

AVERAGING OPERATORS ON NORMED
FUNCTION SPACES

by

Richard A. Alo and Andre de Korvin

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Errata Sheet For

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- Page 6, Line 3 up: Delete "locally compact"
- Page 10, Line 6: Change to: $U(f) = \int f d\gamma$
- Page 14, Line 7 up: Read as "for $f \in L_{\rho}$ ".
- Page 15, Line 4 up: Insert b for " $C_b(T)$ into $C_b(S)$ ".
- Page 17, line 10: Insert ρ for " $L_{\rho} = L^{\psi}$ ".
- Page 19, Line 6: Delete l in $C(B_1)$
- Page 21, Line 9: Make ρ a superscript in M_{ρ} .
- Page 24, Line 1: Add at end of line, "to $L_{\rho}(\hat{S}, \hat{\Sigma}, \hat{\mu})$ ".

AVERAGING OPERATORS ON NORMED FUNCTION SPACES

by

Richard A. Alo¹ and Andre de Korvin

1. Introduction.

A general problem raised by S. Banach may be formulated as follows. For $1 < p < \infty$ and $p \neq 2$, let \mathfrak{M} be a closed subspace of the Lebesgue space $\mathcal{L}^p = \mathcal{L}^p(S, \Sigma, \mu)$. Does there exist a bounded projection from \mathcal{L}^p onto \mathfrak{M} ? It is known that for $p \neq 2$ there always exists a subspace \mathfrak{M} of \mathcal{L}^p for which the answer is negative. In fact, the same is also true for \mathcal{L}^1 and \mathcal{L}^∞ . However by changing the norm, the above problem may be restated as a query asking when \mathfrak{M} is the range of a contractive projection. This problem has many important applications. As pointed out in [17], in non-linear prediction (and approximation) theory the (not necessarily linear) prediction operator P relative to a Tshebyshev subspace⁽¹²⁾ \mathfrak{M} of a Banach function space $L_\rho(S, \Sigma, \mu)$ is linear if and only if the operator $I - P = Q$ is a contractive projection. This is one of the motivations for studying the projection problem.

Very closely related to the existence of projections (as pointed out below) is the existence of averaging operators. Let φ be a measurable map from the measure space (S, Σ, μ) into the arbitrary Hausdorff topological space T .⁽¹⁷⁾ Let us assume that φ induces a continuous map from the space $C_b(T)$ of bounded real valued continuous functions on T into the (as defined below) Banach function space $L_\rho = L_\rho(S, \Sigma, \mu)$ defined by $\varphi^e(f) = f \circ \varphi$ for all $f \in C_b(T)$. If U is a bounded linear operator from L_ρ

into $C_b(T)$ then U is called an averaging operator for the measurable map φ or it is said that φ admits the averaging operator U if

$$U \circ \varphi^e(f) = f$$

for all $f \in C_b(T)$. (1)

It is clear that if U is an averaging operator for φ (then U is surjective) and if P is the map from L_ρ into L_ρ defined by

$$P = \varphi^e \circ U$$

then $P^2(f) = P(P(f)) = P(f)$. That is, P is the linear projection of L_ρ onto the range $R(\varphi^e)$ in L_ρ of φ^e . Conversely if φ^e is injective (that is, $\rho(\varphi^e(f)) = 0$ implies $f \equiv 0$ for all $f \in C_b(T)$) and if P is a projection onto $R(\varphi^e)$, then

$$(\varphi^e)^{-1} P(\varphi^e(f)) = f.$$

Consequently $(\varphi^e)^{-1} P$ is an averaging map for φ . That is, when φ^e is injective there is a one-to-one correspondence between projections from L_ρ onto the range $R(\varphi^e)$ in L_ρ of φ^e and averaging operators from L_ρ into $C_b(T)$. Or more briefly, φ admits an averaging operator if and only if φ has a left inverse.

In Theorem 5 we show that the existence of an averaging operator for the measurable map φ from S into T , with φ^e injective, is determined by the existence of an averaging operator for φ_1 which is the restriction of φ to a certain measurable subset B_1 of S . This extends the literature⁽¹³⁾ to a larger class of spaces.

As a first result leading to Theorem 3, we show how any bounded linear operator U from L_ρ into $C_b(T)$ may define a function taking each t in T to the linear functional U_t in the Banach dual space $(L_\rho)^* = C_\rho(\mu)$. This is called the integral representation of U and is defined by

$$U_t(f) = \langle f, U^*(\delta_t) \rangle$$

for all $f \in L_\rho$. The operator U^* represents the adjoint of U which takes the dual space $M(T)$ ⁽¹⁸⁾ of $C_b(T)$ into the dual space $(L_\rho)^*$. For example, if T were also locally compact then $M(T)$ would consist of all regular Borel measures on the Stone-Cech compactification of X . For each point $t \in T$, the unit point mass $\delta_t \in M(T)$ is defined by $\delta_t(f) = f(t)$ for all $f \in C_b(T)$.

It should be noted that in the ensuing discussion $C_b(T)$ could be replaced by any appropriate Banach space of functions defined on T where the point evaluation map ⁽¹⁸⁾ is continuous. For example, it could be taken as the space $C_c(T)$ of all continuous functions on T with compact support or it could be taken as the space $C_0(T)$ of all continuous functions on T which vanish at infinity. The salient feature in all of this is that the point evaluation map ⁽¹⁸⁾ must be continuous. It is due to the lack of this that $C_b(T)$ may not be replaced by $L_\rho(T, \Sigma, \mu)$.

By using a representation of $L_\rho(S, \Sigma, \mu)$ (actually M^ρ) as a Banach function space over a compact extremally disconnected

Hausdorff space \hat{S} (see Theorem 6), we are able to give, for the present context, an appropriate definition for the concepts of plural points and of an irreducible map (see Definition 7). This representation theorem also tells us that, in essence, the finitely additive set functions $\{\gamma_t\}_{t \in T} \in G_{\rho'}(\mu)$ used in the integral representation of the operator U may be replaced (via an isomorphism) by regular Borel measures $\{\beta_t\}_{t \in T}$ over \hat{S} . These ideas lead then to an upper bound for the γ_t , namely $|\gamma_t|(S) < \rho(\chi_S)\|U\| - 1$ (see Theorem 8). With these we obtain, as a corollary, a relation between the averaging operator U and its associated U_1 , namely

$$\|U_1(f)\| \leq \|\bar{f}\|_{\mathfrak{D}} [\rho(\chi_S)\|U\| - 1]$$

We are finally led to the consideration of conditions on φ under which no bounded projection from L_{ρ} onto the range of φ^e may exist. Our conditions include that L_{ρ} be reflexive, that $(L_{\rho}^{\alpha})^* \cong L_{\rho'}^{\alpha}$, that the range $R(\varphi^e)$ of φ^e be closed, that ρ' has the weak leveling property, and that there be an appropriate $f_E \in C_b(T)$ representing the operator U_E . Then either φ^e is surjective or no bounded projection P from L_{ρ}^{α} onto $R(\varphi^e)$ exists such that P^* commutes with A_E .

2. Preliminary Results and Definitions.

Let (S, Σ, μ) be a measure space and let M^+ be the collection of non-negative scalar valued μ -measurable functions on S .

As is usual functions differing on μ -null sets will be identified. As is done in [14], a mapping ρ from M^+ into the extended real numbers \mathbb{R}^+ is called a function norm if for all f and $g \in M^+$

- (i) $0 \leq \rho(f) \leq \infty$; $\rho(f) = 0$ if and only if $f \equiv 0$ (μ almost everywhere) where 0 is the constant function on S taking all $s \in S$ to 0 .
- (ii) $\rho(\alpha f) = |\alpha| \rho(f)$ for all finite scalars α .
- (iii) $\rho(f+g) \leq \rho(f) + \rho(g)$
- (iv) $f \leq g$ (μ almost everywhere) on M^+ implies that $\rho(f) \leq \rho(g)$.

The function norm ρ may now be extended to the collection M of all μ -measurable functions by setting $\rho(f) = \rho(|f|)$ for all $f \in M$. We will denote by $L_\rho = L_\rho(S, \Sigma, \mu)$ the normed linear space of all functions $f \in M$ with $\rho(f) < \infty$. The norm on L_ρ is given by $\|f\|_\rho = \rho(|f|)$ and is called the ρ -norm of L_ρ . The spaces L_ρ are called normed Köthe spaces.⁽²⁾

In general the spaces L_ρ are not complete. However under rather weak conditions, such as the weak Fatou property they may be made complete. We will say that L_ρ has the weak Fatou property if whenever a monotonically increasing sequence $\{f_n\}_{n \in \mathbb{N}}$ of functions in L_ρ is pointwise convergent to f and for which the $\sup\{\rho(f_n) : n \in \mathbb{N}\} < \infty$ then $\rho(f) < \infty$, that is $f \in L_\rho$. We will assume that L_ρ has this property, that is, the spaces L_ρ are complete in the ρ -norm. Such complete normed Köthe spaces are called Banach function spaces (see [14], [10], and [2])

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spaces include as examples the well-known
 $= \mathcal{L}^p(S, \Sigma, \mu)$ for $1 \leq p \leq \infty$ and the less
important Orlicz spaces (see [15], [14]).

Generalizations in that they permit many
to be naturally couched in their setting.

More general than the spaces L_p are the
or the ordered topological linear spaces.

Along with the theory of vector measures
influence on the theory of Banach func-

[3], [8], and [14]).

In the Lebesgue spaces \mathcal{L}^p , it is natural
associate norm ρ' as

$$\sup\{\int |fg| d\mu : g \in L_p, \rho(g) \leq 1\}$$

as in [9]. It follows readily that ρ'

and it has the weak Fatou property

(is the stronger Fatou property). Conse-

quently, for any normed linear space L_p , called

\tilde{L}_p , defined by

$$\rho' = \{f \in L_p : \rho'(f) < \infty\}$$

is a normed linear space (5).

Introduction we let φ denote a measurable

locally compact Hausdorff space T . Unless

measurable subsets of T will be its collec-

tions. Let $C_b(T)$ be the collection of

bounded continuous real valued functions on T with the sup norm topology. We assume that φ induces a continuous map φ^e from $C_b(T)$ into $L_\rho(S, \Sigma, \mu)$ by

$$\varphi^e(f) = f \circ \varphi$$

for all $f \in C_b(T)$. Thus $\rho(f \circ \varphi) < \infty$ and hence φ^e is a bounded linear operator from $C_b(T)$ to L_ρ .

Such a situation is rather easy to construct. For example if φ is a measurable map then the map φ^e defined above from $C_b(T)$ into the Lebesgue space $L^\infty(S, \Sigma, \mu)$ is a bounded linear operator. If $\mu\varphi^{-1}$ is of finite variation (with respect to Borel partitions of T) then the map φ^e defined from $C_b(T)$ into the Lebesgue space $L^p(S, \Sigma, \mu)$, $1 \leq p < \infty$, is again a bounded linear operator.

To be able to deal with the function spaces L_ρ we need some terminology from the papers ([2], [11], and [17]). Let $\Sigma_0 = \{A \in \Sigma: \rho(\chi_A) < \infty\}$ where χ_A is the characteristic function for the measurable set A . By a partition \mathcal{E} of Σ_0 we mean a finite pairwise disjoint subcollection of Σ_0 consisting of non μ -null sets of finite measure such that $\rho(\chi_A) < \infty$ for $A \in \mathcal{E}$. For a partition \mathcal{E} of Σ_0 and for $f \in L_\rho$ we may define the averaged step function $f_\mathcal{E}$ to be

$$f_\mathcal{E} = \sum_{E \in \mathcal{E}} \left(\int_E \frac{|f|}{\mu(E)} d\mu \right) \chi_E.$$

The function norm ρ is said to be weakly leveling if for each partition \mathcal{E} in Σ_0 , $\rho(f_{\mathcal{E}}) \leq \rho(f)$. All well known Banach function spaces such as the Orlicz spaces (and in particular the Lebesgue spaces) have weakly leveling function norms. In [11] this concept was referred to as ρ having property (J). The present terminology appears more appropriate in comparison to the concept of leveling as discussed in [10].

An important closed linear subspace of L_{ρ} is the space M^{ρ} which is the closure of the linear span of bounded functions in L_{ρ} with support in Σ_0 . Another way of viewing this, is to make the appropriate definition of a Σ_0 -simple function in L_{ρ} , that is, $f = \sum \alpha_i \chi_{A_i}$ where $(A_i)_{i \in I}$ is a finite partition in Σ and $(\alpha_i)_{i \in I}$ is a finite set of scalars. Then

$$M^{\rho} = \text{cl span } \{f \in L_{\rho} : f \text{ is a } \Sigma_0\text{-simple function}\}.$$

For scalar valued finitely additive set functions γ on Σ_0 , an analogous function norm ρ' may be defined as was done in [17]. This is done by setting

$$\rho'(\gamma) = \sup\{|\int f d\gamma| : f \in M_1^{\rho}\}.$$

(For any linear space X we designate by X_1 its unit ball).

Throughout this paper, the integrals given relative to finitely additive set functions are in the sense of Chapter III in [9].⁽⁶⁾

With regard to our previous definition for $g \in L_{\rho}$, we do have compatibility in the sense that if $d\gamma = g d\mu$ then $\rho'(\gamma) = \rho'(g)$.

The present definition of ρ' has the following significance.

Let $A_{\rho'}(\mu) = A_{\rho'}(S, \Sigma_0, \mu)$ denote all finitely additive scalar valued set functions γ defined on Σ_0 which vanish on μ -null sets and for which $\rho'(\gamma) < \infty$. Then with this ρ' , $A_{\rho'}(\mu)$ is a normed linear space.

A positive finitely additive scalar valued set function η is said to be purely finitely additive if $0 \leq \nu \leq \eta$ with ν countably additive then $\nu \equiv 0$. If η is not positive then η is said to be purely finitely additive if its variation is.

Let $B_{\rho'}(\mu) = B_{\rho'}(S, \Sigma, \mu)$ be the set of all purely finitely additive scalar valued set functions defined on Σ which vanish on μ -null sets and which have their supports contained in the support of some $f \in L_{\rho} \setminus M^{\rho}$.

Then as in [17], by letting

$$G_{\rho'}(\mu) = A_{\rho'}(\mu) \oplus B_{\rho'}(\mu)$$

the Banach dual $(L_{\rho})^*$ is isometrically-isomorphic to $G_{\rho'}(\mu)$. More precisely if $\gamma \in G_{\rho'}(\mu)$ then γ may be decomposed uniquely as $\gamma_1 + \gamma_2$ for $\gamma_1 \in A_{\rho'}(\mu)$ and $\gamma_2 \in B_{\rho'}(\mu)$. An appropriate norm is given for γ by

$$\|\gamma\|_{\rho'} = \rho'(\gamma_1) + |\gamma_2|(S).$$

For $f \in L_{\rho}$, there is defined

$$\int f d\gamma = \int f d\gamma_1 + \int F(f) d\gamma_2$$

where F is the canonical map of L_ρ onto L_ρ/M^ρ as given in [11]. The first integral is a Bochner type in the sense of [9] whereas the second is as defined in [11].

Theorem 1. (see [17]). If $U \in (L_\rho)^*$ then there is a unique $\gamma \in G_{\rho, (\mu)}$ such that

$$U(f) = \int d\gamma$$

for all $f \in L_\rho$. Moreover $\|U\| = \|\gamma\|_{\rho, (\mu)}$. Thus $(L_\rho)^*$ is isometric and (lattice) isomorphic to $G_{\rho, (\mu)}$ (see [11] for a slightly different version).

Another useful result from [17] is a spectral type theorem as we now begin to describe. Let $\hat{\Sigma}$ be the σ -field generated by the compact subsets of the locally compact Hausdorff space \hat{S} . Let $\hat{\mu}$ be a measure on $\hat{\Sigma}$ which is finite on compact sets. An adequate function norm $\hat{\rho}$ is defined (see below) so that one may consider the appropriate function space $L_{\hat{\rho}}(\hat{S}, \hat{\Sigma}, \hat{\mu})$. Every element in $L_{\hat{\rho}}$ has σ -compact support. Further let B_ρ be the algebra of essentially bounded functions in $L_\rho(S, \Sigma, \mu)$ and let $\text{cl}_\infty B_\rho$ be its closure in $L_\infty(S, \Sigma, \mu)$ (where $L_\infty = L_\rho$ for $\rho = \rho_\infty$ as discussed in footnote). In [17], it is shown that

Proposition 2. If (S, Σ, μ) is a measure space then

(1) there is a measure space $(\hat{S}, \hat{\Sigma}, \hat{\mu})$ (as defined above) such that $L_\rho(S, \Sigma, \mu)$ is isometric and (lattice) isomorphic to $L_{\hat{\rho}}(\hat{S}, \hat{\Sigma}, \hat{\mu})$.

Moreover if there is an $f_0 \in L_\rho(S, \Sigma, \mu)$ such that $f_0 > 0$, μ almost everywhere, then

(2) statement (1) holds where now \hat{S} is a compact extremally disconnected Hausdorff space and where $\hat{\mu}$ is a regular Borel measure ($\hat{\mu}(S) < \infty$) on the σ -field generated by the clopen subsets of \hat{S} .

(3) there is an isomorphism Φ from $cl_{\infty} B_\rho$ onto $C_b(\hat{S})$. Also $f \equiv 0$, μ almost everywhere if and only if $\Phi(f) \equiv 0$, $\hat{\mu}$ almost everywhere and

$$\|f\|_{\infty, \mu} = \|\Phi(f)\|_{\infty, \hat{\mu}}$$

In addition Φ takes characteristic functions in $cl_{\infty} B_\rho$ into characteristic functions in $C_b(\hat{S})$.

Let us note that the condition on $f_0 \in L_\rho$ given in the above proposition to demonstrate statements (2) and (3), does hold rather generally, for example, in any σ -finite measure space. Also in the above proposition, the norms are taken as was previously defined for $\|\cdot\|_\rho$ where now $\rho = \rho_\infty$. We have utilized the appropriate measure in indexing to further emphasize the underlying structure. The proof follows from that given in Theorem 2.1 of [17] once one realizes that B_ρ is a vector lattice, $cl_{\infty} B_\rho$ in L_∞ is an abstract M space (in the sense of Kakutani, see [13]), $f_n = \min\{f, n\} \in B_\rho$ for all f and $f_n > 0$, μ almost everywhere. The topology on $C_b(\hat{S})$ is just that of the supremum norm. The function norm $\hat{\rho}$ is defined

for $f \in B_\rho$ as

$$\hat{\rho}(\Phi(f)) = \rho(f).$$

If $0 \leq f \in L_\rho(S, \Sigma, \mu)$, let $f_n = f \wedge \frac{1}{n} \in B_\rho$, then $\hat{\rho}(\Phi(f_n)) = \rho(f_n)$ and $\hat{\rho}(\Phi(f)) = \rho(f)$ since $\{\Phi(f_n)\}_{n \in \mathbb{N}}$ is monotonically increasing and pointwise convergent to $\Phi(f) = \hat{f}$, $\hat{\mu}$ almost everywhere (further details may also be seen in Theorem 1.1 of [17]).

The case where $f_0 \in L_\rho(S, \Sigma, \mu)$ exists as in Proposition 2 is of the most interest. Consequently we will assume throughout that such is the case.

A linear subspace A of L_ρ is called an order ideal in L_ρ if whenever $f \in A$ and g is a measurable function on S such that $|g| \leq |f|$ then $g \in A$. An important order ideal in L_ρ will now be defined as it will be utilized in the development later.

A function $f \in L_\rho$ is said to be of absolutely continuous norm if the sequence $\{\rho(f_n)\}_{n \in \mathbb{N}}$ is monotonically decreasing and convergent to zero whenever the sequence $\{f_n\}_{n \in \mathbb{N}} \in L_\rho$ is monotonically decreasing and pointwise convergent μ almost everywhere to zero with $f_1 \leq |f|$.

Let L_ρ^a represent all $f \in L_\rho$ which are of absolutely continuous norm. It can be shown that L_ρ^a is a norm closed order ideal in L_ρ .⁽¹⁰⁾ For our purposes, its significance will be in its determination of the reflexivity of L_ρ .

The function norm ρ is said to be absolutely continuous if $L_\rho = L_\rho^a$. The space L_ρ is reflexive if and only if both ρ and ρ' are absolutely continuous and ρ has the weak Fatou property⁽¹¹⁾.

3. Averaging Operators.

The main problem as we have stated in the Introduction is to find conditions under which there exists (or fails to exist) a bounded projection from L_ρ onto the range of φ^e . Let us first give some general results about bounded linear operators and their integral representations.

Let U be a bounded linear operator from L_ρ into $C_b(T)$. Then for each $t \in T$, the operator U gives rise to the point linear functionals $U_t \in (L_\rho)^*$ defined by

$$U_t(f) = (U(f))(t)$$

for all $f \in L_\rho$. Then as in Theorem 1 there must be a unique $\gamma_t \in \mathcal{G}_\rho(S, \Sigma, \mu)$ such that

$$U_t(f) = \int f d\gamma_t$$

for all $f \in L_\rho$. If δ_t represents the point mass at $t \in T$ then in [17], it is shown that for all $f \in L_\rho$

$$U_t(f) = (U(f))(t) = \langle U(f), \delta_t \rangle = \langle f, U^*(\delta_t) \rangle.$$

Thus

$$\langle f, U^*(\delta_t) \rangle = \langle f, \gamma_t \rangle$$

for all $f \in L_\rho$, that is, $U^*(\delta_t) = \gamma_t$. It is easy to see that the map now defined taking $t \in T$ to $\gamma_t \in \mathcal{G}_\rho(\mu)$ is continuous when the weak* topology is placed on the dual space $\mathcal{G}_\rho(\mu)$.

If U is now an averaging operator for the measurable function φ from S into T then for $f \in L_\rho$,

$$\langle f, \delta_t \rangle = \langle U(\varphi^e(f)), \delta_t \rangle = \langle \varphi^e(f), U^*(\delta_t) \rangle = \langle f, [\varphi^e]^*(\gamma_t) \rangle$$

Consequently $[\varphi^e]^*(\gamma_t) = \delta_t$ for all $t \in T$. Thus U is an averaging operator for φ if and only if $[\varphi^e]^*(\gamma_t) = \delta_t$ for all $t \in T$.

For such γ_t let us designate its decomposition as yielded from Theorem 1 by $\gamma_{t,1} + \gamma_{t,2}$ where $\gamma_{t,1} \in A_{\rho,1}(\mu)$ and $\gamma_{t,2} \in B_{\rho,1}(\mu)$. We then have almost shown completely the following theorem.

Theorem 3. If U is a bounded linear operator from L_ρ into $C_b(T)$, then for each $t \in T$ there is a unique $\gamma_t \in G_{\rho,1}(\mu)$ such that $\gamma_t = U^*(\delta_t)$. The operator U and the integral representation from t to γ_t are related by

$$(U(f))(t) = \gamma_t(f) \quad \text{for } t \in L_\rho \text{ and } t \in T$$

with

$$\|U\| = \sup\{\|U_t\| : t \in T\}.$$

This map from t to γ_t is weak* continuous.

If φ^e is continuous then U is an averaging operator for the measurable function φ from S into T if and only if $[\varphi^e]^*(\gamma_t) = \delta_t$. If the range $R(\varphi^e)$ of φ^e is contained in M^ρ

then U is an averaging operator for φ if and only if
 $\gamma_{t,1} \circ \varphi^{-1} = \delta_t$ (as elements of the dual space of $C_b(T)$).

Proof. The last part is all that is left to check.

If $R(\varphi^e) \subset M^0$ then

$$\int F(\varphi^e(f)) d\gamma_{t,2} = 0$$

(see Theorem 1). Thus for $f \in L_\rho$

$$\langle f, \delta_t \rangle = \langle \varphi^e(f), \gamma_t \rangle = \int f(\varphi(s)) d\gamma_{t,1} = \int f(t) d(\gamma_{t,1} \circ \varphi^{-1})$$

Thus as elements of the dual of $C(T)$, $\delta_t = \gamma_{t,1} \circ \varphi^{-1}$. This completes our proof.

We should remark, that the above result is somewhat similar to that of Pelczynski in [16]. Also the above proof will be established for even more general situations. In particular $C_b(T)$ may be replaced by even more general spaces, for example, by a Banach space $F(T)$ of functions on T where the map from f to $f(t)$ is continuous. Such is the case for the space of bounded functions on T under the supremum norm.

For the case where φ^e maps just $C(T)$ into $C(S)$ as in [18] where φ is a continuous map of S onto T , the points $t \in T$ for which the fiber $\varphi^{-1}(t)$ is a subset in S of more than one point play an important role in studying averaging operators.

Such points t have been called plural points in T . In our study where φ_e is defined on L_ρ and φ is a measurable map, the concept as defined, is not satisfactory. Shortly we will redefine this taking into account the measure μ . For the time being let P_φ be that subset of T such that

$$P_\varphi = \{t \in T: \text{card}(\varphi^{-1}(t)) > 1\}.$$

Let B be a Borel subset of T containing P_φ and let $B_1 = \varphi^{-1}(B)$. We assume that φ is measurable with respect to Σ_0 , that is $\int f d\gamma_{t,1}$ exists (as defined in [9]). We now may give some formulas to compute $\langle \varphi^e(f), \gamma_t \rangle$ when φ^e has values in M^p .

Proposition 4. Assume that $R(\varphi^e) \subset M^p$ and that φ admits an averaging operator. Then the following hold.

$$(1) \text{ If } t \in B \text{ then } \langle \varphi^e(f), \gamma_t \rangle = \int_{B_1} \varphi^e(f) d\gamma_{t,1}.$$

$$(2) \text{ If } \varphi \text{ is surjective and if } t \notin B \text{ then } \\ \langle \varphi^e(f), \gamma_t \rangle = \int_{B_1} \varphi^e(f) d\gamma_{t,1} + f(t).$$

Proof. If $t \in B$, then $\delta_t(CB) = 0$.⁽¹⁴⁾ Thus $\gamma_{t,1}(\varphi^{-1}(CB)) = 0$. Moreover if H is measurable and $B \cap H = \emptyset$ then $\gamma_{t,1}(\varphi^{-1}(H)) = 0$. Consequently for $t \in B$,

$$\langle \varphi^e(f), \gamma_t \rangle = \int_{B_1} \varphi^e(f) d\gamma_{t,1} + \int_{CB_1} \varphi^e(f) d\gamma_{t,1} = \int_{B_1} \varphi^e(f) d\gamma_{t,1}.$$

This shows statement (1). For statement (2), if $t \notin B$, then $\varphi^{-1}(t)$ is a singleton in S . Partition T into the Borel sets B , $\{t\}$, and the set A . Then S is partitioned into sets $B_1, B_2 = \{\varphi^{-1}(\{t\})\}$, and $\varphi^{-1}(A)$. On A , $\gamma_{t,1} \circ \varphi^{-1} \equiv \underline{0}$, so

$$\langle \varphi^e(f), \gamma_t \rangle = \int_{B_1} \varphi^e(f) d\gamma_{t,1} + \int_{B_2} \varphi^e(f) d\gamma_{t,1}.$$

The last integral is just $f(t)$ for $\gamma_{t,1} \circ \varphi^{-1} \equiv \underline{1}$ on $\{t\}$. This completes the proof.

The assumption in our proposition (and in other results) that $R(\varphi^e)$ be contained in M^p is reasonable. For example in the class $L = L^\psi$ of Orlicz spaces where ψ satisfies the so called Δ_2 condition, one has that $M^p = L_\rho$ (see [15]).

The result in Proposition 4 for $\varphi^e(f)$ may be given more generally for any $h \in L_\rho$. If $t \in B$, then

$$\langle h, \gamma_t \rangle = \int_{B_1} h d\gamma_{t,1} + h\varphi^{-1}(t).$$

Let us now consider the question of the existence of an averaging operator for φ in terms of the existence of an averaging operator for the restriction φ_1 of φ to B_1 .

In particular let B be a Borel subset of T (it need not contain p_φ at all!) and let $B_1 = \varphi^{-1}(B)$. Since φ is measurable, $B_1 \in \Sigma$. A new Banach function space $L_{\rho_1}(B_1, \Sigma_1, \mu_1)$

may be defined as follows. Let $\Sigma_1 = \{A \cap B_1 : A \in \Sigma\}$ and let μ_1 be the restriction of μ to Σ_1 . For f a function defined on B_1 and measurable with respect to μ_1 , we may define \bar{f} on S by $\bar{f} \equiv f$ on B_1 and $\bar{f} \equiv 0$ on CB_1 . Now ρ_1 may be defined for such f by

$$\rho_1(f) = \rho(\bar{f}).$$

Clearly L_{ρ_1} is a Banach function space. Let φ_1 mapping B_1 into B be the restriction of φ to B_1 .

For U a bounded linear operator from L_{ρ} into $C_b(T)$, we will need the following two conditions for the next theorem. We will say that the operator U is B extendable if for every $g \in C_b(B)$, the map $U(\overline{\varphi_1^e(g)})$ in $C_b(T)$ is an extension of g . In particular if U has B -tight range then $\overline{\varphi_1^e(g)} \in L_{\rho}$. Motivated by this we will say that φ is determined by B_1 if for every $f \in L_{\rho}$ such that $f|_{B_1} = \varphi_1^e(g)$ for some $g \in C_b(B)$ there is $g' \in C_b(T)$ such that $f = \varphi^e(g')$. Note that in this case g' need not be an extension of g .

What may be said if B does definitely contain the subset P_{φ} of T ? In this case, as we will see in the next theorem, the fact that φ is determined by B_1 , may be replaced by the following somewhat weaker statement. We will say that φ is weakly determined by B_1 if for all $f \in L_{\rho}(S, \Sigma, \mu)$ there is $g \in C_b(B)$ such that $f|_{B_1} = \varphi_1^e(g)$ and if g' is defined on T to be $g'(t) = g(t)$ for $t \in B$ and $g'(t) = f \varphi^{-1}(t)$ for

$t \in CB$ then $g' \in C_b(T)$. Let us note that if S and T were both compact spaces and if B is a closed Borel subset of T containing P_φ then φ is always weakly determined by B_1 .

Theorem 5. Assume that $R(\varphi^e) \subset M^p$. If φ admits an averaging operator U that is B -extendable, then φ_1 admits an averaging operator U_1 from $L_{\rho_1}(B_1, \Sigma_1, \mu_1)$ into $C(B_1)$.

Conversely if φ_1 admits an averaging operator U_1 and if φ^e is injective with φ determined by B_1 then φ admits an averaging operator. However if $P_\varphi \subset B$ then φ need not be determined by B_1 but need only be weakly determined by B_1 .

Proof. Let U be an averaging operator for φ that is B -extendable and let $\{\gamma_t\}_{t \in T}$ be the family of associated set functions as described for Theorem 3. The operator U from $L_\rho(S, \Sigma, \mu)$ into $C_b(T)$ induces an operator U_1 from $L_{\rho_1}(B_1, \Sigma_1, \mu_1)$ into $C_b(B)$ defined by

$$(U_1(f))(t) = \int_{B_1} \bar{f} d\gamma_t$$

for all $f \in L_{\rho_1}$ and $t \in B$. Since $\rho_1(f) \leq 1$ implies $\rho(\bar{f}) \leq 1$, it follows that $\|U_1\| \leq \|U\|$, that is U_1 is a bounded linear operator. It is clear that $(U_1(f))(t) = (U(\bar{f}))(t)$ for all $t \in B$. Thus we have

$$U_1[\varphi_1^e(g)] = U[\overline{\varphi_1^e(g)}] \Big|_B$$

where the right side represents restriction to B . Since ψ is B -extendable, it follows that $U_1[\varphi_1^e(g)] = g$ for all $g \in C(B)$. Thus U_1 is an averaging operator for φ_1 .

If it is assumed now that φ^e is injective then to show that φ has an averaging operator U , it is sufficient to show the existence of a projection P from $L_\rho(S, \Sigma, \mu)$ onto the range of φ^e . If U_1 is an averaging operator for φ_1 , define P_1 to be $\varphi_1^e \circ U_1$. Clearly P_1 is a bounded projection operator from $L_{\rho_1}(B_1, \Sigma_1, \mu_1)$ onto $R(\varphi_1^e)$. Define a bounded linear operator T from $L_\rho(S, \Sigma, \mu)$ into $L_{\rho_1}(B_1, \Sigma_1, \mu_1)$ by

$$T(f) = f|_{B_1}$$

for all $f \in L_\rho$. We now may define the required projection P . For $f \in L_\rho(S, \Sigma, \mu)$, define

$$P(f) = f - \overline{P_1 T(f) - T(f)}.$$

Now $P^2(f) = P(f) - \overline{P_1 T(P(f)) - T(P(f))}$. For all $h \in L_{\rho_1}(B_1, \Sigma_1, \mu_1)$, if $s \in CB_1$, then $\bar{h}(s) = 0$. Consequently $(P^2(f))(s) = (P(f))(s)$ for all $s \in CB_1$. Moreover for $s \in B_1$, $\bar{h}(s) = h(s)$. Thus $(P_1 T P(f))(s) = (\varphi_1^e U_1 T P(f))(s) = (T P(f))(s)$. Consequently $(P^2(f))(s) = (P(f))(s)$ for all $s \in S$, that is, P is a projection.

The range of P and the range of φ^e coincide. For if $P(f) = f$ then $P_1 T(f) - T(f) = 0$. Consequently $T(f) \in R(\varphi_1^e)$.

If now φ is determined by B_1 then $f \in R(\varphi^e)$. Conversely if $f = \varphi^e(h)$ for some $h \in C(T)$, then $P(f) = f - \overline{P_1 T(f) - T(f)}$. In this case $P_1 T(f) = T(f)$ and thus $P(f) = f$. Therefore $R(\varphi^e) = R(P)$.

If we now assume that the subset P_φ of T is contained in B , then the above arguments show there is a $g \in C(B)$ such that $T(f) = \varphi_1^e(g)$. If φ is now weakly determined by B_1 , let g' be the function in $C_b(T)$ as defined in the definition. Hence $f = \varphi_1^e(g')$ and φ is now determined by B_1 . This completes our proof.

Thus the existence of an averaging operator has somewhat been characterized in terms of a smaller, so to speak, averaging operator defined on an appropriate function space. Let us now consider more of a reduction type theorem where the set functions $\{\gamma_t\}_{t \in T}$ may be replaced, in some cases, by regular Borel measures $\{\beta_t\}_{t \in T}$ defined over a compact space.

We need to assume that $L_\rho = M_\rho^{(15)}$ and that $\rho(\chi_S) < \infty$. The last condition is needed to insure that there is an $f_0 \in L_\rho(S, \Sigma, \mu)$ such that $f_0 > 0$ μ almost everywhere. With this we may invoke Proposition 2. Utilizing notation from that proposition we continue with the following definitions.

Let $\hat{\Sigma}_c$ denote those clopen subsets in $\hat{\Sigma}$ contained in the compact extremally disconnected space \hat{S} , which are in one-to-one correspondence with the sets $A \in \Sigma$. For every $t \in T$ we may define the finitely additive scalar valued set functions β_t

by

$$\beta_t(\hat{A}) = \gamma_t(A)$$

for $\hat{A} \in \hat{\Sigma}_C$, $A \in \Sigma_0 = \Sigma$. Since we are assuming that $M_\rho = L_\rho$, it is clear that $\gamma_t = \gamma_{t,1}$. Note also that $\rho'(\beta_t) = \rho'(\gamma_t)$ where

$$\rho'(\beta_t) = \sup\left\{\left|\int \hat{f} d\beta_t\right| : \hat{f} \text{ is a } \hat{\Sigma}_C \text{ simple function; } \hat{\rho}(\hat{f}) \leq 1\right\}.$$

Now for $\hat{A} \in \hat{\Sigma}_C$, $\hat{\mu}(\hat{A}) = 0$ if and only if $\mu(A) = 0$. Consequently $\hat{\mu}(\hat{A}) = 0$ implies $\beta_t(\hat{A}) = 0$.

Let $|\beta_t|$ represent the variation of β_t , that is,

$$|\beta_t|(\hat{S}) = \sup\left\{\left|\sum \gamma_t(A_i)\right| : (A_i)_{i \in I} \text{ finite partition in } \Sigma\right\}.$$

Now this variation is finite. In fact if $(\alpha_i)_{i \in I}$ are a finite set of scalars such that $|\alpha_i| = 1$ and such that $\alpha_i \beta_t(A_i) = |\beta_t(A_i)|$ then

$$\sum |\gamma_t(A_i)| = \int (\sum \alpha_i \chi_{A_i}) d\gamma_t \leq \rho(\chi_S) \rho'(\gamma_t) < \infty.$$

It is also clear that β_t is regular on $\hat{\Sigma}_C$. Let us see now how β_t may be extended to a regular Borel measure on $\hat{\Sigma}$. Since $\rho(\chi_S) < \infty$ the ring $\hat{\Sigma}_C$ is dense in the power set of \hat{S} , that is, if K and G are respectively compact and open subsets of \hat{S} , then there is $\hat{A} \in \hat{\Sigma}$ such that $K \subset \hat{A} \subset G$. In [8], it shows that such a situation yields β_t as countably additive

on $\hat{\Sigma}_C$ and that a unique extension to $\hat{\Sigma}$ of β_t exists as a regular Borel measure. Furthermore the variation of the extension (considered as a Borel measure) is finite and coincides on $\hat{\Sigma}_C$ with the variation of β_t . For simplicity let us retain β_t as notation for this extension.

Let ψ be the correspondence that takes Σ simple functions into $\hat{\Sigma}_C$ simple functions as now

$$\int (\sum \alpha_i \chi_{A_i}) d\gamma_t = \int (\sum \alpha_i \chi_{\hat{A}_i}) d\beta_t.$$

Since $M^p = L_\rho$, ψ may be extended to all $f \in L_\rho$ as f is then in the closure of Σ -simple functions. Since $\rho'(\gamma_t) = \rho'(\beta_t) < \infty$, a final limit argument will show that

$$\int f d\gamma_t = \int \hat{f} d\beta_t$$

where $\hat{f} = \psi(f)$.

Note that what we have just proceeded to do, could be applied to more general situations. What is crucial here is that in addition to $M^p = L_\rho$, we need the variation $|\beta_t|$ finite, the field $\hat{\Sigma}_C$ dense in the power set of \hat{S} and the existence of an $f_0 \in L_\rho$ such that $f_0 > 0$ μ almost everywhere. If $M^p \neq L_\rho$, then the above arguments may be applied to $\beta_{t,1}$. More formally we have shown

Theorem 6. If $L_\rho = M^p$ and if $\rho(\chi_S) < \infty$ then there is an extremally disconnected compact Hausdorff space \hat{S} with $\hat{\Sigma}$, its field of clopen subsets, and $\hat{\mu}$, a regular Borel measure on

$\hat{\Sigma}$, such that $L_\rho(S, \Sigma, \mu)$ is isometric and lattice isomorphic with $L_\rho(\hat{S}, \hat{\Sigma}, \hat{\mu})$. If ψ is this isomorphism and if U is an averaging operator for φ then for every $t \in T$ there exists a regular Borel measure β_t on $\hat{\Sigma}$ such that

$$\int f d\gamma_t = \int \hat{f} d\beta_t$$

where $\hat{f} = \psi(f)$ for $f \in L_\rho$ and γ_t is the additive set function associated with U as determined for Theorem 3.

We are now in a position to give a reasonable definition of plurality as was indicated earlier. The above result also leads to a definition, for the present context, of the concept of an irreducible map (see [18] for the concepts in the more restricted cases).

Again we need to assume that $L_\rho = M^0$ and the existence of an $f_0 \in L_\rho$ such that $f_0 > 0$ μ almost everywhere. Let U be an averaging operator for φ .

Definition 7. For $t \in T$, if $\varphi^{-1}(t) \in \Sigma_0$, let $\hat{\varphi}^{-1}(t)$ be the associated clopen set in the Stone space \hat{S} . The point $t \in T$ is called a plural point if

- (a) whenever β_t is positive on subsets of $\hat{\varphi}^{-1}(t)$ then there is a set $\hat{A} \in (\hat{\Sigma}_c)_0$ such that $\hat{A} \subset \hat{\varphi}^{-1}(t)$ and $0 < \beta_t(\hat{A}) < 1$.
- (b) whenever β_t is not positive on $\hat{\varphi}^{-1}(t)$ then for the part N of $\hat{\varphi}^{-1}(t)$ on which β_t is negative there is a subset $\hat{A} \in (\hat{\Sigma}_c)_0$ such that $\hat{A} \subset N$ and $0 < -\beta_t(\hat{A}) < 1$.

Let Pl_φ be the set of plural points of T . The measurable map φ will be called irreducible⁽¹⁶⁾ if for $A \subset S$, with $\mu(A) > 0$ there is a $t \in T$ such that $\emptyset \neq \varphi^{-1}(t) \subset A$.

Let us recall that since φ has an averaging operator and since $M^\rho = L_\rho$, Theorem 3 says that $\beta_t(\hat{\varphi}^{-1}(t)) = 1$. Also let us note that as we have defined it, saying that t is a plural point amounts to saying that $\hat{\varphi}^{-1}(t)$ is not an atom for β_t . An interesting relation between non atomicity and the Darboux property may be found in [8].

We now make use of our ideas to establish an upper bound for the variation of the set functions γ_t in terms of the norm for U .

Theorem 8. Let φ have an averaging operator U and assume that

- (1) $M^\rho = L_\rho$
- (2) $\rho(\chi_S) < \infty$
- (3) t is plural
- (4) φ is irreducible

then $|\gamma_t|(S) < \rho(\chi_S) \|U\| - 1$.

Proof. To simplify the notation in this proof we will replace the operator $\hat{\varphi}^{-1}$ by ξ . As noted above $\beta_t(\xi(t)) = 1$. Plurality of t finds a clopen set $\hat{A} \subset \hat{S}$ such that

$$1 = \beta_t(\hat{A}) + \beta_t(\xi(t) - \hat{A})$$

where $0 \neq |\beta_t(\hat{A})| \neq 1$, $0 \neq \beta_t[\xi(t) - \hat{A}] \neq 1$. Thus, in short, there is a clopen set \tilde{A} (which may be either \hat{A} or $\xi(t) - \hat{A}$) such that for $\epsilon > 0$, $\beta_t(\tilde{A}) < \epsilon + 1/2$. Actually there is a compact set $K \subset \tilde{A}$ such that $0 < \beta_t(K) < 1/2 + \epsilon$ and $|\beta_t(K)| = |\beta_t|(K)$. If β_t is positive on $\xi(t)$, the regularity of β_t assures the existence of a compact set $K \subset \tilde{A}$ such that

$$|\beta_t(K) - \beta_t(\tilde{A})| < \epsilon + 1/2 - \beta_t(\tilde{A}).$$

Since $0 < \beta_t(\tilde{A}) < 1$, K may be chosen so that $\beta_t(K) \neq 0$. In addition $\beta_t(K) = \beta_t(\tilde{A}) + \beta_t(K) - \beta_t(\tilde{A}) < \epsilon + 1/2$. Clearly $\beta_t(K) = |\beta_t(K)| = |\beta_t|(K)$ as β_t is countably additive. If β_t is not positive on $\xi(t)$, let N be the negative part as in the definition. Again by the regularity of β_t , a compact set $K \subset N$ may be obtained so that

$$|\beta_t(K) - \beta_t(N)| < 1/2.$$

Again it may be assumed that $\beta_t(K) \neq 0$. Now

$$\beta_t(K) \leq \beta_t(N) + 1/2 < 1/2.$$

Since $-\beta_t$ is positive on N , $-\beta_t(K) = |\beta_t(K)| = |\beta_t|(K)$.

Now the regularity of $|\beta_t|$ permits us to pick a clopen set $C \subset \hat{S}$ such that $K \subset C$ and $|\beta_t|(C \setminus K) < \epsilon$. Incidentally $\chi_C \in C(\hat{S})$ and $\hat{\rho}(\chi_C) < \omega$. A finite pairwise disjoint family of

clopen sets $C_i \subset \hat{S} \setminus C$, $i \in I$, may be chosen such that

$$|\beta_t|(\hat{S} \setminus K) - \epsilon < \sum_{i=1}^n |\beta_t(C_i)|.$$

Of course $\hat{\rho}(\chi_{C_i}) < \infty$.

Let α_i be scalars such that $|\alpha_i| = 1$ and $\alpha_i \beta_t(C_i) = |\beta_t(C_i)|$. Now

$$(1) \quad |\beta_t|(\hat{S} \setminus C) - \epsilon < \int (\sum \alpha_i \chi_{C_i}) d\beta_t$$

$$(2) \quad \int \chi_C d\beta_t \leq \beta_t(K) + |\beta_t|(C \setminus K) < \beta_t(K) + \epsilon^{(7)}$$

Since the map from t to γ_t is weak* continuous (Theorem 3),⁽⁸⁾ it follows that there is some neighborhood V of t such that for all $y \in V$

$$(3) \quad |\beta_t|(\hat{S} \setminus C) - \epsilon < \int (\sum \alpha_i \chi_{C_i}) d\beta_y$$

$$(4) \quad \int \chi_C d\beta_y < \beta_t(K) + \epsilon$$

If $D \in \Sigma$ is the correspondent of C , then $\mu[\varphi^{-1}(V) \cap D] > 0$. In fact

$$\varphi^{-1}(V) \cap D \supset \varphi^{-1}(t) \cap D.$$

The last set corresponds to $\xi(t) \cap C$ which contains K . Now $\hat{\mu}(K) > 0$ or else $\beta_t(K) = 0$ which is a contradiction. Thus

$$\hat{\mu}[\xi(t) \cap C] > 0 \quad \text{and} \quad \mu[\varphi^{-1}(V) \cap D] > 0.$$

The irreducibility of φ assures $q \in T$ such that $\varphi^{-1}(q) \neq \emptyset$ and $\varphi^{-1}(q) \subset \varphi^{-1}(V) \cap D$. Hence $q \in V$ and

$$|\beta_t|(\hat{S} \setminus C) - \epsilon < \int \sum \alpha_i \chi_{C_i} d\beta_q$$

$$\int \chi_C d\beta_q < \beta_t(K) + \epsilon.$$

Since $\varphi^{-1}(q) \subset D$, $\xi(q) \subset C$ and

$$\int \chi_C d\beta_q = \int_{C \setminus \xi(q)} \chi_C d\beta_q + \beta_q[\xi(q)]$$

Consequently $\beta_q(C) = \beta_q(C \setminus \xi(q)) + 1$, and by (4)

$$\int_{C \setminus \xi(q)} \chi_C d\beta_q = \beta_q(C) - 1 < \beta_t(K) + \epsilon - 1.$$

Recall that $0 < |\beta_t(K)| = |\beta_t|(K) < 1/2 + \epsilon$. If $\beta_t(K) < 0$ for sufficiently small $\epsilon > 0$, then

$$|\beta_q|[C \setminus \xi(q)] > |\beta_t(K) + \epsilon - 1| > |\beta_t(K)| - 3\epsilon$$

If $0 < \beta_t(K) < \epsilon + 1/2$ for sufficiently small $\epsilon > 0$, then

$$|\beta_q|(C \setminus \xi(q)) \geq |\beta_q|[C \setminus \xi(q)] > 1/2 - 3\epsilon > |\beta_t(K)| - 3\epsilon$$

since

$$\beta_q[C \setminus \xi(q)] < \beta_t(K) + \epsilon - 1 < 1/2 + 2\epsilon$$

Thus in all cases for sufficiently small $\epsilon > 0$

$$|\beta_q| [C \setminus \xi(q)] > |\beta_t| (K) - 3\epsilon.$$

Now

$$\|U\| > \sup\{|\int f d\gamma_q| : f \in M_1^0\} = \sup\{|\sum \beta_q(\hat{A}_i) \alpha_i| : \hat{f} = \sum \alpha_i \chi_{\hat{A}_i} \\ \text{a } \hat{\Sigma}_c\text{-step function, } \rho(\hat{f}) < \infty\}.$$

Picking scalars β_i , $|\beta_i| = 1$ and $\beta_i \gamma_q(A_i) = |\gamma_q(A_i)|$ we have

$$\rho(\sum \chi_{\hat{A}_i} \cdot \beta_i) \leq \hat{\rho}(\chi_S^*) = \rho(\chi_S).$$

Since $\hat{\Sigma}_c$ is dense in the power set of \hat{S} , we have

$$\|U\| > \frac{1}{\rho(\chi_S)} [|\beta_q| (\hat{S} \setminus C) + |\beta_q| (C \setminus \xi(q)) + 1]$$

Thus

$$\|U\| > \frac{1}{\rho(\chi_S)} [\int (\sum \chi_{C_i} \alpha_i) d\beta_q + 1 - 4\epsilon] > \frac{1}{\rho(\chi_S)} [|\beta_t| (\hat{S}) + 1 - 5\epsilon]$$

It finally follows that $|\gamma_t| (S) \leq \rho(\chi_S) \|U\| - 1$ and our proof is now complete.

We now have the answer to a rather natural question. What is the relation between the norm of the averaging operator U and the norm of U_1 as defined above? We quickly obtain it below.

Corollary 9. Assuming the hypotheses of Theorem 8 and assuming that φ and φ_1 admit the averaging operators U

and U_1 respectively, then

$$\|U_1(f)\| \leq \|\bar{f}\|_{\infty} [\rho(\chi_S)\|U\| - 1]$$

where $B = \text{cl } Pl_{\varphi}$ as needed for U_1 .

Proof. Following through the proof of Theorem 5 we see that

$$(U_1(f))(t) = \int_{B_1} \bar{f} d\gamma_t.$$

Then pick $\hat{f} \in C(\hat{S})$, $0 \leq \hat{f} < 1$ with

$$\int \hat{f} d\beta_t > \|\beta_t\| - \epsilon$$

where $\epsilon > 0$ is given, t is a fixed point in B_1 , and $\|\beta_t\| = |\beta_t|(\hat{S})$. Now if G is an open set containing t and if $r \in G \cap Pl_{\varphi}$, then the theorem yields

$$\rho(\chi_S)\|U\| - 1 > |\beta_r|(\hat{S}) \geq \int \hat{f} d\beta_r.$$

The weak* continuity of the map that takes r to γ_r (restricting to G if necessary) yields

$$\rho(\chi_S)\|U\| - 1 > \int \hat{f} d\beta_r > \|\beta_t\| - \epsilon$$

for all $r \in G$. Now for every $t \in \text{cl } Pl_{\varphi}$ it follows that

$$\|\beta_t\| \leq \rho(\chi_S) \|U\| - 1.$$

Since

$$(U_1(f))(t) = \int_{B_1} \bar{f} d\gamma_t = \int_{B_1} \hat{\bar{f}} d\beta_t$$

it follows that

$$\|U_1(f)(t)\| \leq \|\bar{f}\|_{\infty} \sup\{\|\beta_t\| : t \in \text{cl } P1_{\varphi}\} \leq \|\bar{f}\|_{\infty} [\rho(\chi_S) \|U\| - 1]$$

This completes the proof.

We are now finally led to the consideration of obtaining conditions on φ with which we will know that no bounded projection will exist onto the range of φ^e . For this remaining section we will need to assume that L_{ρ} is reflexive. For this case, as we have already mentioned, $L_{\rho} = L_{\rho}^a$ and $L_{\rho'} = L_{\rho'}^a$. We need also to assume that the dual space of L_{ρ}^a may be identified with $L_{\rho'}^a$. We have already stated conditions under which this will hold. Thus from now on $R(\varphi^e)$ may be considered as a subset of $L[L_{\rho}^a, \mathbb{C}]$ the set of bounded linear operators from L_{ρ}^a into the complex scalars \mathbb{C} , or as a subset of $L_{\rho'}^a$. The following operators will be needed.

Let U be an arbitrary element of $L[L_{\rho}^a, \mathbb{C}]$. For E in a partition \mathcal{E} and for $f \in L_{\rho'}^a$, define the operator $U_E \in L[L_{\rho}^a, \mathbb{C}]$ by

$$U_E(f) = U\left[\left(\int_E f d\mu\right) \chi_E\right]$$

and define the linear operator $A_E \in L[L_\rho^a, L_\rho^a]$ by

$$A_E(f) = f \chi_E .$$

We may also define

$$A_E(f) = \frac{1}{\mu(E)} \left(\int_E f d\mu \right) \chi_E .$$

If $R(\varphi^e)$ is closed, we will let P be a bounded projection of $L[L_\rho^a, \mathbb{C}]$ onto $R(\varphi^e)$. Thus if P^* is the adjoint of P , then P^* is a bounded linear map from L_ρ^a into L_ρ^a .

Asking that $R(\varphi^e)$ be closed is not much of an assumption. Specifically this occurs when φ admits an averaging operator.

Any linear operator K from L_ρ^a into L_ρ^a may be written (see [19]) as

$$K(f) = \int g f d\mu$$

for some $g \in L_\rho^a = L_\rho^a$. The important assumption here is that for a certain class of operators K , g may be chosen in $R(\varphi^e)$.

Theorem 10. Assume the following conditions

- (1) L_ρ is reflexive with $(L_\rho^a)^* \cong L_\rho^a$.
- (2) $R(\varphi^e)$ is closed

(3) ρ' has the weak leveling property

(4) For every $\mathcal{E} \in \mathcal{E}$ there is $f_{\mathcal{E}} \in C_b(T)$ such that

$$U_{\mathcal{E}}(f) = \langle \varphi^{\mathcal{E}}(f_{\mathcal{E}}), f \rangle (= \int \varphi^{\mathcal{E}}(f_{\mathcal{E}}) f d\mu).$$

Then either $\varphi^{\mathcal{E}}$ is surjective or no bounded projection P exists from $L_{\rho}(S, \Sigma, \mu)$ onto $R(\varphi^{\mathcal{E}})$ such that

$$P[UA_{\mathcal{E}}] = P(U)A_{\mathcal{E}}.$$

In particular either $\varphi^{\mathcal{E}}$ is surjective or no bounded projection P from L_{ρ}^a onto $R(\varphi^{\mathcal{E}})$ exists such that P^* commutes with $A_{\mathcal{E}}$.

Proof. If $\varphi^{\mathcal{E}}$ is not surjective, let P be a bounded projection of $L[L_{\rho}^a, \mathbb{C}]$ onto $R(\varphi^{\mathcal{E}})$ with $P[UA_{\mathcal{E}}] = P(U)A_{\mathcal{E}}$.

Now

$$\begin{aligned} UA_{\mathcal{E}}(f) &= U\left[\sum_{\mathcal{E}} \left(\frac{1}{\mu(E_i)} \int_{E_i} f d\mu\right) \chi_{E_i}\right] \\ &= \sum_{\mathcal{E}} \frac{1}{\mu(E_i)} U_{E_i}(f) = \sum_{\mathcal{E}} \frac{1}{\mu(E_i)} \langle \varphi^{\mathcal{E}}(f_{E_i}), f \rangle \end{aligned}$$

Thus if $h = \sum_{\mathcal{E}} \frac{1}{\mu(E_i)} f_{E_i}$ then $UA_{\mathcal{E}}(f) = \langle \varphi^{\mathcal{E}}(h), f \rangle$. Consequently $UA_{\mathcal{E}} \in R(\varphi^{\mathcal{E}})$ and $P[UA_{\mathcal{E}}] = UA_{\mathcal{E}}$. Since ρ' has the weak leveling property, we obtain from [11], that $A_{\mathcal{E}}(f)$ converges to f in the ρ norm as \mathcal{E} gets finer. Thus

$$\lim_{\mathcal{E}} P[UA_{\mathcal{E}}](f) = U(f)$$

and so

$$\lim_{\mathcal{E}} P[UA_{\mathcal{E}}](f) = \lim_{\mathcal{E}} P(U)A_{\mathcal{E}}(f) = P(U)(f).$$

Hence $P(U) = U$ which contradicts the assumption of φ^e being not surjective.

To complete the proof we need show that if P^* commutes with A_E then $P[UA_{\mathcal{E}}] = P(U)A_{\mathcal{E}}$. Now

$$\begin{aligned} \langle P[UA_{\mathcal{E}}], f \rangle &= \langle UA_{\mathcal{E}}, P^*(f^*) \rangle \\ &= \langle U, \sum_{\mathcal{E}} \left(\frac{1}{\mu(E_i)} \int_{E_i} P^*(f) d\mu \chi_{E_i} \right) \rangle \\ &= \langle U, \sum_{\mathcal{E}} \frac{1}{\mu(E_i)} A_{E_i} P^*(f) \rangle \\ &= \langle U, P^* \left[\sum_{\mathcal{E}} \frac{1}{\mu(E_i)} A_{E_i} (f) \right] \rangle \\ &= \langle P(U), \sum_{\mathcal{E}} \frac{1}{\mu(E_i)} \int_{E_i} f d\mu \chi_{E_i} \rangle = \langle P(U)A_{\mathcal{E}}, f \rangle. \end{aligned}$$

Thus $P[UA_{\mathcal{E}}] = P(U)A_{\mathcal{E}}$.

Footnotes

- (1) This definition could readily be made by replacing $C(T)$ or L_ρ by other spaces of functions defined on T or S respectively. For example as in [18], the case is studied for L_ρ replaced by $C(S)$ where S and T are compact Hausdorff spaces.
- (2) A few remarks are pertinent here regarding the definition. It follows immediately that if f is μ -measurable and if $g \in L_\rho$, with $|f| \leq |g|$ on Ω then $\rho(f) \leq \rho(g) < \infty$ and $f \in L_\rho$. If $f \in L_\rho$ and if $E = \{x: |f(x)| = \infty\}$ then the characteristic function χ_E satisfies $\rho(n\chi_E) \leq \rho(f) < \infty$ for all $n \in \mathbb{N}$. Consequently $\rho(\chi_E) = 0$ which by (i) of the axioms for ρ implies $\chi_E \equiv 0$ almost everywhere, that is, $\mu(E) = 0$ or more succinctly f is finite almost everywhere on Ω . In another vein, the axioms do not exclude the existence of a positive measure set $A \in \Sigma$ such that not only $\rho(\chi_A) = \infty$ but also $\rho(\chi_B) = \infty$ for all $B \subset A$ and B of positive measure. Such sets A are called, in the literature, unfriendly sets. Using the above argument, it follows that if A is an unfriendly set then any $f \in L_\rho$ is identically zero on A . Consequently to investigate L_ρ -spaces, it is worthwhile to remove the unfriendly sets A . Throughout we will assume that this has been done. It is shown in [14] that there is a largest unfriendly set A_{\max} and once removed the remaining set again designated by Ω contains no unfriendly sets and $\mu(\Omega)$ is still positive. (See [17] for additional remarks). It seems more appropriate to call such unfriendly sets A above purely infinite.
- (3) That ρ' is actually a norm and not a semi-norm follows from the fact that unfriendly sets have been removed (see previous footnote on unfriendly sets).

(4) For any real number p , $1 \leq p < \infty$ we may define a function norm ρ_p for $f \in M^+$ by $\rho_p(f) = \left(\int f^p d\mu \right)^{\frac{1}{p}}$. If $p = 1$ we have simply the integral of f over Ω . A function norm ρ_∞ may also be defined as follows. For any α , $0 \leq \alpha < \infty$, set $A_\alpha = \{x \in \Omega : f(x) > \alpha\}$. If $\mu(A_\alpha) > 0$ for all α define $\rho_\infty(f) = +\infty$; if, $\mu(A_\alpha) = 0$ for some value of α , define $\rho_\infty(f) = \alpha_0$ where $\alpha_0 = \inf\{\alpha : \mu(A_\alpha) = 0\}$. The number $\rho_\infty(f)$ is called the essential upper bound of f (for $\rho_\infty(f) < \infty$ there is $f_1 \in M^+$, almost equal to f , such that f_1 is bounded on X with its least upper bound equal to $\rho_\infty(f)$). The triangle inequality for these norms ρ_p , $1 \leq p \leq \infty$, follows from the Holder inequality (see below). By L_∞ we will mean L_ρ for $\rho = \rho_\infty$.

(5) In particular if $f \in L_\rho$ and $g \in L_{\rho'}$, then a Holder inequality (as in the case $L_\rho = \mathcal{L}^p$, $1 \leq p \leq \infty$, and $L_{\rho'} = \mathcal{L}^q$, $\frac{1}{p} + \frac{1}{q} = 1$) holds as

$$\left| \int fg d\mu \right| \leq \int |fg| d\mu \leq \rho(f) \rho'(g) < \infty.$$

Moreover if $\rho'(g) < \infty$ then $\rho'(g) = \sup\{ \left| \int fg d\mu \right| : \rho(f) \leq 1 \}$. From this it follows that if G is defined for all $f \in L_\rho$ by

$$G(f) = \int fg d\mu$$

then G is a bounded linear functional on L_ρ and $\|G\| = \rho'(g)$. Consequently $L_{\rho'}$ is isometrically and algebraically embedded in the first Banach dual space L_ρ^* of L_ρ , that is, $L_{\rho'}$ is a closed linear subspace of L_ρ^* .

- (6) The unsubscripted \int will always designate integration over the whole space.
- (7) It should be emphasized that $|\beta_t|$ while referring to the variation of β_t as a Borel measure coincides on $\hat{\Sigma}_C$ with the variation of β_t viewed as a set function defined by $\beta_t(\hat{A}) = \gamma_t(A)$.
- (8) Also recall that the simple function $\sum \alpha_i \chi_{C_i}$ and characteristic function χ_C may be identified with corresponding elements in $L_\rho(S, \Sigma, \mu)$.
- (9) It is clear that its name comes from the fact that if f is of absolutely continuous norm then for every $\epsilon > 0$ there is a $\delta > 0$ such that $\mu(E) < \delta$ implies $\rho(f\chi_E) < \epsilon$. If $L_\rho = \mathcal{L}^p$ for any p , $1 \leq p < \infty$ then every $f \in L_\rho$ is of absolutely continuous norm. However for $L_\rho = \mathcal{L}^\infty$ for Lebesgue measure μ on \mathbb{R} , the only function of absolutely continuous norm is the null function. On the other hand if $L_\rho = \mathcal{L}^\infty$ for discrete measure μ on \mathbb{N} , then these functions are just the sequences converging to zero, that is the elements of the subspace c_0 .
- (10) The significance of L_ρ^a lies in the fact that it is exactly the inverse annihilator of the closed linear subspace $L_{\rho, S}^*$ (of L_ρ^*) of all singular bounded linear functionals on L_ρ . In turn $L_{\rho, S}^*$ is also the complementary closed linear subspace obtained when L_ρ is considered as a closed linear subspace of L_ρ^* as discussed in the previous footnote (see [19], Chapter 15).

- (11) If ρ is absolutely continuous and if ρ has the weak Fatou property then ρ has the stronger (sequential) Fatou property. All of the above results for L_ρ^α may be found in Chapter 15 of [19].
- (12) This is defined as a subspace of L_ρ such that for each $x \in L_\rho$ there is a unique $x_0 \in M$ such that $\rho(x - x_0) = \min\{\rho(x - y) : y \in M\}$.
- (13) For example in [18], (where the author studies the existence of some projections from $C_b(S)$ onto the range of φ^e), it was assumed that φ be continuous and that S and T both be compact. In such a situation φ^e is always an isometric embedding induced by φ . It was shown that an averaging operator for φ exists if there exists an averaging operator for φ_1 where φ_1 is the restriction of φ to a certain closed subset F of S and if there exists an extension operator ψ from a certain subspace \mathfrak{S} of $C(F)$ into $C(S)$. See also [4], [5], [6], [12], and [18]. The analogy of course with our work, is that we have under investigation, the existence of projections from $L_\rho(S, \Sigma, \mu)$ onto $R(\varphi^e)$. Results for the historically interesting $L_\infty(S, \Sigma, \mu)$ case are herein obtained.
- (14) By CB we mean $T \setminus B$.
- (15) As pointed out previously this is a reasonable assumption. The Orlicz spaces $L_\rho = L^\psi$ with Δ_2 condition have this property (see [15]).
- (16) Our measure theoretical concept has its topological analogue as the following: The continuous map φ from the topological space S onto the topological space T is irreducible if for every non empty open set G in S there is a point $t \in T$ such that $\emptyset \neq \varphi^{-1}(t) \subset G$.
- (17) However as we will see later (in Theorem 6 and Proposition 2) S is not "as arbitrary as" T . Of especial interest are those S which are extremally disconnected compact Hausdorff spaces. They help in considerably reducing the study of more general S .

- (18) Any appropriate $M(T)$ will do (dependent on the topological structure of T , naturally) just as long as the point mass $\delta_t \in M(T)$. Of course $\delta_t \in M(T)$ if and only if the point evaluation map ξ_t from $C(T)$ into the scalars, defined by $\xi_t(f) = f(t)$, is continuous.

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