GEOMETRY OF LEBESGUE-BOCHNER FUNCTION SPACES - SMOOTHNESS

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I. E. Leonard and K. Sundaresan

Research Report 73-11

April, 1973

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ABSTRACT

This paper contains a complete solution to the problem of the higher order differentiability of the norm function in the Lebesgue-Bochner function spaces $L_p(E,\mu)$, $1 \leq p \leq \infty$, where E is a real Banach and μ is an extended real-valued measure defined on the measurable space (T,Σ). The order of smoothness of $L_p(E,\mu)$ can be summarized as follows: if $1 < p < \infty$, then (i) the norm in $L_p(E,\mu)$ is differentiable away from zero if and only if the norm in E is differentiable away from zero. (ii) If $p = 2$, the norm in $L_p(E,\mu)$ is twice continuously differentiable away from zero if and only if E is a Hilbert space. (iii) If p is an even integer, $p \neq 2$, the norm in $L_p(E,\mu)$ is p-times continuously differentiable away from zero if and only if the pth-power of the norm in E is a continuous homogeneous polynomial of degree p. (iv) If p is an odd integer, the norm in $L_p(E,\mu)$ is $(p-1)$-times continuously differentiable away from zero if and only if the norm in E is $(p-1)$-times continuously differentiable away from zero and the $(p-1)$-st derivative of the norm in E is uniformly bounded on the unit sphere in E. (v) If p is not an integer, and $I(p)$ is the integral part of p, the norm in $L_p(E,\mu)$ is $I(p)$-times continuously differentiable away from zero if and only if the norm in E is $I(p)$-times differentiable away from zero and the $I(p)$-th derivative of the norm in E is uniformly bounded on the unit sphere in E.

Geometry of Lebesgue-Bochner Function Spaces - Smoothness

I. E. Leonard* and K. Sundaresan

Introduction

The class of Lebesgue-Bochner function spaces, introduced by Bochner and Taylor [4] in 1938, have been found to be of considerable importance in various branches of mathematics, and are discussed at length in Dinculeanu [11], Dunford and Schwartz [12], and Edwards [13]. The study of the geometric properties of the Lebesgue-Bochner function spaces dates back about three decades: Day [8] and McShane [17], respectively, characterized uniform convexity and smoothness of these spaces. In fact, the only known result concerning the smoothness of the Lebesgue-Bochner function spaces is due to McShane, and his result concerns only the directional derivative (Gâteaux derivative) of the norm in this class of Banach spaces. Even the Fréchet differentiability of the norm has not been considered anywhere. It might be mentioned in this connection that the first systematic study of higher-order differentiability of the norm in the classical Banach spaces was made by Bonic and Frampton [5] and Sundaresan [18]. In [5] and [18], the order of differentiability of the norm in the classical L_p spaces, $1 \leq p \leq \infty$, is obtained; while in Sundaresan [20], the smoothness of the norm in $C(X,E)$ is discussed.

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*The material in this paper is a portion of a doctoral dissertation submitted to Carnegie-Mellon University. An abstract of this paper will appear in the Bulletin of the American Mathematical Society, May, 1973.

For an elegant up-to-date account of smooth Banach spaces, and related concepts one might refer to the Lecture Notes by S. Yamamuro [23].

This paper contains the first systematic investigation of the higher-order differentiability of the norm function in the Lebesgue-Bochner function spaces $L_p(E,T,\Sigma,\mu)$, $1 \leq p \leq \infty$, where E is a real Banach space and μ is a non-negative extended real-valued measure defined on the measurable space (T,Σ). The paper is divided into four sections. Section 1 contains the basic definitions of the various geometric and analytic properties of Banach spaces to be studied as well as the definition of the Lebesgue-Bochner function spaces $L_p(E,\mu)$. Section 2 contains the results on Fréchet differentiability of the norm in $L_p(E,\mu)$. In Section 3, the results concerning the higher-order differentiability of the norm in $L_p(E,\mu)$ are discussed. Section 4 contains a counterexample which shows that even if the norm in E is of class C^{∞}, the norm in $L_p(E,\mu)$, for any p, $1 \leq p \leq \infty$, need not be even twice differentiable; thus pointing out the importance of the characterizations in Sections 2 and 3.

1. Definitions and Notation

The definitions and notation used throughout the paper are collected in this section for easy reference. In the following definitions, E denotes a real Banach space and E^* is the dual of E.

1.1. Definition. The unit ball of E is $U = \{x \in E \mid \|x\| \leq 1\}$ and its boundary $S = \{x \in E \mid \|x\| = 1\}$ is the unit sphere of E. In the dual space E^*, $U^* = \{f \in E^* \mid \|f\| \leq 1\}$ is the unit ball of E^* and $S^* = \{f \in E^* \mid \|f\| = 1\}$ is the unit sphere of E^*. The conjugate norm will be denoted by $\|\cdot\|$ since there will be no occasion for confusion. The unit ball and unit sphere of E^{**} are defined analogously and are denoted by U^{**} and S^{**} respectively.

1.2. Definition. A Banach space E is said to be smooth at $x \in S$ if and only if there exists a unique hyperplane of support at x, that is, there exists only one continuous linear functional $\ell_x \in E^*$ with $\|\ell_x\| = 1$ such that $\ell_x(x) = 1$. Such a linear functional $\ell_x \in E^*$ is called the support functional of U at x, and $\ell_x^{-1}(\{1\})$ is called the hyperplane of support of U at x. A Banach space E is said to be a smooth Banach space if it is smooth at every $x \in S$.

1.3. Definition. The norm $\|\cdot\| : E \longrightarrow \mathbb{R}^+$ is said to be Gâteaux differentiable at $x \in E$ if and only if there exists a functional $G_x \in E^*$ such that

$$\lim_{t\to 0} \left| \frac{\|x+th\|-\|x\|}{t} - G_x(h) \right| = 0$$

for every $h\in E$. G_x is called the Gâteaux derivative of the norm at $x\in E$. A Banach space E is said to be uniformly smooth if and only if

$$\lim_{t\to 0} \frac{\|x+th\|-\|x\|-tG_x(h)}{t} = 0$$

uniformly for $(x,h)\in S \times S$. The norm $\|\cdot\| : E \longrightarrow \mathbb{R}^+$ is said to be differentiable (Fréchet differentiable) at $x\in E$ if and only if there exists a functional $G_x\in E^*$ such that

$$\lim_{\|h\|\to 0} \frac{\left| \|x+h\|-\|x\|-G_x(h) \right|}{\|h\|} = 0.$$

The norm $\|\cdot\| : E \longrightarrow \mathbb{R}^+$ is said to be uniformly Fréchet differentiable if and only if

$$\lim_{\|h\|\to 0} \left| \frac{\|x+h\|-\|x\|-G_x(h)}{\|h\|} \right| = 0$$

uniformly for $x\in S$. The norm $\|\cdot\| : E \longrightarrow \mathbb{R}^+$ is of class C^1 or continuously differentiable if and only if the mapping $G : E \sim \{0\} \longrightarrow E^* \sim \{0\}$ given by $G(x) = G_x$ is continuous.

1.4. Remark. Before turning to the definitions of higher-order differentiability, note that the norm in E is:

(i) Gâteaux differentiable at $x\in E$ if and only if

$$\lim_{t\to 0} \frac{\|x+th\|-\|x\|}{t} = G_x(h)$$

exists for all $h \in E$.

(ii) Smooth at $x \in E$ if and only if it is Gâteaux differentiable at $x \in E$.

(iii) Fréchet differentiable if and only if it is of class C^1 away from zero.

(iv) Uniformly smooth if and only if it is uniformly Fréchet differentiable.

Note that (i) is Mazur's theorem and can be found in Mazur [16], while (ii), (iii), and (iv) can be found in Day [9].

<u>1.5. Definition</u>. Let E and F be Banach spaces, then $\mathcal{L}(E,F)$ denotes the Banach space of <u>continuous linear mappings</u> from E into F with the usual operator norm. $\mathcal{B}^k(E,F)$ denotes the Banach space of <u>continuous</u> k-<u>multilinear mappings</u> $v : E \times \ldots \times E \longrightarrow F$ with the norm

$$\|v\| = \sup_{\|x_1\| = \ldots = \|x_k\| = 1} \|v(x_1, \ldots, x_k)\|.$$

The spaces $\mathcal{B}^k(E,F)$ may be identified with the spaces defined inductively as follows:

$$\mathcal{B}^0(E,F) = F, \quad \mathcal{B}^k(E,F) = \mathcal{L}(E, \mathcal{B}^{k-1}(E,F)) = \mathcal{B}^{k-1}(E, \mathcal{L}(E,F)).$$

A mapping $\varphi : E \longrightarrow F$ is said to be a <u>homogeneous polynomial of degree</u> k if there exists a k-multilinear mapping $f : E \times \ldots \times E \longrightarrow F$ such that

$$\varphi(x) = f(x, \ldots, x)$$

for all $x \in E$.

1.6. Definition. Let E and F be Banach spaces and let A be an open set in E. A mapping $f : A \to F$ is said to be differentiable at $x \in A$ if there exists a mapping $f'(x) \in \mathcal{L}(E,F)$ such that

$$\lim_{h \to 0} \frac{\|f(x+h) - f(x) - f'(x) \cdot h\|}{\|h\|} = 0.$$

In this case f is continuous at $x \in A$ and $f'(x)$, which is unique, is called the derivative of f at x. The higher-order derivatives $f^{(k)} : A \to \mathcal{B}^k(E,F)$ are defined in the usual manner (see Dieudonné [10]). The mapping $f : A \to F$ is said to be of class C^k or k-times continuously differentiable if it is k-times differentiable and the kth derivative $f^{(k)} : A \to \mathcal{B}^k(E,F)$ is continuous. The mapping $f : A \to F$ is said to be of class C^∞ if it is indefinitely continuously differentiable.

1.7. Remark.

(i) Let E and F be Banach spaces and let A be an open subset of E. If the mapping $f : A \to F$ is k-times differentiable on A, then the multilinear mapping $f^{(k)}(x) \in \mathcal{B}^k(E,F)$ is symmetric for each $x \in A$.

(ii) Any continuous k-multilinear mapping is indefinitely differentiable, and all its derivatives of order $\geq k + 1$ are zero.

A proof can be found in either Cartan [6] or Dieudonné [10].

Next the Lebesgue-Bochner function spaces are defined. In the following (T,Σ,μ) is an arbitrary measure space, that is,

T is a non-empty set, Σ is a σ-algebra of subsets of T and $\mu : \Sigma \longrightarrow \overline{\mathbb{R}}^+$ is a countably additive measure. (Here μ is assumed to be non-trivial, that is, μ is not supported by finitely many atoms and the range of μ contains at least one non-zero real number.)

1.8. Definition. Let E be a normed linear space. If $f : T \longrightarrow E$, then f is μ-measurable if and only if:

(i) $f^{-1}(G) \in \Sigma$ for every open set $G \subset E$, and

(ii) there exists a set $N \in \Sigma$, with $\mu(N) = 0$, and a countable set $H \subset E$, such that $f(T \sim N) \subset \overline{H}$.

1.9. Definition. If $1 \leq p \leq \infty$, the Lebesgue-Bochner function spaces $L_p(E,\mu)$ are defined as follows:

$$L_p(E,\mu) = \{f \mid f : T \longrightarrow E \text{ is measurable, and } \int_T \|f(t)\|^p d\mu(t) < \infty\}$$

for $1 \leq p < \infty$, and

$$L_\infty(E,\mu) = \{f \mid f : T \longrightarrow E \text{ is measurable, and } \operatorname*{ess\,sup}_{t \in T} \|f(t)\| < \infty\}$$

(as usual, identifying functions which agree μ-a.e.). When T is the set of positive integers and μ is the counting measure, $L_p(E,\mu)$ is usually denoted by $\ell_p(E)$.

2. Fréchet Differentiability of the Norm in $L_p(E,\mu)$, $1 < p < \infty$

In this section a complete characterization of the Fréchet differentiability of the norm in the Lebesgue-Bochner function spaces $L_p(E,\mu)$ is given. It is shown that the norm in $L_p(E,\mu)$, $1 < p < \infty$, is differentiable away from zero if and only if the norm in E is differentiable away from zero. Before proceeding to the theorem a few useful lemmas are stated. The first can be found in Vainberg [21, p. 43].

2.1. Definition. Let E,F be Banach spaces and A an open subset of E.

(i) A mapping $f : A \to F$ is said to be locally bounded on A if and only if for each $x_0 \in A$ there exists a $\rho_0 = \rho_0(x_0) > 0$ such that $B(x_0,\rho_0) = \{x \in E \mid \|x-x_0\| < \rho_0\} \subset A$ and f is bounded on $B(x_0,\rho_0)$.

(ii) A mapping $f : A \to F$ is said to have a locally uniform derivative f' on A if and only if given any $\epsilon > 0$ and $x_0 \in A$, there exist $\eta = \eta(x_0,\epsilon) > 0$ and $\delta = \delta(x_0,\epsilon) > 0$ such that

$$f(x + h) - f(x) = f'(x) \cdot h + \Theta_x(h)$$

where $\|\Theta_x(h)\| < \epsilon\|h\|$ for all $x \in A$ with $\|x-x_0\| < \eta(x_0,\epsilon)$, whenever $\|h\| < \delta(x_0,\epsilon)$.

2.2. Lemma [Vainberg]. Let E,F be Banach spaces and $f : E \to F$ be differentiable. Then $f' : E \to \mathcal{L}(E,F)$ is continuous in the ball $B_r = \{x \in E \mid \|x\| < r\}$ if and only if

(i) f has a locally uniform derivative in B_r, and

(ii) $f' : E \to \mathcal{L}(E,F)$ is locally bounded in B_r.

Note that the above lemma is valid if the ball B_r is replaced by any bounded set in E.

2.3. Lemma. Let E and F be Banach spaces and $g : E \to F$ be continuously differentiable. If $C \subset E$ is compact, then g is uniformly differentiable on C.

The proof is a direct consequence of Lemma 2.2, and standard compactness arguments.

The next lemma is known, and is stated here for completeness.

2.4. Lemma. Let E be a Banach space and (T,Σ,μ) a measure space. If $f : T \longrightarrow E$ is a measurable function, then f is locally almost compact-valued; that is, if $P \in \Sigma$, $0 < \mu(P) < \infty$, and $\epsilon > 0$, then there exists $Q \in \Sigma$, $Q \subset P$, with $0 < \mu(Q) < \epsilon$, such that $f(P \sim Q)$ is precompact.

Proof: Let $f_n : T \longrightarrow E$, $n \geq 1$, be a sequence of measurable simple functions such that $f_n \longrightarrow f$ μ-a.e., and let $P \in \Sigma$, $0 < \mu(P) < \infty$. From Egoroff's theorem, there exists $Q \in \Sigma$, $Q \subset P$, with $0 < \mu(Q) < \epsilon$, such that $f_n \longrightarrow f$ uniformly on $P \sim Q$.

If $\delta > 0$, choose n sufficiently large that

$$\|f(t) - f_n(t)\| < \delta$$

for all $t \in P \sim Q$. Let the range of $f_n : T \longrightarrow E$ be $\{b_1, b_2, \ldots, b_k\}$, and let $U(b_i, \delta) = \{x \in E \mid \|b_i - x\| < \delta\}$, $i = 1,2,\ldots,k$. Then

$$f(P \sim Q) \subset \bigcup_{i=1}^{k} U(b_i, \delta),$$

hence $f(P \sim Q)$ is precompact, that is, f is locally almost compact-valued. q.e.d.

2.5. Theorem. Let E be a Banach space, (T,Σ,μ) a measure space and $1 < p < \infty$. The norm $\|\cdot\| : L_p(E,\mu) \longrightarrow \mathbb{R}^+$ is differentiable away from zero if and only if the norm $\|\cdot\| : E \longrightarrow \mathbb{R}^+$ is differentiable away from zero.

Proof: 1. Suppose that the norm in E is differentiable away from zero. Define $g : E \longrightarrow \mathbb{R}^+$ and $\hat{g} : L_p(E,\mu) \longrightarrow \mathbb{R}^+$ by

$$g(x) = \frac{1}{p}\|x\|^p \qquad \text{for } x \in E$$

and

$$\hat{g}(f) = \frac{1}{p}\|f\|^p = \frac{1}{p}\int_T \|f(t)\|^p d\mu(t) \qquad \text{for } f \in L_p(E,\mu).$$

Since the norm in E is continuously differentiable away from zero (Cudia [7, Corollary 4.11]), and since $p > 1$, then g is continuously differentiable at all $x \in E$ and

$$g'(x) = \|x\|^{p-1} G_x \qquad \text{for } x \neq 0$$
$$g'(0) = 0$$

where $G_x \in E^*$ is the derivative of the norm in E at x. Now, $g(x + y) = g(x) + g'(x) \cdot y + \Theta_x(y)$ where $\frac{\|\Theta_x(y)\|}{\|y\|} \longrightarrow 0$ as $\|y\| \longrightarrow 0$. Let $f \in L_p(E,\mu)$ with $\|f\| = 1$, and let $h \in L_p(E,\mu)$; then

$$g(f(t)+h(t)) = g(f(t)) + g'(f(t)) \cdot h(t) + \Theta_{f(t)}(h(t)) \qquad (2.1)$$

for all $t \in T$. The mapping $\gamma : T \longrightarrow E^*$, given by $\gamma(t) = g'(f(t))$ for $t \in T$, being the composition of a measurable function and a continuous function, is measurable; and from the definition of g', it follows that $\gamma \in L_q(E^*,\mu)$. The measurability of γ, together with (2.1), imply that the mapping $t \longmapsto \Theta_{f(t)}(h(t))$ is measurable. Thus,

$$\int_T g(f(t)+h(t))\,d\mu(t) = \int_T g(f(t))\,d\mu(t) + \int_T g'(f(t)) \cdot h(t)\,d\mu(t) + \int_T \Theta_{f(t)}(h(t))\,d\mu(t). \qquad (2.2)$$

Defining $\hat{g}'(f)\cdot h = \int_T g'(f(t))\cdot h(t)\,d\mu(t)$ for all $h\in L_p(E,\mu)$, then $\hat{g}'(f) \in L_p(E,\mu)^*$, and in order to show that $\hat{g}'(f)$ is the derivative of $\hat{g}$ at $f\in L_p(E,\mu)$, from (2.2) it suffices to verify that

$$\frac{\int_T \Theta_{f(t)}(h(t))\,d\mu(t)}{(\int_T \|h(t)\|^p d\mu(t))^{1/p}} \longrightarrow 0$$

as $\|h\| = (\int_T \|h(t)\|^p d\mu(t))^{1/p} \longrightarrow 0$. From (2.1) and the mean-value theorem (Cartan [6]), noting that $\|g'(f(t))\| = \|f(t)\|^{p-1}$, it follows that

$$|\Theta_{f(t)}(h(t))| \leq 2\|h(t)\| \sup_{0\leq\varphi\leq 1} \|f(t)+\varphi h(t)\|^{p-1} \tag{2.3}$$

for all $t\in T$. The rest of the discussion is divided into two cases.

<u>Case 1</u>: $\mu(T) < \infty$ (assume $\mu(T) = 1$). The mapping $\lambda : \Sigma \longrightarrow \mathbb{R}^+$ defined by

$$\lambda(G) = \int_G \|f(t)\|^p d\mu(t) \qquad \text{for } G\in\Sigma$$

is a finite, positive measure, which is absolutely continuous with respect to μ. Therefore, given $\epsilon > 0$, there exists a $\delta_1 = \delta_1(f,\epsilon) > 0$ such that

$$\int_G \|f(t)\|^p d\mu(t) < \frac{1}{2}\left(\frac{\epsilon}{6}\right)^q \tag{2.4}$$

whenever $G \in \Sigma$ and $\mu(G) < \delta_1$. (Here $\frac{1}{p} + \frac{1}{q} = 1$.)

Let $0 \leq \varphi \leq 1$, from the mean-value theorem,

$$\left| \|f(t)+\varphi h(t)\|^p - \|f(t)\|^p \right| \leq 2p\|\varphi h(t)\| \sup_{0\leq\psi\leq 1} \|f(t)+\psi\varphi h(t)\|^{p-1},$$

and it follows immediately that

$$\left| \sup_{0\leq\varphi\leq 1} \|f(t)+\varphi h(t)\|^p - \|f(t)\|^p \right| \leq 2p\|h(t)\| [\|f(t)\|+\|h(t)\|]^{p-1}.$$

From Hölder's inequality,

$$\int_T \left| \sup_{0\leq\varphi\leq 1} \|f(t)+\varphi h(t)\|^p - \|f(t)\|^p \right| d\mu(t)$$

$$\leq 2p\left(\int_T \|h(t)\|^p d\mu(t)\right)^{1/p}\left(\int_T [\|f(t)\|+\|h(t)\|]^p d\mu(t)\right)^{1/q},$$

and thus,

$$\int_T \left| \sup_{0\leq\varphi\leq 1} \|f(t)+\varphi h(t)\|^p - \|f(t)\|^p \right| d\mu(t) \longrightarrow 0 \quad \text{as } \|h\| \longrightarrow 0.$$

Since for arbitrary $G \in \Sigma$,

$$\left| \int_G [\sup_{0\leq\varphi\leq 1} \|f(t)+\varphi h(t)\|^p - \|f(t)\|^p] d\mu(t) \right|$$

$$\leq \int_T \left| \sup_{0\leq\varphi\leq 1} \|f(t)+\varphi h(t)\|^p - \|f(t)\|^p \right| d\mu(t),$$

given any $\epsilon > 0$, there exists a $\delta_2 = \delta_2(f,\epsilon) > 0$ such that

$$\left|\int_G \left[\sup_{0\leq\varphi\leq 1} \|f(t)+\varphi h(t)\|^p - \|f(t)\|^p\right] d\mu(t)\right| < \frac{1}{2}\left(\frac{\epsilon}{6}\right)^q \tag{2.5}$$

for all $G\in\Sigma$, whenever $\|h\| < \delta_2$.

Also, from Lemma 2.4, there exists $T_1\in\Sigma$, with $0 < \mu(T_1) < \delta_1$, such that $f(T \sim T_1)$ is precompact; hence given any $\epsilon > 0$, there exists by Lemma 2.3, $\delta_3 = \delta_3(f,\epsilon) > 0$ such that

$$|\Theta_x(y)| < \frac{\epsilon}{3}\|y\| \tag{2.6}$$

for all $x \in f(T \sim T_1)$, whenever $\|y\| < \delta_3$. Let

$$\delta_0 = \min(\delta_2, \delta_3, \delta_1^{1/p}\cdot\delta_3)$$

and let $h\in L_p(E,\mu)$ with $\|h\| < \delta_0$.

Since $\mu(T_1) < \delta_1$, (2.4) implies that

$$\int_{T_1} \|f(t)\|^p d\mu(t) < \frac{1}{2}\left(\frac{\epsilon}{6}\right)^q,$$

and thus from (2.5) it is inferred that

$$\int_{T_1} \sup_{0\leq\varphi\leq 1} \|f(t)+\varphi h(t)\|^p d\mu(t) < \left(\frac{\epsilon}{6}\right)^q.$$

From (2.3) and Hölder's inequality, it is verified that

$$\int_{T_1} |\Theta_{f(t)}(h(t))|\, d\mu(t) < \frac{\epsilon}{3}\|h\|. \tag{2.7}$$

Now let $T_2 = T \sim T_1$ and define

$$T_3(h) = \{t\in T_2 \mid \|h(t)\| \geq \delta_3\},$$

then

$$\delta_3^p \mu(T_3(h)) \le \int_{T_3(h)} \|h(t)\|^p d\mu(t) \le \int_T \|h(t)\|^p d\mu(t) < \delta_0^p \le \delta_1 \cdot \delta_3^p,$$

that is, $\mu(T_3(h)) < \delta_1$. The preceding inequalities together with (2.4) imply that

$$\int_{T_3(h)} \|f(t)\|^p d\mu(t) < \frac{1}{2}\left(\frac{\epsilon}{6}\right)^q$$

and since $\|h\| < \delta_0 \le \delta_2$, (2.5) implies that

$$\int_{T_3(h)} \sup_{0 \le \varphi \le 1} \|f(t) + \varphi(t) h(t)\|^p d\mu(t) < \left(\frac{\epsilon}{6}\right)^q.$$

From (2.3) and Hölder's inequality,

$$\int_{T_3(h)} |\Theta_{f(t)}(h(t))| d\mu(t) < \frac{\epsilon}{3}\|h\|. \tag{2.8}$$

Now let $T_4(h) = T_2 \sim T_3(h)$, then $T_4(h) \subset T \sim T_1$, and since $f(T \sim T_1)$ is precompact, and $\|h(t)\| < \delta_3$ for all $t \in T_4(h)$, (2.6) implies that

$$\int_{T_4(h)} |\Theta_{f(t)}(h(t))| d\mu(t) < \frac{\epsilon}{3} \int_{T_4(h)} \|h(t)\| d\mu(t)$$

$$\le \frac{\epsilon}{3} \int_T \|h(t)\| d\mu(t).$$

From Hölder's inequality, since $\mu(T) = 1$,

$$\int_{T_4(h)} |\Theta_{f(t)}(h(t))| d\mu(t) < \frac{\epsilon}{3}\|h\|. \tag{2.9}$$

Combining (2.7), (2.8), and (2.9), since $T = T_1 \cup T_3(h) \cup T_4(h)$, (disjoint union), then

$$\int_T |\varphi_{f(t)}(h(t))|d\mu(t) < \epsilon\|h\|.$$

What has been shown is that given any $\epsilon > 0$, there exists a $\delta_0 = \delta_0(f,\epsilon) > 0$, such that

$$\int_T |\varphi_{f(t)}(h(t))|d\mu(t) < \epsilon\|h\| \tag{2.10}$$

whenever $h \in L_p(E,\mu)$ and $\|h\| < \delta_0$.

Case 2: $\mu(T) = \infty$. In this case, since the support of f is σ-finite, it can be assumed without loss of generality that the measure space (T,Σ,μ) is σ-finite. Let $\{A_n\}_{n\geq 1}$ be a pairwise disjoint sequence of sets from Σ such that $\mu(A_n) < \infty$ for all $n \geq 1$ and $T = \bigcup_{n=1}^{\infty} A_n$.

Let $f \in L_p(E,\mu)$ with $\|f\| = (\int_T \|f(t)\|^p d\mu(t))^{1/p} = 1$, then

$$\int_T \|f(t)\|^p d\mu(t) = \int_{\bigcup_{n=1}^{\infty} A_n} \|f(t)\|^p d\mu(t) = \sum_{n=1}^{\infty} \int_{A_n} \|f(t)\|^p d\mu(t) = 1.$$

Given $\epsilon > 0$, there exists a positive integer N_0 such that

$$\sum_{n=N_0+1}^{\infty} \int_{A_n} \|f(t)\|^p d\mu(t) < (\tfrac{\epsilon}{4})^q,$$

that is,

$$\int_{\bigcup_{n=N_0+1}^{\infty} A_n} \|f(t)\|^p d\mu(t) < (\tfrac{\epsilon}{4})^q.$$

Since the mapping $\beta : L_p(E,\mu) \longrightarrow \mathbb{R}^+$ given by

$$\beta(h) = \int_{\bigcup_{n=N_0+1}^{\infty} A_n} \sup_{0\leq\varphi\leq 1} \|f(t)+\varphi h(t)\|^p d\mu(t)$$

is continuous, there exists a $\delta_1 = \delta_1(f,\epsilon) > 0$ such that

$$\int_{\bigcup_{n=N_0+1}^{\infty} A_n} \sup_{0\leq\varphi\leq 1} \|f(t)+\varphi h(t)\|^p d\mu(t) < (\tfrac{\epsilon}{4})^q$$

whenever $\|h\| < \delta_1$. Proceeding exactly as in the previous case, it is obtained that

$$\int_{\bigcup_{n=N_0+1}^{\infty} A_n} |\Theta_{f(t)}(h(t))|d\mu(t) < \tfrac{\epsilon}{2}\|h\| \tag{2.11}$$

whenever $\|h\| < \delta_1$.

Now, $\mu(\bigcup_{n=1}^{N_0} A_n) = \sum_{n=1}^{N_0} \mu(A_n) < \infty$, and from the result of Case 1, there exists a $\delta_2 = \delta_2(f,\epsilon) > 0$ such that

$$\int_{\bigcup_{n=1}^{N_0} A_n} |\Theta_{f(t)}(h(t))|d\mu(t) < \tfrac{\epsilon}{2}\|h\| \tag{2.12}$$

whenever $\|h\| < \delta_2$.

Letting $\delta_0 = \min(\delta_1,\delta_2)$, then

$$\int_T |\Theta_{f(t)}(h(t))|\,d\mu(t) < \epsilon\|h\|$$

whenever $\|h\| < \delta_0$.

Thus, in either case,

$$\lim_{h \to 0} \frac{\int_T |\Theta_{f(t)}(h(t))|\,d\mu(t)}{\|h\|} = 0,$$

and $\hat{g} : L_p(E,\mu) \longrightarrow \mathbb{R}^+$ is differentiable at $f \in L_p(E,\mu)$, with

$$\hat{g}'(f)\cdot h = \int_T g'(f(t))\cdot h(t)\,d\mu(t)$$

for all $h \in L_p(E,\mu)$. Therefore, the norm $\|\cdot\| : L_p(E,\mu) \longrightarrow \mathbb{R}^+$ is differentiable away from zero.

2. Conversely, suppose the norm $\|\cdot\| : L_p(E,\mu) \longrightarrow \mathbb{R}^+$ is differentiable away from zero; since E is isometrically isomorphic to a closed subspace of $L_p(E,\mu)$, the norm $\|\cdot\| : E \longrightarrow \mathbb{R}^+$ is differentiable away from zero. q.e.d.

3. Higher-Order Differentiability of the Norm in $L_p(E,\mu)$

In this section a complete characterization of the higher-order differentiability of the norm in the Lebesgue-Bochner function spaces $L_p(E,\mu)$ is given. The first result shows that if $p > k$, then the norm in $L_p(E,\mu)$ is k-times continuously differentiable away from zero if and only if the norm in E is k-times continuously differentiable away from zero and the kth derivative of the norm in E is uniformly bounded on the unit sphere in E. This is proved first for the case $k = 2$, and then a straightforward induction argument completes the proof for $k > 2$. (Note that the induction cannot start with $k = 1$, since the first derivative of the norm in E is automatically bounded on the unit sphere.)

3.1. Theorem. Let E be a Banach space, (T,Σ,μ) a measure space and $p > 2$. If the norm $\|\cdot\| : E \longrightarrow \mathbb{R}^+$ is twice continuously differentiable away from zero and the second derivative of the norm in E is uniformly bounded on the unit sphere in E, then the norm $\|\cdot\| : L_p(E,\mu) \longrightarrow \mathbb{R}^+$ is twice continuously differentiable away from zero.

Proof: Suppose the norm in E is twice continuously differentiable away from zero and the second derivative of the norm in E is uniformly bounded on S. Define $g : E \longrightarrow \mathbb{R}^+$ and $\hat{g} : L_p(E,\mu) \longrightarrow \mathbb{R}^+$ by

$$g(x) = \frac{1}{p}\|x\|^p \qquad \text{for } x \in E$$

and

$$\hat{g}(f) = \frac{1}{p}\|f\|^p = \frac{1}{p}\int_T \|f(t)\|^p d\mu(t) \qquad \text{for} \quad f \in L_p(E,\mu).$$

Since the norm in E is twice differentiable away from zero and $p > 2$, then g is twice differentiable at all $x \in E$ and $g''(0) = 0$.

If $\|g''(x)\| \leq M$ for all $x \in E$, $\|x\| = 1$, then

$$g''(\lambda x) = |\lambda|^{p-2} g''(x) \qquad \text{for} \quad \lambda \neq 0$$

implies that $\|g''(x)\| \leq M\|x\|^{p-2}$ for all $x \in E$, $x \neq 0$. Hence g is twice continuously differentiable at all $x \in E$. Now,

$$g'(x + y) = g'(x) + g''(x) \cdot y + \Theta_x(y)$$

where $\dfrac{\|\Theta_x(y)\|}{\|y\|} \longrightarrow 0$ as $\|y\| \to 0$.

Let $f \in L_p(E,\mu)$ with $\|f\| = (\int_T \|f(t)\|^p d\mu(t))^{1/p} = 1$, and let $h \in L_p(E,\mu)$; then

$$g'(f(t) + h(t)) = g'(f(t)) + g''(f(t)) \cdot h(t) + \Theta_{f(t)}(h(t)) \tag{3.1}$$

for all $t \in T$. The mapping $\gamma : T \longrightarrow \mathcal{L}(E,E^*)$ given by $\gamma(t) = g''(f(t))$ for $t \in T$, being the composition of a measurable function and a continuous function, is measurable; and from (3.1) this implies that the mapping $t \longmapsto \Theta_{f(t)}(h(t))$ is measurable.

Define the mapping $\hat{\Theta}_f(h) : T \longrightarrow E^*$ by

$$\hat{\Theta}_f(h)(t) = \Theta_{f(t)}(h(t)) \qquad \text{for} \quad t \in T.$$

From (3.1) and the mean-value theorem (Cartan [6]), since $\|g''(x)\| \leq M\|x\|^{p-2}$ for all $x \in E$, it follows that

$$\|\Theta_{f(t)}(h(t))\| \leq 2M\|h(t)\| \sup_{0 \leq \varphi \leq 1} \|f(t) + \varphi h(t)\|^{p-2}$$

for all $t \in T$. Therefore, from Hölder's inequality, it follows that

$$\int_T \|\Theta_{f(t)}(h(t))\|^q d\mu(t) \leq (2M)^q \left(\int_T \left| \sup_{0 \leq \varphi \leq 1} \|f(t) + \varphi h(t)\| \right|^p d\mu(t)\right)^{q(p-2)/p} \cdot \left(\int_T \|h(t)\|^p d\mu(t)\right)^{q/p} \tag{3.2}$$

and hence, $\hat{\Theta}_f(h) \in L_q(E^*, \mu)$ with

$$\|\hat{\Theta}_f(h)\| = \left(\int_T \|\Theta_{f(t)}(h(t))\|^q d\mu(t)\right)^{1/q}.$$

Now let $\hat{g}''(f) : L_p(E, \mu) \times L_p(E, \mu) \longrightarrow \mathbb{R}$ be defined by

$$\hat{g}''(f) \cdot (h_1, h_2) = \int_T g''(f(t)) \cdot (h_1(t), h_2(t)) \, d\mu(t)$$

for all $h_1, h_2 \in L_p(E, \mu)$. Then from Hölder's inequality it follows that $\|\hat{g}''(f)\| \leq M\|f\|^{p-2}$. Hence $\hat{g}''(f) : L_p(E, \mu) \times L_p(E, \mu) \longrightarrow \mathbb{R}$ is a bounded, symmetric bilinear form.

From (3.1) it follows that

$$\int_T g'(f(t)+h_1(t))\cdot h_2(t)\,d\mu(t) - \int_T g'(f(t))\cdot h_2(t)\,d\mu(t)$$

$$- \int_T g''(f(t))\cdot(h_1(t),h_2(t))\,d\mu(t) = \int_T \Theta_{f(t)}(h_1(t))\cdot h_2(t)\,d\mu(t),$$

that is,

$$\hat{g}'(f+h_1)\cdot h_2 - \hat{g}'(f)\cdot h_2 - \hat{g}''(f)\cdot(h_1,h_2) = \hat{\Theta}_f(h_1)\cdot h_2.$$

Taking the supremum over those $h_2 \epsilon L_p(E,\mu)$ for which $\|h_2\| = 1$, then

$$\|\hat{g}'(f+h_1) - \hat{g}'(f) - \hat{g}''(f)\cdot h_1\| = \|\hat{\Theta}_f(h_1)\|. \tag{3.3}$$

In order to show that $\hat{g}''(f) \in \mathcal{B}^2(L_p(E,\mu), \mathbb{R})$ is the second derivative of $\hat{g}$ at $f \epsilon L_p(E,\mu)$, from (3.3) it suffices to show that

$$\frac{\|\hat{\Theta}_f(h)\|}{\|h\|} = \frac{(\int_T \|\Theta_{f(t)}(h(t))\|^q d\mu(t))^{1/q}}{(\int_T \|h(t)\|^p d\mu(t))^{1/p}} \longrightarrow 0$$

as $\|h\| = (\int_T \|h(t)\|^p d\mu(t))^{1/p} \longrightarrow 0$.

Proceeding exactly as in Theorem 2.5, two cases are considered:

<u>Case 1</u>: $\mu(T) < \infty$ (assume $\mu(T) = 1$). From (3.2), for any $G \epsilon \Sigma$,

$$\int_G \|\Theta_{f(t)}(h(t))\|^q d\mu(t) \leq (2M)^q \left(\int_G \|f(t)+\varphi(t)h(t)\|^p d\mu(t)\right)^{q(p-2)/p} \|h\| \tag{3.3$'$}$$

(where $0 \leq \varphi(t) \leq 1$ is chosen so that $\sup_{0\leq\varphi\leq 1} \|f(t)+\varphi h(t)\| = \|f(t)+\varphi(t)h(t)\|$). Given $\epsilon > 0$, choose $\delta_1 = \delta_1(f,\epsilon) > 0$ such that

$$\int_G \|f(t)\|^p d\mu(t) < \frac{1}{2}\left(\frac{\epsilon}{2M\cdot 3^{1/q}}\right)^{p/p-2} \tag{3.4}$$

whenever $\mu(G) < \delta_1$.

Choose $\delta_2 = \delta_2(f,\epsilon) > 0$ such that

$$\left|\int_G \|f(t)+\varphi(t)h(t)\|^p - \|f(t)\|^p)\,d\mu(t)\right| < \frac{1}{2}\left(\frac{\epsilon}{2M\cdot 3^{1/q}}\right)^{p/p-2} \tag{3.5}$$

for all $G\epsilon\Sigma$ whenever $\|h\| < \delta_2$.

Choose $T_1\epsilon\Sigma$ with $0 < \mu(T_1) < \delta_1$ such that $f(T \sim T_1)$ is precompact, and choose $\delta_3 = \delta_3(f,\epsilon) > 0$ such that

$$\|\Theta_x(y)\| < \frac{\epsilon}{3^{1/q}} \|y\| \tag{3.6}$$

for all $x\epsilon f(T \sim T_1)$ whenever $\|y\| < \delta_3$.

Now let $\delta_0 = \min(\delta_2, \delta_3, \delta_1^{1/p}\cdot\delta_3)$ and let $h\epsilon L_p(E,\mu)$ with $\|h\| < \delta_0$. Proceeding as in the proof of Theorem 2.5, it is seen that

$$\left(\int_T \|\Theta_{f(t)}(h(t))\|^q d\mu(t)\right)^{1/q} < \epsilon\|h\|$$

whenever $\|h\| < \delta_0$.

<u>Case 2</u>: $\mu(T) = \infty$. Again, since the support of f is σ-finite, it can be assumed that (T,Σ,μ) is σ-finite. Let $\{A_n\}_{n\geq 1}$ be a sequence of pairwise disjoint sets from Σ such that $\mu(A_n) < \infty$ for all $n \geq 1$ and $T = \bigcup_{n=1}^{\infty} A_n$. Since

$\|f\|^p = \sum_{n=1}^{\infty} \int_{A_n} \|f(t)\|^p d\mu(t) = 1$, choose N_0 such that

$$\int_{\bigcup_{n=N_0+1}^{\infty} A_n} \|f(t)\|^p d\mu(t) < \frac{1}{2}\left(\frac{\epsilon}{2M\cdot 2^{1/q}}\right)^{p/p-2} .$$

By the same procedure as above, there exists a $\delta_1 = \delta_1(f,\epsilon) > 0$ such that

$$\int_{\bigcup_{n=N_0+1}^{\infty} A_n} \|\Theta_{f(t)}(h(t))\|^q d\mu(t) < \frac{\epsilon^q}{2} \|h\|^q$$

whenever $h \in L_p(E,\mu)$ and $\|h\| < \delta_1$. Since $\mu(\bigcup_{n=1}^{N_0} A_n) < \infty$, from Case 1, there exists a $\delta_2 = \delta_2(f,\epsilon) > 0$ such that

$$\int_{\bigcup_{n=1}^{N_0} A_n} \|\Theta_{f(t)}(h(t))\|^q d\mu(t) < \frac{\epsilon^q}{2} \|h\|^q$$

whenever $h \in L_p(E,\mu)$ and $\|h\| < \delta_2$.

Let $\delta_0 = \min(\delta_1, \delta_2)$, then

$$\left(\int_T \|\Theta_{f(t)}(h(t))\|^q d\mu(t)\right)^{1/q} < \epsilon\|h\|$$

whenever $\|h\| < \delta_0$.

Thus, $\hat{g} : L_p(E,\mu) \longrightarrow \mathbb{R}^+$ is twice differentiable at $f \in L_p(E,\mu)$ with

$$\hat{g}''(f)\cdot(h_1,h_2) = \int_T g''(f(t))\cdot(h_1(t),h_2(t))\,d\mu(t)$$

for all $h_1,h_2 \in L_p(E,\mu)$. It remains only to show that the mapping $\hat{g}'' : L_p(E,\mu) \longrightarrow \mathcal{L}(L_p(E,\mu),L_p(E,\mu)^*)$ is continuous. Again, two cases are considered.

<u>Case 1</u>: $\mu(T) < \infty$ (assume $\mu(T) = 1$). Let $f,f_n \in L_p(E,\mu)$ for $n \geq 1$, and suppose $f_n \longrightarrow f$ in $L_p(E,\mu)$. It must be shown that $\hat{g}''(f_n) \longrightarrow \hat{g}''(f)$ in $\mathcal{L}(L_p(E,\mu),L_p(E,\mu)^*)$. Since $f_n \longrightarrow f$ in $L_p(E,\mu)$, there exists a subsequence $\{f_{n_i}\}_{i\geq 1}$ such that $f_{n_i}(t) \longrightarrow f(t)$ in E μ-a.e., and since $g'' : E \longrightarrow \mathcal{L}(E,E^*)$ is continuous, $g''(f_{n_i}(t)) \longrightarrow g''(f(t))$ in $\mathcal{L}(E,E^*)$ μ-a.e. For $h_1,h_2 \in L_p(E,\mu)$,

$$\hat{g}''(f)\cdot(h_1,h_2) = \int_T g''(f(t))\cdot(h_1(t),h_2(t))\,d\mu(t),$$

and therefore,

$$|\hat{g}''(f_{n_i})\cdot(h_1,h_2) - \hat{g}''(f)\cdot(h_1,h_2)|$$

$$\leq \int_T \|g''(f_{n_i}(t)) - g''(f(t))\| \|h_1(t)\| \|h_2(t)\|\,d\mu(t). \tag{3.7}$$

From the Vitali convergence theorem (Dunford and Schwartz [12]), since $f_{n_i} \longrightarrow f$ in $L_p(E,\mu)$, given any $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that

$$\int_G \|f_{n_i}(t)\|^p d\mu(t) < (\tfrac{\epsilon}{4M})^{p/p-2} \quad \text{for all} \quad i \geq 1$$

and (3.8)

$$\int_G \|f(t)\|^p d\mu(t) < (\tfrac{\epsilon}{4M})^{p/p-2},$$

whenever $\mu(G) < \delta(\epsilon)$. (Here $\|g''(x)\| \leq M$ for all $x \in S$.) From Egoroff's theorem, there exists an $F \in \Sigma$ with $0 < \mu(F) < \delta(\epsilon)$ such that

$$\|g''(f_{n_i}(t)) - g''(f(t))\| \to 0, \quad \text{as} \quad n_i \to \infty$$

uniformly on $T \sim F$; that is, given any $\epsilon > 0$, there exists an $N(\epsilon)$ such that

$$\|g''(f_{n_i}(t)) - g''(f(t))\| < \frac{\epsilon}{2} \tag{3.9}$$

for all $t \in T \sim F$, whenever $n_i > N(\epsilon)$.

Therefore, from (3.7) and Hölder's inequality,

$$\int_{T\sim F} \|g''(f_{n_i}(t))-g''(f(t))\| \|h_1(t)\| \|h_2(t)\| d\mu(t) < \frac{\epsilon}{2}\|h_1\| \|h_2\| \tag{3.10}$$

for all $h_1, h_2 \in L_p(E,\mu)$, whenever $n_i > N(\epsilon)$.

Now, since $\|g''(x)\| \leq M\|x\|^{p-2}$,

$$\int_F \|g''(f_{n_i}(t))-g''(f(t))\| \|h_1(t)\| \|h_2(t)\| d\mu(t)$$

$$\leq M \int_F \{\|f_{n_i}(t)\|^{p-2} + \|f(t)\|^{p-2}\} \|h_1(t)\| \|h_2(t)\| d\mu(t),$$

and from Hölder's inequality,

$$\int_F \|g''(f_{n_i}(t))-g''(f(t))\|\|h_1(t)\|\|h_2(t)\|d\mu(t)$$

$$\leq M[(\int_F \|f_{n_i}(t)\|^p d\mu(t))^{p-2/p} + (\int_F \|f(t)\|^p d\mu(t))^{p-2/p}]\|h_1\|\|h_2\|. \tag{3.11}$$

From (3.8), since $\mu(F) < \delta(\epsilon)$,

$$\int_F \|g''(f_{n_i}(t))-g''(f(t))\|\|h_1(t)\|\|h_2(t)\|d\mu(t) < \frac{\epsilon}{2}\|h_1\|\|h_2\| \tag{3.12}$$

for all n_i, and for all $h_1,h_2 \in L_p(E,\mu)$.

Combining (3.10) and (3.12) if $n_i > N(\epsilon)$, then

$$\int_T \|g''(f_{n_i}(t))-g''(f(t))\|\|h_1(t)\|\|h_2(t)\|d\mu(t) < \epsilon\|h_1\|\|h_2\|$$

for all $h_1,h_2 \in L_p(E,\mu)$; that is,

$$|\hat{g}''(f_{n_i})\cdot(h_1,h_2)-\hat{g}''(f)\cdot(h_1,h_2)| < \epsilon\|h_1\|\|h_2\|$$

for all $h_1,h_2 \in L_p(E,\mu)$, whenever $n_i > N(\epsilon)$.

Taking the supremum over all $h_1,h_2 \in L_p(E,\mu)$ with $\|h_1\| = \|h_2\| = 1$, then, $\|\hat{g}''(f_{n_i}) - \hat{g}''(f)\| \leq \epsilon$ whenever $n_i > N(\epsilon)$. Thus, $\hat{g}''(f_{n_i}) \to \hat{g}''(f)$ in $\mathcal{L}(L_p(E,\mu),L_p(E,\mu)^*)$ as $n_i \to \infty$.

What has actually been shown is that every subsequence of the sequence $\{\hat{g}''(f_n)\}_{n\geq 1}$ contains a convergent subsequence, and they all converge to the same limit, namely, $\hat{g}''(f)$. Therefore the entire sequence $\{\hat{g}''(f_n)\}_{n\geq 1}$ converges to $\hat{g}''(f)$, that is, $\hat{g}''(f_n) \to \hat{g}''(f)$ in $\mathcal{L}(L_p(E,\mu),L_p(E,\mu)^*)$.

Case 2: $\mu(T) = \infty$. Let $f_n, f \in L_p(E,\mu)$, $n \geq 1$, with $f_n \to f$ in $L_p(E,\mu)$. Again, there exists a subsequence $\{f_{n_i}\}_{i\geq 1}$ such that $f_{n_i}(t) \to f(t)$ in E μ-a.e.; and by continuity of g'', $g''(f_{n_i}(t)) \to g''(f(t))$ in $\mathcal{L}(E,E^*)$ μ-a.e.

By the Vitali convergence theorem (Dunford and Schwartz [12]), given $\epsilon > 0$, there exists a set $E_\epsilon \in \Sigma$ with $0 \leq \mu(E_\epsilon) < \infty$ such that

$$\int_{T\sim E_\epsilon} \|f_{n_i}(t)\|^p d\mu(t) < (\tfrac{\epsilon}{4M})^{p/p-2} \quad \text{for all } n_i$$

and (3.13)

$$\int_{T\sim E_\epsilon} \|f(t)\|^p d\mu(t) < (\tfrac{\epsilon}{4M})^{p/p-2}.$$

By the previous case, since $\mu(E_\epsilon) < \infty$, given any $\epsilon > 0$, there exists an $N(\epsilon)$ such that

$$\int_{E_\epsilon} \|g''(f_{n_i}(t)) - g''(f(t))\| \|h_1(t)\| \|h_2(t)\| d\mu(t) < \tfrac{\epsilon}{2}\|h_1\|\|h_2\|$$

for all $h_1, h_2 \in L_p(E,\mu)$ whenever $n_i > N(\epsilon)$.

From (3.13), reasoning as above,

$$\int_{T\sim E_\epsilon} \|g''(f_{n_i}(t)) - g''(f(t))\| \|h_1(t)\| \|h_2(t)\| d\mu(t) < \tfrac{\epsilon}{2}\|h_1\|\|h_2\| \quad (3.14)$$

for all $h_1, h_2 \in L_p(E,\mu)$, and for all n_i.

Thus,

$$|\hat{g}''(f_{n_i}) \cdot (h_1,h_2) - \hat{g}''(f) \cdot (h_1,h_2)| < \epsilon\|h_1\|\|h_2\|$$

for all $h_1, h_2 \in L_p(E,\mu)$ whenever $n_i > N(\epsilon)$. Therefore, $\|\hat{g}''(f_{n_i}) - \hat{g}''(f)\| \leq \epsilon$ whenever $n_i > N(\epsilon)$; that is, $\hat{g}''(f_{n_i}) \to \hat{g}''(f)$ in $\mathcal{L}(L_p(E,\mu), L_p(E,\mu)^*)$. Reasoning as before, this implies that $\hat{g}''(f_n) \to \hat{g}''(f)$ in $\mathcal{L}(L_p(E,\mu), L_p(E,\mu)^*)$ as $n \to \infty$.

Therefore, the norm $\|\cdot\| : L_p(E,\mu) \to \mathbb{R}^+$ is twice continuously differentiable away from zero. q.e.d.

The converse of this theorem is also true:

<u>3.2. Theorem</u>. Let E be a Banach space, (T,Σ,μ) a measure space and $p > 2$. If the norm $\|\cdot\| : L_p(E,\mu) \to \mathbb{R}^+$ is twice continuously differentiable away from zero, then the norm $\|\cdot\| : E \to \mathbb{R}^+$ is twice continuously differentiable away from zero and the second derivative of the norm in E is uniformly bounded on the unit sphere in E.

<u>Proof</u>: Suppose the norm in $L_p(E,\mu)$ is twice continuously differentiable away from zero; since E is isometrically isomorphic to a closed subspace of $L_p(E,\mu)$, it follows immediately that the norm in E is twice continuously differentiable away from zero. The main difficulty here is showing that the second derivative of the norm in E is uniformly bounded on the unit sphere in E. In this case, since $\ell_p(E)$ is isometrically isomorphic to a closed subspace of $L_p(E,\mu)$, it is sufficient to work with the sequence spaces $\ell_p(E)$. Define $g : E \to \mathbb{R}^+$ and $\hat{g} : \ell_p(E) \to \mathbb{R}^+$ in the usual manner:

$$g(x) = \frac{1}{p}\|x\|^p \quad \text{for} \quad x \in E$$

and $\hat{g}(a) = \frac{1}{p}\|a\|^p = \frac{1}{p}\sum_{n=1}^{\infty}\|a_n\|^p$ for $a = \{a_n\}_{n \geq 1} \in \ell_p(E)$.

Since $p > 2$, then g and $\hat{g}$ are twice differentiable at $x = 0$ and $a = 0$, respectively, with $g'(0) = g''(0) = 0$ and $\hat{g}'(0) = \hat{g}''(0) = 0$. Note also that

$$\hat{g}(a) = \sum_{n=1}^{\infty} g(a_n) \qquad \text{and} \qquad \hat{g}'(a)\cdot h = \sum_{n=1}^{\infty} g'(a_n)\cdot h_n$$

for all $a,h \in \ell_p(E)$ (see Theorem 2.5 with $T = \mathbb{N}$ and μ the counting measure).

Now if $a,b \in \ell_p(E)$, from Taylor's formula,

$$\hat{g}(a + b) = \hat{g}(a) + \hat{g}'(a)\cdot b + \frac{1}{2}\hat{g}''(a)\cdot(b,b) + \hat{\theta}_a(b)$$

where $\dfrac{\hat{\theta}_a(b)}{\|b\|^2} \to 0$ as $\|b\| \to 0$, and hence

$$\hat{g}''(a)\cdot(b,b) = 2(\hat{g}(a+b) - \hat{g}(a)) - 2\hat{g}'(a)\cdot b - 2\hat{\theta}_a(b). \tag{3.15}$$

Again, from Taylor's formula, for all $k \geq 1$,

$$g(a_k+b_k) = g(a_k) + g'(a_k)\cdot b_k + \frac{1}{2}g''(a_k)\cdot(b_k,b_k) + \theta_{a_k}(b_k) \tag{3.16}$$

where $\dfrac{\theta_x(y)}{\|y\|^2} \to 0$ as $\|y\| \to 0$.

If $n \geq 1$, let $b^n = (b_1,b_2,\ldots,b_n,0,0,\ldots)$, then from (3.15)

$$\hat{g}''(a)\cdot(b^n,b^n) = \frac{2}{p}\sum_{k=1}^{n}(g(a_k+b_k)-g(a_k)) - 2\sum_{k=1}^{n} g'(a_k)\cdot b_k - 2\hat{\Theta}_a(b^n),$$

while from (3.16),

$$\hat{g}''(a)\cdot(b^n,b^n) = \sum_{k=1}^{n} g''(a_k)\cdot(b_k,b_k) + 2\sum_{k=1}^{n}\Theta_{a_k}(b_k) - 2\hat{\Theta}_a(b^n). \tag{3.17}$$

If $\lambda \neq 0$, then

$$\hat{g}''(a)\cdot(\lambda b^n,\lambda b^n) = \sum_{k=1}^{n} g''(a_k)\cdot(\lambda b_k,\lambda b_k) + 2\sum_{k=1}^{n}\Theta_{a_k}(\lambda b_k) - 2\hat{\Theta}_a(\lambda b^n),$$

and therefore,

$$\hat{g}''(a)\cdot(b^n,b^n) = \sum_{k=1}^{n} g''(a_k)\cdot(b_k,b_k) + 2\sum_{k=1}^{n}\frac{\Theta_{a_k}(\lambda b_k)}{\lambda^2} - \frac{2\hat{\Theta}_a(\lambda b^n)}{\lambda^2}.$$

Letting $\lambda \to 0$, then

$$\hat{g}''(a)\cdot(b^n,b^n) = \sum_{k=1}^{n} g''(a_k)\cdot(b_k,b_k)$$

for all $b \in \ell_p(E)$ and for all $n \geq 1$. Since $\hat{g}''(a)$ is symmetric, it is determined by its values on the diagonal, hence

$$\hat{g}''(a)\cdot(b^n,c^n) = \sum_{k=1}^{n} g''(a_k)\cdot(b_k,c_k) \tag{3.18}$$

for all $b,c \in \ell_p(E)$ and for all $n \geq 1$.

Now $\|b^n-b\| = \left(\sum_{k=n+1}^{\infty}\|b_k\|^p\right)^{1/p}$, $\|c^n-c\| = \left(\sum_{k=n+1}^{\infty}\|c_k\|^p\right)^{1/p}$ and $b^n \to b$, $c^n \to c$ in $\ell_p(E)$ as $n \to \infty$; since $\hat{g}''(a)$ is a continuous bilinear form, then

$$\lim_{n \to \infty} \hat{g}''(a)\cdot(b^n, c^n) = \hat{g}(a)\cdot(b,c) < \infty .$$

Therefore, $\lim_{n \to \infty} \sum_{k=1}^{n} g''(a_k)\cdot(b_k, c_k) = \hat{g}''(a)\cdot(b,c) < \infty$, that is,

$$\hat{g}''(a)\cdot(b,c) = \sum_{k=1}^{\infty} g''(a_k)\cdot(b_k, c_k) < \infty \tag{3.19}$$

for all $b, c \in \ell_p(E)$. This implies that there exists an $M > 0$ such that $\|g''(x)\| \leq M$ for all $x \in S$. If the assertion is false, then there exists a sequence $\{x_n\}_{n\geq 1}$ in E with $\|x_n\| = 1$ for all $n \geq 1$, such that

$$\|g''(x_n)\| > 2^n \quad \text{for all} \quad n \geq 1. \tag{3.20}$$

Given $\epsilon > 0$, there exist sequences $\{b_n\}_{n\geq 1}$, $\{c_n\}_{n\geq 1}$ in E with $\|b_n\| = \|c_n\| = 1$ for all $n \geq 1$, such that

$$g''(x_n)\cdot(b_n, c_n) > \|g''(x_n)\| - \epsilon \quad \text{for all} \quad n \geq 1. \tag{3.21}$$

Therefore,

$$g''(x_n)\cdot\left(\frac{b_n}{2^{n/p}}, \frac{c_n}{2^{n/p}}\right) > \frac{1}{2^{2n/p}}\|g''(x_n)\| - \frac{\epsilon}{2^{2n/p}}, \tag{3.22}$$

and since $g''(x_n) = (2^{n/p})^{p-2} g''\left(\frac{x_n}{2^{n/p}}\right)$, then (3.22) implies that

$$g''\left(\frac{x_n}{2^{n/p}}\right)\cdot\left(\frac{b_n}{2^{n/p}}, \frac{c_n}{2^{n/p}}\right) > \frac{\|g''(x_n)\|}{2^n} - \frac{\epsilon}{2^n} \tag{3.23}$$

for all $n \geq 1$.

Let $a = \{\frac{x_n}{2^{n/p}}\}_{n\geq 1}$, $b = \{\frac{b_n}{2^{n/p}}\}_{n\geq 1}$, and $c = \{\frac{c_n}{2^{n/p}}\}_{n\geq 1}$, then $a,b,c \in \ell_p(E)$ and $\|a\| = \|b\| = \|c\| = 1$; but

$$\hat{g}''(a)\cdot(b,c) = \sum_{n=1}^{\infty} g''(\frac{x_n}{2^{n/p}})\cdot(\frac{b_n}{2^{n/p}}, \frac{c_n}{2^{n/p}}) > \sum_{n=1}^{\infty} \frac{\|g''(x_n)\|}{2^n} - \epsilon$$

$$> \sum_{n=1}^{\infty} \frac{2^n}{2^n} - \epsilon = \infty,$$

a contradiction, since by hypothesis $\hat{g}''(a)\cdot(b,c) < \infty$. q.e.d.

Theorem 3.1 and Theorem 3.2 together yield the following theorem:

<u>3.3. Theorem</u>. If E is a Banach space, (T,Σ,μ) a measure space and $p > 2$, then the norm $\|\cdot\| : L_p(E,\mu) \longrightarrow \mathbb{R}^+$ is twice continuously differentiable away from zero if and only if:

(i) the norm $\|\cdot\| : E \longrightarrow \mathbb{R}^+$ is twice continuously differentiable away from zero, and

(ii) the second derivative of the norm in E is uniformly bounded on the unit sphere in E.

Suppose now that k is a positive integer, $p > k$, and E is a Banach space whose norm is k-times continuously differentiable away from zero. The mappings $g : E \longrightarrow \mathbb{R}^+$ and $\hat{g} : L_p(E,\mu) \longrightarrow \mathbb{R}^+$ are defined as usual:

$$g(x) = \frac{1}{p}\|x\|^p \qquad \text{for } x \in E$$

$$\hat{g}(f) = \frac{1}{p}\|f\|^p = \frac{1}{p}\int_T \|f(t)\|^p d\mu(t) \qquad \text{for } f \in L_p(E,\mu).$$

<u>3.4. Lemma</u>. (i) If $g^{(k)} : E \longrightarrow \mathcal{B}^k(E, \mathbb{R})$ is uniformly bounded on S, then $g^{(\ell)} : E \longrightarrow \mathcal{B}^\ell(E, \mathbb{R})$ is uniformly bounded on S for $1 \leq \ell \leq k-1$.

(ii) If $x \in E$, $x \neq 0$, and $\lambda \neq 0$, then for $1 \leq \ell \leq k$,

$$g^{(\ell)}(\lambda x) = (\operatorname{sgn} \lambda)^\ell |\lambda|^{p-\ell} g^{(\ell)}(x).$$

(iii) If $p > k$, and $\|g^{(k)}(x)\| \leq M$ for all $x \in S$, then g is k-times continuously differentiable at $x = 0$, and

$$g^{(\ell)}(0) = 0$$

for $1 \leq \ell \leq k$.

Statements (i) and (ii) follow from the definition of g, and (iii) is a consequence of (i) and (ii).

Assume now that the norm $\|\cdot\| : L_p(E,\mu) \longrightarrow \mathbb{R}^+$ is $(k-1)$-times continuously differentiable away from zero, and that for $f \in L_p(E,\mu)$,

$$\hat{g}^{(k-1)}(f)\cdot(h^2,h^3,\ldots,h^k) = \int_T g^{(k-1)}(f(t))\cdot(h^2(t),h^3(t),\ldots,h^k(t))\,d\mu(t)$$

for all $h^2,h^3,\ldots,h^k \in L_p(E,\mu)$. Writing

$$g^{(k-1)}(x+y) = g^{(k-1)}(x) + g^{(k)}(x)\cdot y + \Theta_x(y),$$

where $\frac{\|\Theta_x(y)\|}{\|y\|} \to 0$ as $\|y\| \to 0$, a straightforward induction argument then shows that the norm $\|\cdot\| : L_p(E,\mu) \to \mathbb{R}^+$ is k-times continuously differentiable away from zero and that for $f \in L_p(E,\mu)$,

$$\hat{g}^{(k)}(f)\cdot(h^1,h^2,\ldots,h^k) = \int_T g^{(k)}(f(t))\cdot(h^1(t),h^2(t),\ldots,h^k(t))\,d\mu(t)$$

for all $h^1,h^2,\ldots,h^k \in L_p(E,\mu)$.

Hence the following theorem holds:

3.5. Theorem. Let E be a Banach space, (T,Σ,μ) a measure space, and k a positive integer with $p > k$. If the norm $\|\cdot\| : E \to \mathbb{R}^+$ is k-times continuously differentiable away from zero and the kth-derivative of the norm in E is uniformly bounded on the unit sphere in E, then the norm $\|\cdot\| : L_p(E,\mu) \to \mathbb{R}^+$ is k-times continuously differentiable away from zero.

In order to prove the converse of this theorem, suppose that $p > k$ and the norm $\|\cdot\| : L_p(E,\mu) \to \mathbb{R}^+$ is k-times continuously differentiable away from zero. Since E is isometrically isomorphic to a closed subspace of $L_p(E,\mu)$, it is clear that the norm $\|\cdot\| : E \to \mathbb{R}^+$ is k-times continuously differentiable away from zero. Again, as in the case of the second derivative, the main difficulty is showing that the kth-derivative of the norm in E is uniformly bounded on the unit sphere in E. Also, as in the case of the second derivative, it suffices to consider the sequence spaces $\ell_p(E)$. Proceeding by induction, it is assumed first that whenever $a \in \ell_p(E)$ and $1 \leq \ell \leq k-1$, the following are true:

$$(i)_\ell \quad \hat{g}^{(\ell)}(a)\cdot(b^1,b^2,\ldots,b^\ell) = \sum_{n=1}^{\infty} g^{(\ell)}(a_n)\cdot(b_n^1,b_n^2,\ldots,b_n^\ell)$$

for all $b^1,b^2,\ldots,b^\ell \in \ell_p(E)$, and

$(ii)_\ell$ there exists an $M_\ell > 0$ such that

$$\|g^{(\ell)}(x)\| \leq M_\ell$$

for all $x \in S$.

It is then a simple matter to show that $(i)_\ell$ and $(ii)_\ell$ imply that whenever $a \in \ell_p(E)$, the following are true:

$$(i)_k \quad \hat{g}^{(k)}(a)\cdot(b^1,b^2,\ldots,b^k) = \sum_{n=1}^{\infty} g^{(k)}(a_n)\cdot(b_n^1,b_n^2,\ldots,b_n^k)$$

for all $b^1,b^2,\ldots,b^k \in \ell_p(E)$, and

$(ii)_k$ there exists an $M_k > 0$ such that

$$\|g^{(k)}(x)\| \leq M_k$$

for all $x \in S$.

Hence the following theorem is true:

3.6. Theorem. Let E be a Banach space, (T,Σ,μ) a measure space, and k a positive integer with $p > k$. If the norm $\|\cdot\| : L_p(E,\mu) \longrightarrow \mathbb{R}^+$ is k-times continuously differentiable away from zero, then the norm $\|\cdot\| : E \longrightarrow \mathbb{R}^+$ is k-times continuously differentiable away from zero and the kth-derivative of the norm in E is uniformly bounded on the unit sphere in E.

Combining Theorem 3.5 and Theorem 3.6:

3.7. Theorem. Let E be a Banach space, (T,Σ,μ) a measure space, and k a positive integer with $p > k$. The norm $\|\cdot\| : L_p(E,\mu) \longrightarrow \mathbb{R}^+$ is k-times continuously differentiable away from zero if and only if:

(i) the norm $\|\cdot\| : E \longrightarrow \mathbb{R}^+$ is k-times continuously differentiable away from zero, and

(ii) the k-th derivative of the norm in E is uniformly bounded on the unit sphere in E.

As a corollary of this theorem:

3.8. Corollary. Let E be a Banach space, (T,Σ,μ) a measure space, and $1 < p < \infty$.

(i) If p is not an integer and $I(p)$ is the integral part of p, then the norm $\|\cdot\| : L_p(E,\mu) \longrightarrow \mathbb{R}^+$ is $I(p)$-times continuously differentiable away from zero if and only if the norm $\|\cdot\| : E \longrightarrow \mathbb{R}^+$ is $I(p)$-times continuously differentiable away from zero and the $I(p)$-th derivative of the norm in E is uniformly bounded on the unit sphere in E.

(ii) If p is an odd integer, then the norm $\|\cdot\| : L_p(E,\mu) \longrightarrow \mathbb{R}^+$ is $(p-1)$-times continuously differentiable away from zero if and only if the norm $\|\cdot\| : E \longrightarrow \mathbb{R}^+$ is $(p-1)$-times continuously differentiable away from zero and the $(p-1)$-st derivative of the norm in E is uniformly bounded on the unit sphere in E.

Proof: The proof of (i) is immediate from Theorem 3.7, by taking $k = I(p) < p$. The proof of (ii) also follows from Theorem 3.7, by taking $k = p-1 < p$. The fact that $L_p(\mathbb{R},\mu)$ is isometrically isomorphic to a closed subspace of $L_p(E,\mu)$, and the fact that p is an odd integer, imply that $L_p(E,\mu)$ is not p-times continuously differentiable away from zero (see Sundaresan [18]). q.e.d.

The only remaining case is when p is an even integer. The case when $p = 2$ is not as difficult as the previous case, even though the results are very surprising:

3.9. Theorem. If E is a Banach space, and (T,Σ,μ) is a measure space, then the norm $\|\cdot\| : L_2(E,\mu) \longrightarrow \mathbb{R}^+$ is twice continuously differentiable away from zero if and only if E is a Hilbert space.

Proof: If E is a Hilbert space, then $L_2(E,\mu)$ is a Hilbert space and the result follows. Conversely, suppose the norm in $L_2(E,\mu)$ is twice continuously differentiable away from zero. As in the previous theorem, it is sufficient to work with the sequence space $\ell_2(E)$. Note that with $g(x) = \frac{1}{2}\|x\|^2$ for $x \in E$,

$$\frac{\frac{1}{2}\|0+y\|^2 - \frac{1}{2}\|0\|^2}{\|y\|} = \frac{1}{2}\|y\| \longrightarrow 0$$

as $\|y\| \longrightarrow 0$; therefore $g : E \longrightarrow \mathbb{R}^+$ is differentiable at $x = 0$ and $g'(0) = 0$.

From Taylor's formula, if $x,y \in E$, $x \neq 0$; then

$$\|x+y\|^2 = \|x\|^2 + 2g'(x)\cdot y + g''(x)\cdot(y,y) + \Theta_x(y) \tag{3.24}$$

where $\frac{\Theta_x(y)}{\|y\|^2} \longrightarrow 0$ as $\|y\| \longrightarrow 0$.

Let $x_0 \in E$, $x_0 \neq 0$, be arbitrary. Let $a = (x_0, 0, 0, \ldots)$, then $a \in \ell_2(E)$ and $\|a\| = \|x_0\|$. If $h = \{h_n\}_{n \geq 1} \in \ell_2(E)$, from Taylor's formula,

$$\|a+h\|^2 = \|a\|^2 + 2\hat{g}'(a)\cdot h + \hat{g}''(a)\cdot(h,h) + \hat{\Theta}_a(h) \tag{3.25}$$

where $\frac{|\hat{\Theta}_a(h)|}{\|h\|^2} \longrightarrow 0$ as $\|h\| \longrightarrow 0$.

Since $\hat{g}'(a)\cdot h = \sum_{k=1}^{\infty} g'(a_k)\cdot h_k = g'(x_0)\cdot h_1$ $(g'(0) = 0)$, from (3.25) it follows that

$$\|x_0+h_1\|^2 + \sum_{n=2}^{\infty} \|h_n\|^2 = \|x_0\|^2 + 2g'(x_0)\cdot h_1 + \hat{g}''(a)\cdot(h,h) + \hat{\Theta}_a(h). \tag{3.26}$$

From (3.24), with $x = x_0$ and $y = h_1$, it follows that

$$\|x_0+h_1\|^2 = \|x_0\|^2 + 2g'(x_0)\cdot h_1 + g''(x_0)\cdot(h_1,h_1) + \Theta_{x_0}(h_1). \tag{3.27}$$

Combining (3.26) and (3.27),

$$\hat{g}''(a)\cdot(h,h) - \|h\|^2 = g''(x_0)\cdot(h_1,h_1) - \|h_1\|^2 + \Theta_{x_0}(h_1) - \hat{\Theta}_a(h). \tag{3.28}$$

Let $t \neq 0$, then

$$\hat{g}''(a)\cdot(th,th) - \|th\|^2 = g''(x_0)\cdot(th_1,th_1) - \|th_1\|^2$$
$$+ \Theta_{x_0}(th_1) - \hat{\Theta}_a(th),$$

that is,

$$\hat{g}''(a)\cdot(h,h) - \|h\|^2 = g''(x_0)\cdot(h_1,h_1) - \|h_1\|^2 + \frac{\Theta_{x_0}(th_1)}{t^2} - \frac{\hat{\Theta}_a(th)}{t^2}.$$

Letting $t \rightarrow 0$, it follows that

$$\hat{g}''(a)\cdot(h,h) - \|h\|^2 = g''(x_0)\cdot(h_1,h_1) - \|h_1\|^2 \tag{3.29}$$

for $a = (x_0,0,0,\ldots)$, $h = (h_1,h_2,\ldots,h_n,\ldots) \in \ell_2(E)$.

Now let $\xi\in E$ be arbitrary and let $h = (0,\xi,0,0,\ldots)$. From (3.29), it follows that

$$\hat{g}''(a)\cdot(h,h) - \|h\|^2 = g''(x_0)\cdot(h_1,h_1) - \|h_1\|^2 = 0,$$

since now $h_1 = 0$. But $\|h\|^2 = \|\xi\|^2$, and therefore

$$\|\xi\|^2 = \hat{g}''(a)\cdot((0,\xi,0,0,\ldots),(0,\xi,0,0,\ldots)).$$

Since $\xi\in E$ is arbitrary and $\hat{g}''(a) : \ell_2(E) \times \ell_2(E) \longrightarrow \mathbb{R}$ is a positive, symmetric, bounded bilinear form, then E is a Hilbert space. q.e.d.

If p is an even integer, $p \neq 2$, using an argument almost identical to that in the proof of the preceding theorem, the following theorem is obtained:

<u>3.10. Theorem</u>. Let E be a Banach space, (T,Σ,μ) a measure space, and p an even integer, $p \neq 2$. The norm $\|\cdot\| : L_p(E,\mu) \longrightarrow \mathbb{R}^+$

is p-times continuously differentiable away from zero if and only if the pth-power of the norm in E is a continuous homogeneous polynomial of degree p. (In which case the norm in E is of class C^{∞}, and from Theorem 3.7 this implies that the norm in $L_p(E,\mu)$ is of class C^{∞}.)

Combining Theorem 2.5, Corollary 3.8, and Theorem 3.10, the order of smoothness of the Lebesgue-Bochner function spaces can be summarized in the following theorem:

3.11. Theorem. Let E be a Banach space, (T,Σ,μ) a measure space, and $1 < p < \infty$, then:

(i) The norm in $L_p(E,\mu)$ is differentiable away from zero if and only if the norm in E is differentiable away from zero.

(ii) If $p = 2$, the norm in $L_p(E,\mu)$ is twice continuously differentiable away from zero if and only if E is a Hilbert space.

(iii) If p is an even integer, $p \neq 2$, the norm in $L_p(E,\mu)$ is p-times continuously differentiable away from zero if and only if the pth-power of the norm in E is a continuous homogeneous polynomial of degree p.

(iv) If p is an odd integer, the norm in $L_p(E,\mu)$ is (p-1)-times continuously differentiable away from zero if and only if the norm in E is (p-1)-times continuously differentiable away from zero and the (p-1)-st derivative of the norm in E is uniformly bounded on the unit sphere in E.

(v) If p is not an integer, and I(p) is the integral part of p, the norm in $L_p(E,\mu)$ is I(p)-times continuously differentiable away from zero if and only if the norm in E is I(p)-times continuously differentiable away from zero and the I(p)-th derivative of the norm in E is uniformly bounded on the unit sphere in E.

<u>3.12. Remark</u>. The results on the order of differentiability of the norm in $L_p(E,\mu)$ are exactly the same as the preceding results on the continuous differentiability of the norm (except that "continuous differentiability" is to be replaced by "differentiability") and will not be discussed here.

4. Example

The purpose of this section is to discuss an example which brings forth the importance of the theorems characterizing the order of differentiability of the norm function in the Lebesgue-Bochner function spaces $L_p(E,\mu)$. Also, it is this example that led to a deeper study of the order of smoothness of the norm in $L_p(E,\mu)$, $1 < p < \infty$. Certain results in Bonic and Frampton [5] were found to be of considerable importance, especially the result by N. H. Kuiper (added in proof in [5], p. 896). Before starting this example, a few lemmas are necessary.

4.1. Lemma. Let E be a Banach space whose norm $\|\cdot\| : E \longrightarrow \mathbb{R}^+$ is twice differentiable away from zero, and define $g : E \longrightarrow \mathbb{R}^+$ by $g(x) = \frac{1}{2}\|x\|^2$ for $x \in E$. If there exists an $M > 0$ such that

$$g''(x)\cdot(y,y) \leq M\|y\|^2$$

for all $x \in S$ and for all $y \in E$, then E is uniformly smooth.

Proof: Recall that E is uniformly smooth if and only if given any $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that

$$\left|\frac{\|x+ty\|-\|x\|-tG_x(y)}{t}\right| < \epsilon$$

for all $(x,y) \in S \times S$, whenever $|t| < \delta$.

Let $x,h \in E$, $\|x\| = \|h\| = 1$, and let B be the open annulus

$$B = \{y \in E \mid \tfrac{1}{2} < \|y\| < \tfrac{3}{2}\}.$$

Note that if $|\alpha| < \frac{1}{2}$, then the line segment joining x and $x + \alpha h$ lies in B, that is

$$[x,x+\alpha h] = \{y\epsilon E \mid y = x + \lambda\cdot\alpha h,\ 0 \leq \lambda \leq 1\} \subset B.$$

Hence, $[x,x+\alpha h] \subset B$ for all $|\alpha| < \frac{1}{2}$ and for all $(x,h)\epsilon S \times S$.

If $T^{(2)}(x) : E \longrightarrow E^*$ is the second derivative of the norm at $x \neq 0$, then

$$g''(x)\cdot(y,y) = G_x(y)^2 + \|x\| T^{(2)}(x)\cdot(y,y) \tag{4.1}$$

for all $x,y\epsilon E$, $x \neq 0$. Therefore,

$$T^{(2)}(x)\cdot(y,y) \leq \frac{M+1}{\|x\|}\|y\|^2 \tag{4.2}$$

for all $x,y\epsilon E$, $x \neq 0$; hence

$$T^{(2)}(x)\cdot(y,y) \leq 2(M+1)\|y\|^2$$

for all $y\epsilon E$ and for all $x\epsilon B$.

Since B is open, from Taylor's formula (Cartan [6, p. 70, Theorem 5.6.2]),

$$\big|\,\|x+\alpha h\| - \|x\| - \alpha G_x(h)\,\big| \leq (M+1)|\alpha|^2 \tag{4.3}$$

for all $(x,h)\epsilon S \times S$ and for all α, $|\alpha| < \frac{1}{2}$. Therefore,

$$\left|\frac{\|x+\alpha h\| - \|x\| - \alpha G_x(h)}{\alpha}\right| \leq (M+1)|\alpha| \tag{4.4}$$

for all $(x,h)\epsilon S \times S$ and for all α, $0 < |\alpha| < \frac{1}{2}$. Let $\delta = \delta(\epsilon) = \min(\frac{\epsilon}{M+1}, \frac{1}{2})$, then

$$\left|\frac{\|x+\alpha h\| - \|x\| - \alpha G_x(h)}{\alpha}\right| < \epsilon \tag{4.5}$$

for all $(x,h) \in S \times S$, whenever $0 < |\alpha| < \delta$. Thus, E is uniformly smooth. q.e.d.

4.2. Lemma. [Kuiper] The Banach space c_0 is isomorphic to a Banach space whose norm is of class C^∞ away from zero.

The proof of this lemma can be found on p. 896 of Bonic and Frampton [5].

4.3. Example. Let E be the Banach space whose norm is of class C^∞ away from zero and which is isomorphic to c_0. Let (T,Σ,μ) be a measure space such that the measure μ is not supported by finitely many atoms, and such that the range of μ contains at least one non-zero real number. If p is an arbitrary positive real number, $1 \leq p \leq \infty$, then the norm $\|\cdot\| : L_p(E,\mu) \longrightarrow \mathbb{R}^+$ is not even twice differentiable away from zero.

The cases $p = 1$ and $p = \infty$ are trivial, since in these cases the norm in $L_p(E,\mu)$ is not even once differentiable away from zero. In the case $1 < p < 2$, from the previous theorems, the norm in $L_p(E,\mu)$ is not twice differentiable away from zero. If $p = 2$, and the norm in $L_p(E,\mu)$ is twice differentiable away from zero, from Theorem 3.9, E is a Hilbert space and hence reflexive; a contradiction, since this implies c_0 is reflexive. The remaining case, $2 < p < \infty$, is handled as follows: Suppose $2 < p < \infty$ and the norm in $L_p(E,\mu)$ is twice differentiable away from zero, this implies by Theorem 3.1 that the second derivative of the norm in E is uniformly bounded on the unit sphere in E. However, this implies (by Lemma 4.1) that E is uniformly smooth and hence reflexive, hence c_0 is reflexive, a contradiction.

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