.5), we find that

$$-\int_{\mathcal{n}} aV^2\,ds \tag{5.5}$$

represents the left side of (4.1) minus the right. Further, if we take R in this difference to be a region containing $\Omega(t)$ in its interior, so that $\mathcal{R}(t)=\Omega(t)$ and $\mathcal{n}(t)=\partial\Omega(t)$, then, in view of (4.2), we are led to the *global growth relation:*

$$(d/dt)\Big\{\int_{\partial\Omega} f\,ds + F\,\mathrm{area}(\Omega(t))\Big\} = -\int_{\partial\Omega} aV^2\,ds \leq 0, \tag{5.6}$$

so that *the total energy is a Lyapunov function for interfacial evolution.*

## 6. Evolution equations for the interface.

The evolution equations of the theory are:

$$M_{ss} + m_s - fK + MK^2 + F + aV = 0,$$
$$f = f\hat{}(\theta,K), \qquad m = -f\hat{}_\theta(\theta,K), \tag{6.1}$$
$$M = f\hat{}_K(\theta,K), \qquad a = a\hat{}(\theta,K,V).$$

The general equations (6.1) are difficult, and for that reason we consider a model in which the interfacial energy has a relatively simple dependence on curvature and the kinetic coefficient depends only on the angle:

$$f = f_0(\theta) + \tfrac{1}{2}\varepsilon K^2, \qquad a = a_0(\theta), \tag{6.2}$$

with $f_0(\theta), a_0(\theta), \varepsilon > 0$. Then

$$m = -f_0'(\theta), \quad M = \varepsilon K, \tag{6.3}$$

and, writing $a(\theta) = a_0(\theta)$, (6.1) reduce to

$$a(\theta)V = g(\theta)K - \varepsilon(K_{ss} + \tfrac{1}{2}K^3) - F, \tag{6.4}$$
$$g(\theta) = f_0(\theta) + f_0''(\theta).$$

The theory of [AG] concerns the nonregularized equation

$$a(\theta)V = g(\theta)K - F, \tag{6.5}$$

which exhibits backward-parabolic behavior when $\theta$ lies in an *interface spinodal;* that is, an angle interval within which $g(\theta) < 0$. The spinodals were excluded in [AG] with the aid of corners. The location[15] of such corners depends on the shape of the *Frank diagram* $\mathfrak{F}$, which is the polar diagram of $f(\theta)^{-1}$: given a corner $c$, the interfacial stress must be continuous across $c$ to ensure compliance with balance of forces $(3.3)_1$; this continuity is ensured provided the two angles that define $c$ correspond to *bitangency angles* (of a bitangent to $\mathfrak{F}$). We use the term *globally unstable segment,* or more

[15] Cf. [AG], Part III.

succintly GUS, for any maximally connected portion of $\mathfrak{F}$ that does not lie on the boundary $\mathfrak{B}$ of the convex hull of $\mathfrak{F}$. Then each spinodal belongs to a GUS, and the initial and terminal angles of any GUS are bitangency angles. A given spinodal can therefore be eliminated by a corner whose angles are the initial and terminal angles of the corresponding GUS. (Such corners are called *admissible.*) This is exactly the proceedure followed in [AG].

The properties of $\mathfrak{F}$ alluded to above suggested the term interface spinodal; these properties also suggest the term *interfacial phase* for each maximally connected portion of $\mathfrak{F}\cap\mathfrak{B}$. Adjacent interfacial phases are then connected by a GUS and therefore correspond to an admissible corner.

The regularized equation (6.4) should be appropriate to study the behavior of the interface within the spinodal intervals. In fact, if $\theta_1$ and $\theta_2$ with $0<\theta_2-\theta_1<\pi$ are the angles that define the corner between two given interfacial phases, then the behavior of the interface within the corner might be modeled by the system

$$\begin{aligned} &a(\theta)V = g(\theta)K - \varepsilon(K_{ss} + \tfrac{1}{2}K^3) - F && -\infty<s<+\infty, \quad t>0,\\ &\theta(-\infty,t) = \theta_1, \qquad \theta(+\infty,t) = \theta_2 && t>0,\\ &\theta(s,0) = \theta_0(s) && -\infty<s<+\infty, \end{aligned}$$

with initial data $\theta_0(s)\in(\theta_1,\theta_2)$ and with the limits at $s=\pm\infty$ in the $C^2$-topology.

The evolution equations simplify when the interface is *convex.* The relevant differential equations are then (2.6) and (6.4); thus, since $K_{ss}=\frac{1}{2}K(K^2)_{\theta\theta}$, the interface evolves according to

$$\begin{aligned} &K_t = K^2(V_{\theta\theta} + V),\\ &a(\theta)V = g(\theta)K - \tfrac{1}{2}\varepsilon K[(K^2)_{\theta\theta} + K^2] - F \end{aligned} \tag{6.6}$$

(with $K_t$ the derivative holding $\theta$ fixed).

For a *steady convex motion,*[16]

[16] Cf. [AG], Sects. 2.2, 6.

$$K = K(\theta), \qquad V = \mathbf{U}\cdot\mathbf{n}(\theta), \tag{6.7}$$

with $\mathbf{U}$ the steady velocity, and (6.6) reduce to

$$g(\theta)K - \tfrac{1}{2}\varepsilon K[(K^2)_{\theta\theta} + K^2] = F + a(\theta)\mathbf{U}\cdot\mathbf{n}(\theta). \tag{6.8}$$

The problem of finding a steady convex solution (with the interface unbounded) consists in solving (6.8) on an angle interval $(\theta_1,\theta_2)$, $0<\theta_2-\theta_1<\pi$, subject to

$$K(\theta_1) = K(\theta_2) = 0, \qquad K(\theta) \neq 0 \quad \text{for} \quad \theta\in(\theta_1,\theta_2). \tag{6.9}$$

By (6.8) and (6.9), $\theta_1$ and $\theta_2$ must be consecutive zeros of $F + a(\theta)\mathbf{U}\cdot\mathbf{n}(\theta)$. Assume that the polar diagram Polar(a) of $a(\theta)$ is convex. Then $\theta_1$ and $\theta_2$ are consecutive zeros of $F + a(\theta)\mathbf{U}\cdot\mathbf{n}(\theta)$ provided the line perpendicular to $\mathbf{U}$ through the tip of the vector $-(F/|U|^2)\mathbf{U}$ intersects Polar(a) at the angles $\theta_1$ and $\theta_2$. For $\varepsilon=0$, if the interval $(\theta_1,\theta_2)$ belongs to a spinodal, then (using the analysis and terminology of [AG]) there is a solution of (6.8), (6.9), and this solution corresponds to an *advancing bump*, emphasizing the instability of the spinodals. It would be interesting to see if such solutions are possible when $\varepsilon>0$.

## 7. Linearized equations. Spinodal decomposition on the interface.

We now discuss an interface that is close to a flat interface at angle $\theta=\pi/2$. We represent the interface as the graph of a function $y=h(x,t)$, (with orientation such that arc length increases with increasing $x$), and let

$$u := h_x, \tag{7.1}$$

so that

$$u\tan\theta = -1, \qquad V = (\sin\theta)h_t, \qquad K = (\sin\theta)^3 h_{xx}, \qquad \partial/\partial s = (\sin\theta)\partial/\partial x. \tag{7.2}$$

We substitute the identities (7.2) into (6.4), and then linearize under the assumption that the derivatives of $h$ are small; the result is

$$h_t = \alpha h_{xx} - \delta h_{xxxx} - \beta, \tag{7.3}$$

where $\alpha = g_0/a_0$, $\beta = F/a_0$, $\delta = \varepsilon/a_0$, and the subscript zero indicates evaluation at $\theta = \pi/2$.

Differentiating (7.3) with respect to $x$ yields

$$u_t = \alpha u_{xx} - \delta u_{xxxx}, \tag{7.4}$$

which is exactly the linear equation used by Cahn [Ca] in his discussion of spinodal decomposition; this further supports the analogy between interfacial phases as defined in Section 6 and the more standard bulk phases. Following Cahn, we note that (7.4) has solutions of the form

$$u(x,t) = Ce^{\rho t}\sin\lambda x \tag{7.5}$$

provided

$$\rho = \rho(\lambda) = -\alpha\lambda^2 - \delta\lambda^4. \tag{7.6}$$

If the angle $\pi/2$ lies in a spinodal interval, then $\alpha < 0$ and spatial oscillations of frequency $\lambda$ between 0 and $(|\alpha|/\delta)^{\frac{1}{2}}$ are unstable, with a maximum of $\rho(\lambda)$ occuring at $\lambda = \lambda_m = (\frac{1}{2}|\alpha|/\delta)^{\frac{1}{2}}$; this indicates that spatial patterns of frequency $\lambda_m$ should be most often observed.

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