LINEAR DIFFERENTIAL EQUATIONS
WITH DELAYS. EXISTENCE, UNIQUENESS,
GROWTH, AND COMPACTNESS UNDER
NATURAL CARATHÉODORY CONDITIONS

by

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1• <u>Introduction</u>.

A typical description of an initial-value problem for linear retarded differential equations "under Carathéodory conditions" might run roughly as follows: for given $h > 0$ and positive integer n, $\underset{\sim}{C} = \pounds([-h,0], R^n)$ is the usual space of continuous functions; $F: R_+ \times \underset{\sim}{C} \rightarrow R^n$ is measurable in the first variable and linear and continuous in the second, and satisfies an inequality of the form

(1.1) $\qquad\qquad l|F(t,v)|| \pounds \text{cp}(t)\sup_{s \in EVJ}||v(s)||$

where (p is a fixed locally integrable function. For each continuous function u defined on an interval containing $[t-h,t]$ and with values in R, $u_t \in \underset{\sim}{C}$ denotes the "slice" of u between $t-h$ and t "transplanted" to $[-h,0]$. Then the initial-value problem with the initial datum $v \in \underset{\sim}{C}$ and given "non-homogeneous term" r (a locally integrable function with values in R) reads

(1.2) $\qquad\qquad ii(t) = F(t,u_t) + r(t), \qquad t \pounds t_0 \qquad u_{t_Q} = v$

(see for instance [2; p.30], adapted to the linear case). Under these conditions, existence and uniqueness theorems, and propositions concerning the growth of solutions (depending on cp and r) can be proved in the usual way.

It seems to us that these conditions are somewhat artificial; most especially (1.1), which imposes on F a "narrowness" in its dependence on v that appears little justified. A bit less artificial is the requirement that the right-hand side of (1.2) depend precisely on u_t, rather than merely being independent of u away

from [t-h,t], a distinction that might - under measurability con-
ditions that preclude pointwise verification of (1.2) - be signifi-
cant a priori. We therefore believe it useful to investigate how
much of the theory will remain if "natural Carathéodory conditions[11]
are imposed.

By "natural Carathéodory conditions" we mean, beyond the line-
arity and the non-dependence on the future, merely the requirement
that the derivative be equated to a locally integrable function, plus
the weakest possible local boundedness condition on the dependence of
this locally integrable function on the presumptive solution. For
greater generality, we consider functions defined for all time, past
and future, and delays that may be unbounded; in Section 5 we demon-
strate the reduction of a more conventional situation to a special
case of this one.

We thus assume given a "memory" M, a linear mapping transforming
continuous functions into locally integrable ones in such a way that
Mu and Mv agree on $]-\infty,t]$ if u and v agree there (non-depend-
ence on the future), and consider the equation

$$(1.3) \qquad\qquad \dot{u} + Mu = r \qquad\qquad t \geq t_0,$$

where u is specified on $]-\infty,t_0]$ and r is locally integrable. The
local boundedness condition we impose states, in effect, that there
are everywhere intervals [a,b], sufficiently small, such that M maps
a bounded set of continuous functions <u>vanishing</u> <u>outside</u> [a,b] into a
set of functions whose $\underset{\sim}{L}^1$-norm <u>when</u> <u>restricted</u> ^o [a,b] is bounded
(and not necessarily small, a priori, for small intervals). Nothing is
assumed about the behavior of the transforms outside [a,b], nor about
the action of M on other continuous functions (see Section 3).

Under these weak conditions the initial-value problem (1.3) always has a strongly unique solution (Theorem 4.4). As a bonus, R^n may be replaced by any Banach space E without modifying the proofs; these rely on nothing more exotic than the Banach Contractive Mapping Principle.

If one imposes somewhat stronger and uniform boundedness conditions on M, one obtains the usual exponential bounds on the growth of the solutions (Theorem 4.6).

In Section 5 we specialize to the situation in which Mu and Mv agree on an interval $[a,b]$ if u and v agree on $[a-1,b]$; i.e., the memory recalls nothing beyond a delay of 1. In this case we have the usual "transition operators" $U(t,t_0)$ that map the slice on $[t_0-1,t_0]$ of a solution of the homogeneous equation into the slice on $[t-1,t]$ of the same solution.

In Section 6 we show that, under the conditions guaranteeing bounds on the growth of solutions, the transition operators $U(t,t_0)$ for $t \geqq t_0+1$ are compact if E is finite-dimensional. This section appears to require results in functional analysis slightly less elementary than those used in the rest of the paper.

2. Notation and terminology.

Throughout this paper, E shall denote a real or complex Banach space; the norm in E is denoted by $\| \ \|$.

The domain of most functions we shall deal with in Sections 3 and 4 is R, the real line, which we suppose provided with its usual metric topology and Lebesgue measure. An interval is a connected subset of R with more than one point. The notations $[a,b]$, $]a,b]$, $[a,\infty[$, etc., are used to denote specific intervals.

the linear space of all continuous functions

the Banach space of all bounded continuous

:h the norm $]f| = \sup_{t \in R}||f(t)||$. We always de-

:(E) by £.

the linear space of all (equivalence classes

asurable functions $f: R \to E$ that are Bochner→

>act interval; and by $\underset{\sim}{M}(E)$ the Banach space

:h $|£_L = \sup \int^{t+1} llf(s)||ds < »$, with this su-

• $\underset{\sim}{L}(E)$ <u>agree</u> on a measurable subset S of R

if the restrictions to S are equal (modulo

E), its <u>support</u> , supp f, is the complement

jubset of R on which f agrees with 0; in

(E) then supp f is the closure of $\{t \; e \; R$:

3. <u>Memories</u>.

;pace E be fixed. A <u>memory</u> is a linear mapping

that: <u>if</u> $t \; e \; R$ <u>and</u> <u>if</u> $u,v \; e \; \underset{\sim}{K}(E)$ <u>agree</u> <u>cm</u>

<u>agree</u> <u>on</u> $]-°°,t]$; equivalently: <u>if</u> $u \; e \; \underset{\sim}{K}(E)$

, <u>then</u> supp$(Mu) \; c \; [t,»[$.

' M and each compact interval $[a,b]$ we define

native number

$\geq pb$

$.\,|$

$;a,b) = \sup J \;\; ||(Mu)(s)||ds: u \; e \; £, \; \text{supp} \; u \; c \; [a,b]f;$

likely^ we omit mention of the memory M and

: definition obviously implies

)

$\% \; k_Q(a,b) \quad if \quad [c,d] \; c \; [a,b].$

3.1. Lemma. Let the memory M and the interval $[a,b]$ be given. Then

(3.3)
$$k_0(a,b) = \sup\left\{\int_a^b \|(Mu)(s)\| ds : u \in \Sigma, \text{ supp } u \subset [a,\infty[\right\},$$

and $\lim_{t \to b-0} k_0(a,t) = k_0(a,b)$.

Proof. Denote the right-hand side of (3.3) by $k_1(a,b)$. Since obviously $k_0(a,b) \leqq k_1(a,b)$, and since $t \mapsto k_0(a,t)$ is isotone by (3.2), it will be enough to prove this: for every number $k < k_1(a,b)$ there exists t, $a < t < b$, such that $k_0(a,t) \geqq k$.

Let such a number k be given; we may choose a fixed $u \in \Sigma$ such that supp $u \subset [a,\infty[$ and $\int_a^b \|(Mu)(s)\| ds > k$. Since $Mu \in \underset{\sim}{L}(E)$, there exists c, $a < c < b$, such that

(3.4)
$$\int_a^c \|(Mu)(s)\| ds \geqq k.$$

Choose c in this manner, and choose t, $a < c < t < b$. We may select $v \in \Sigma$ such that v agrees with u on $]-\infty,c]$ and with 0 on $[t,\infty[$. Then supp $v \subset [a,t]$ and, since M is a memory, Mv agrees with Mu on $]-\infty,c]$. Using (3.4) and the definition of $k_0(a,t)$, we find

$$k_0(a,t) \geqq \int_a^t \|(Mv)(s)\| ds \geqq \int_a^c \|(Mv)(s)\| ds = \int_a^c \|(Mu)(s)\| ds \geqq k,$$

as was to be proved.

3.2. Lemma. Let the memory M and the interval $[a,b]$ be given. If $k_0(a,b) < \infty$, then $\lim_{t \to a+0} k_0(t,b) = k_0(a,b)$.

Proof. Assume $k_0(a,b) < \infty$. Since $t \mapsto k_0(t,b)$ is antitone by (3.2), it will be enough to prove the following: for every number $k < k_0(a,b)$ there exists t, $a < t < b$, such that $k_0(t,b) \geqq k$.

Let such a number k be given; we may then choose a fixed $u \in \Sigma$ such that supp $u \subset [a,b]$ and $\int_a^b \|(Mu)(s)\| ds > k$. Since u is continuous and $u(a) = 0$, we may choose c, $a < c < b$, such that $k_0(a,b) \cdot \|u(t')\| \leqq \int_a^b \|(Mu)(s)\| ds - k$ for all $t' \in [a,c]$. We further

(3.10) indeed hold for $i = k$ too. If, on the other hand, this inequality does not hold, we set $v_k = v_{k-1} + u_k$. By (3.7), (3.9), supp $v_{k-1} \subset [a_0, a_{k-1}]$ (or $= \emptyset$ if $k = 1$) and supp $u_k \subset [a_{k-1}, a_k]$; it follows that $v_k \in \Sigma$ and (3.9) is satisfied for $i = k$ too. Since v_k and v_{k-1} agree on $]-\infty, a_{k-1}]$, so do Mv_k and Mv_{k-1}; thus (3.10) holds for $1 \leqq j < i = k$. Finally, (3.8) and the assumption on v_{k-1} imply

$$\int_{a_{k-1}}^{a_k} \|(Mv_k)(s)\| ds \geqq \int_{a_{k-1}}^{a_k} \|(Mu_k)(s)\| ds - \int_{a_{k-1}}^{a_k} \|(Mv_{k-1})(s)\| ds \geqq$$
$$\geqq k_0(a_{k-1}, a_k) - \varepsilon - \frac{1}{2} k_0(a_{k-1}, a_k) = \frac{1}{2} k_0(a_{k-1}, a_k) - \varepsilon,$$

so that (3.10) also holds for $j = i = k$, and the induction is complete.

From (3.9), (3.10) with $i = n$ we obtain

$$\frac{1}{2} \sum_{j=1}^{n} k_0(a_{j-1}, a_j) - n\varepsilon \leqq \sum_{j=1}^{n} \int_{a_{j-1}}^{a_j} \|(Mv_n)(s)\| ds = \int_{a_0}^{a_n} \|(Mv_n)(s)\| ds \leqq$$
$$\leqq k_0(a_0, a_n).$$

Since $\varepsilon > 0$ was arbitrary, the conclusion follows.

3.4. <u>Theorem</u>. <u>Let the memory</u> M, <u>the interval</u> $[a,b]$, <u>and the positive integer</u> n <u>be given. There exists a sequence</u> (a_i), $i = 0, \ldots, n$, <u>such that</u>

(3.11) $a = a_0 < a_1 < \cdots < a_n = b$,

(3.12) $k_0(a_{i-1}, a_i) \leqq 2n^{-1} k_0(a,b)$, $i = 1, \ldots, n$.

<u>Proof</u>. If $k_0(a,b) = 0$ or $k_0(a,b) = \infty$, every sequence (a_i), $i = 0, \ldots, n$, satisfying (3.11) will also satisfy (3.12) (use (3.2) in the former case). We therefore assume that $0 < k_0(a,b) < \infty$ and set $\rho = 2n^{-1} k_0(a,b)$, so that $0 < \rho < \infty$.

For each $t \in [a,b[$ we set $\tau(t) = \sup\{s \in [t,b]: s > t, k_0(t,s) \leqq \rho\}$, so that $t \leqq \tau(t) \leqq b$. By (3.2), $\tau: [a,b[\to [a,b]$ is isotone. By Lemma 3.2, the function $t \mapsto k_0(t,s): [a,s[\to R$ is right-continuous for each fixed s. It follows at once that the

isotone function τ is right-continuous. From Lemma 3.1 we obtain, on the other hand,

$$(3.13) \qquad k_0(t,\tau(t)) \leqq \rho \quad \text{if} \quad \tau(t) > t.$$

We construct the sequence (a_i) - perhaps terminating, perhaps not - by setting $a_i = \tau^i(a)$, $i = 0,.,...$ (exponents indicate iteration) and continuing so long as $\tau^i(a) < b$. From the definition we have $a = a_0 \leqq a_1 \leqq \cdots$. The sequence ends at $i = k$ if and only if $\tau(a_k) = b$; this may happen even for $k = 0$. On the other hand, if $a_{k+1} = a_k$ for some k, the sequence is constant from $i = k$ on, and does not end at all.

We intend to show that the sequence does indeed end. Let $k \geqq 0$ be some index reached by the sequence, so that $\tau^k(a) < b$. Since τ is isotone and right-continuous, there exists a sequence (b_i), $i = 0,...,k$, such that

$$b_0 = a; \qquad b_i > \tau(b_{i-1}), \ i = 1,...,k; \qquad b_k < b.$$

From the definition of τ it follows that $a = b_0 < b_1 < \cdots < b_k < b$ and $k_0(b_{i-1},b_i) > \rho$, $i = 1,...,k$. Lemma 3.3, (3.2), and the definition of ρ then imply that either $k = 0$, or $k > 0$ and

$$k\rho < \sum_{i=1}^{k} k_0(b_{i-1},b_i) \leqq 2k_0(b_0,b_k) \leqq 2k_0(a,b) = n\rho.$$

In either case, we conclude that $k < n$. Since k was an arbitrary index reached by the sequence, the sequence must end at some index k', $0 \leqq k' < n$. But then $a = a_0 < a_1 < \cdots < a_{k'} < b$ and $\tau(a_{k'}) = b$. We choose $a_{k'+1},...,a_n$ arbitrarily so that $a_{k'} < a_{k'+1} < \cdots < a_n = b$. Thus (3.11) holds. We have $a_i = \tau(a_{i-1}) > a_{i-1}$, $i = 1,...,k'$, and $b = \tau(a_{k'}) > a_{k'}$; therefore, by (3.13) and (3.2), $k_0(a_{i-1},a_i) \leqq \rho$, $i = 1,...,k'$, and $k_0(a_{i-1},a_i) \leqq k_0(a_{k'},b) \leqq \rho$, $i = k'+1,...,n$; this proves (3.12).

The preceding analysis suggests the formulation of the following condition that a memory M may have:

(M_0): <u>For</u> <u>every</u> t e R <u>there</u> <u>are</u> t^1, t'' e R, $t^1 < t < t''$, <u>such</u> <u>that</u> $k_Q(t',t)$, $k_Q(t,t'') < \infty$.

3.5. <u>Corollary</u>. <u>A</u> <u>memory</u> M <u>satisfies</u> (M_0) <u>If</u> <u>and</u> <u>only</u> j^ <u>there</u> <u>exists</u>, <u>for</u> <u>each</u> <u>number</u> p > 0 <u>and each compact interval</u> [a,b], <u>a</u> <u>finite</u> <u>sequence</u> (a_i), i = 0, ... ,n, <u>such</u> <u>that</u> $a = a_0 < \bullet\bullet\bullet < a_n = b$ <u>and</u> $k_0(a_{i-1}, a_i)$ ^ p, *- = 1,...,n.

<u>Proof</u>. The "if" part is a trivial consequence of (3.2). If M satisfies (M_n) and [a,b] is given, an obvious compactness argument implies, with the help of (3.2), that there exists a finite sequence (c_j), j = 1,...,m, say, such that $a = c_0 < \bullet\bullet\bullet < c_m = b$ and ^$_n$^c $_{j-1} > c_j$) ^ ∞, J = 1,...,m« The conclusion follows by Theorem 3.4.

4. <u>Solutions</u>.

When attempting to define and work with solutions of equations such as (1.3) under "Carathéodory conditions" it is most convenient to refer to the corresponding integral equation, as we now do.

Let the Banach space E and the memory M be given and fixed in the sequel. If r e L(E) and a e R are given, a <u>solution</u> of

$(4.1)_{[a]}$ $\qquad\qquad$ $\dot{u} + Mu = r,$ $\qquad$ t £ a

is defined to be a function u e K(E) that satisfies

$(4.2)_{[a]}$ $\qquad$ $u(t) = u(a) - \int_a^t (Mu-r)(s)ds,$ $\qquad$ $t \geq a.$

It is clear that if u is a solution of $(4.1)_r$, and a^T ^ a, then u is also a solution of $(4.1)_{r,n}$.

Our principal aim in this section is to prove a strong existence and uniqueness theorem for the "initial-value problem" for the equations $(4.1)_r$, under the very mild condition (M_0) on the memory M.

4.1. <u>Lemma</u>. <u>Let</u> $f \in \underset{\sim}{L}(E)$ <u>and the interval</u> $[a,b]$ <u>be given</u>, <u>and assume that</u> $k_0(a,b) < 1$. <u>There exists a unique function</u> $y \in \underset{\sim}{C}(E)$ <u>such that</u> y <u>agrees with</u> 0 <u>on</u> $]-\infty,a]$ <u>and with a constant on</u> $[b,\infty[$, <u>and such that</u>

$$(4.3) \qquad y(t) = \int_a^t (f-My)(s)ds, \qquad a \leqq t \leqq b.$$

<u>This function satisfies</u>

$$(4.4) \qquad |y| \leqq (1 - k_0(a,b))^{-1} \int_a^b \|f(s)\| ds.$$

<u>Proof</u>. Let $\underset{\sim}{A}$ be the subspace of $\underset{\sim}{C}(E)$ consisting of those functions that agree with 0 on $]-\infty,a]$ and with a constant on $[b,\infty[$. A function $y \in \underset{\sim}{A}$ satisfies (4.3) if and only if it is a fixed point of the affine mapping $F: \underset{\sim}{A} \to \underset{\sim}{A}$ defined by

$$(4.5) \qquad (Fz)(t) = \int_{\min\{a,t\}}^{\min\{b,t\}} (f-Mz)(s)ds, \qquad t \in R, \quad z \in \underset{\sim}{A}.$$

But such a fixed point exists indeed, and is unique, for F is contractive: by Lemma 3.1,

$$(4.6) \quad |Fz' - Fz| \leqq \int_a^b \|(M(z'-z))(s)\| ds \leqq k_0(a,b) \cdot |z' - z|, \qquad z,z' \in \underset{\sim}{A},$$

and $k_0(a,b) < 1$ by assumption. By (4.5), (4.6), the unique fixed point y satisfies

$$|y| = |Fy| \leqq |Fy - F0| + |F0| \leqq k_0(a,b) \cdot |y| + \int_a^b \|f(s)\| ds,$$

and this implies (4.4).

4.2. <u>Lemma</u>. <u>Let</u> $f \in \underset{\sim}{L}(E)$ <u>and the interval</u> $[a,b]$ <u>be given</u>, <u>and assume that there exists a sequence</u> (a_i), $i = 0,\ldots,n$, <u>such that</u> $a = a_0 < \cdots < a_n = b$ <u>and</u> $k_0(a_{i-1},a_i) < 1$, $i = 1,\ldots,n$. <u>Then there exists a unique function</u> $w_0 \in \underset{\sim}{C}(E)$ <u>such that</u> w_0 <u>agrees with</u> 0 <u>on</u> $]-\infty,a]$ <u>and with a constant on</u> $[b,\infty[$, <u>and such that a function</u> $w \in \underset{\sim}{K}(E)$ <u>agrees with</u> 0 <u>on</u> $]-\infty,a]$ <u>and satisfies</u>

$$(4.7) \qquad w(t) = \int_a^t (f-Mw)(s)ds, \qquad a \leqq t \leqq b$$

<u>if and only if</u> w <u>agrees with</u> w_0 <u>on</u> $]-\infty,b]$.

Proof .We define the sequence (y_i), $i = 1,\ldots,n$, in $\underset{\sim}{C}(E)$ by

induction, as follows: by Lemma 4.1 there exists, for each $i = 1,\ldots,n$,

a unique $y_i \in \underset{\sim}{C}(E)$ that agrees with 0 on $]-\infty, a_{i-1}]$ and with a

constant on $[a_i, \infty[$, and satisfies

(4.8) $y_i(t) = \int^t (f - M(\sum y_i))(s)ds,$ $a_{i-1} \le t \le a_i.$

We set $w_0 = \sum_{i=1}^{n-1} y_i$ and claim that this function satisfies the con-

clusion. Indeed, $w_0 \in \underset{\sim}{C}(E)$ agrees with 0 on $]-\infty, a]$ and with a

constant on $[b, \infty[$, since all y_i have these properties. We observe

that w_0 and $\sum_{i=1}^{i} y_i$ agree on $]-\infty, a_i]$, hence Mw_0 and $M(\sum_{i=1}^{i} y_i)$

agree on the same interval for $i = 1,\ldots,n$; thus (4.8) implies

(4.9) $y_i(t) = \int^t (f - Mw_0)(s)ds,$ $a_{i-1} < t < a_i$ $i = 1,\ldots,n.$

Let $w \in K(E)$ be given. Suppose that w and w_0 agree on

$]-\infty, b]$ (hence w agrees with 0 on $]-\infty, a]$); then Mw and Mw_0

also agree on $]-\infty, b]$. Let $t \in [a,b]$ be given; then $t \in [a_{k-1}, a_k]$

for some index k, $1 \le k \le n$; from (4.9) and the assumptions on the

y_i we find

$$w(t) = w_0(t) = \sum_{i=1}^{k-1} y_i(t) + \sum_{i=k+1}^{n} y_i(t) = \sum_{i=1}^{k-1} y_i(a_i) + y_k(t) + 0 =$$

$$= \sum_{i=1}^{k-1} \int_{a_{i-1}}^{a_i} (f - Mw_0)(s)ds + \int_{a_{k-1}}^{t} (f - Mw_0)(s)ds = \int_a^t (f - Mw_0)(s)ds =$$

$$= \int_a^t (f - Mw)(s)ds,$$

so that indeed w satisfies (4.7); in particular, so does w_0 itself.

Assume conversely that w agrees with 0 on $]-\infty, a]$ and satis-

fies (4.7); since w_0 has the same properties, we may set $z = w - w_0$

and find that

(4.10) $z(t) = - \int_a^t (Mz)(s)ds,$ $a \le t \le b.$

We claim that z agrees with 0 on $]-\infty, a_i]$, $i = 0,\ldots,n$. The claim

is true for $i = 0$; suppose it is true for $i = k-1$ for a given k,

$1 \pounds k \wedge n$. Define z, e $C(E)$ to agree with z on $]-\infty, a,]$ (hence with 0 on $]-\infty^{\wedge}a_{k-\bar{1}}]$) and with a constant on $[a_{K},\infty[$. Since Mz_{k} and Mz agree on $]-\infty^{\wedge}a_{K},]$, (4.10) implies

$$z_{k}(t) = \int_{a_{k-\bar{1}}}^{t} (Mz_{k})(s)ds, \qquad a_{fc-1} \leqslant t \leqslant a_{k};$$

by the uniqueness conclusion of Lemma 4.1, $z_{,}^{K} = 0$, and the claim is proved for $i = k$. Hence it holds for all i, and in particular $z = w - w_{0}$ agrees with $z_{n} = 0$ on $]-\infty, a_{n}] =]-\infty, b]$; thus w and w_{0} agree on this interval, as was to be proved.

4.3. <u>Scholium</u>. The existence of the required sequence (a_{i}) is guaranteed by Theorem 3.4 if $k_{0}(a,b) < \infty$. In this case we can estimate the size of w_{0}; we shall try to do this as precisely as possible. Set $k_{0} = k_{0}(a,b) < \infty$ and choose a positive integer $n > 2k_{0}$. By Theorem 3.4 there exists (a_{i}) such that $a = a_{0} < \bullet\bullet\bullet < a_{n} = b$ and $k_{n}(^{a}{}_{1-i}, {}^{a}{}_{1} \bullet) \wedge 2n^{-1}k_{0} < 1$. Then we construct (y_{i}) and w_{0} as in the proof of Lemma 4.2. By Lemma 4.1 and (4.8),

$$|y_{i}| \leqq (1-2n^{-1}k_{0})^{-1}(\int_{a_{i-i}}^{a_{i}} ||f(s)||ds + \int_{^{\wedge}a_{,-i}}^{} ||(My_{J})(s)||ds), \quad i = 1,\ldots,n,$$

and hence

$$\bigvee_{i=t}|y_{i}| \wedge (1-2n^{-1}k_{0})^{-1}(\int_{0}^{a_{k}}||f(s)||ds + \int^{a_{j}} ||(My.)(s)||ds) =$$

$$= (1-2n^{-1}k_{0})^{-1}\left(\int^{a_{k}}||f(s)||ds + \sum_{j=1}^{}\int_{a_{j-1}}^{a_{j}} ||(My_{i})(s)||ds\right) =$$

$$= (1-2n^{-1}k_{0})^{-1}(\int_{a_{0}}^{a_{k}}||f(s)||ds + \sum_{i=1}^{}\int_{a_{i}}^{}||(My_{i})(s)||ds) \leqq$$

$$\leqq (1-2n^{-1}k_{0})^{-1}(\int_{a}^{b}||f(s)||ds + k_{0}\sum^{k-1}|y_{i}|), \qquad k = 1,\ldots,n.$$

It follows from this sequence of inequalities that $w_{0} = \sum_{i=1} y_{i}$ satisfies

$$|w_{0}| \leqslant (\wedge k_{j}-\wedge 1 - 2n^{-1}\wedge)^{-1})\int_{a}^{} ||f(s)||ds;$$

we note that this inequality holds for every positive integer $n > 2k_{n}$.

4.4. Theorem. Let M be a memory satisfying (M_0). For every $a \in R$ there are linear mappings $P(a)\colon \underset{\sim}{K}(E) \to \underset{\sim}{K}(E)$ and $Q(a)\colon \underset{\sim}{L}(E) \to \underset{\sim}{K}(E)$ such that for every $v \in \underset{\sim}{K}(E)$ and $r \in \underset{\sim}{L}(E)$:

(1): $u = P(a)v + Q(a)r$ is the unique solution of $(4.1)_{[a]}$ that agrees with v on $]-\infty,a]$;

(2): if $u \in \underset{\sim}{K}(E)$ agrees with v on $]-\infty,a]$ and satisfies

(4.11) $u(t) = u(a) - \int_a^t (Mu-r)(s)ds$, $a \leqq t \leqq b$,

for a given $b > a$, then u agrees with $P(a)v + Q(a)r$ on $]-\infty,b]$.

Proof. Let a, v, r be given, and let $v_0 \in \underset{\sim}{K}(E)$ be the function that agrees with v on $]-\infty,a]$ and with a constant on $[a,\infty[$; set $f = r - Mv_0 \in \underset{\sim}{L}(E)$. For given $b > a$, a function $u \in \underset{\sim}{K}(E)$ agrees with v (hence with v_0) on $]-\infty,a]$ and satisfies (4.11) if and only if $w = u - v_0$ agrees with 0 on $]-\infty,a]$ and satisfies

$w(t) = u(t) - v_0(t) = u(t) - u(a) = \int_a^t (r-Mu)(s)ds = \int_a^t (f-Mw)(s)ds$,

$$a \leqq t \leqq b,$$

that is, (4.7). By Corollary 3.5, the assumptions of Lemma 4.2 are satisfied; the conclusion of the theorem then follows from that lemma, from comparison of (4.11) and $(4.2)_{[a]}$, and from the linearity of the problem.

4.5. Scholium. P and Q satisfy certain functional equations. Indeed, let $a,a' \in R$, $a' \geqq a$, and $v \in \underset{\sim}{K}(E)$, $r \in \underset{\sim}{L}(E)$ be given, and set $u = P(a)v + Q(a)r$. Since u is a solution of both $(4.1)_{[a]}$ and $(4.1)_{[a']}$, and of course agrees with itself on $]-\infty,a']$, we have

$P(a)v + Q(a)r = u = P(a')u + Q(a')r = P(a')P(a)v + (P(a')Q(a)+Q(a'))r$.

Since v, r were arbitrary, P and Q satisfy the equations

(4.12) $P(a) = P(a')P(a)$, $Q(a) = P(a')Q(a) + Q(a')$, $a \leqq a'$.

An interesting case to which Theorem 4.4 is applicable is that

in which the memory M satisfies the following condition:

(M): <u>The</u> <u>restriction</u> <u>of</u> M <u>to</u> $\underset{\sim}{C}(E)$ <u>is</u> <u>a</u> <u>bounded</u> <u>linear</u> <u>mapping</u> $M_{\underset{\sim}{C}}: \underset{\sim}{C}(E) \rightarrow \underset{\sim}{M}(E)$.

We note that if M satisfies (M), then

(4.13) $k_0(a,b) \leqq \|M_{\underset{\sim}{C}}\|,$ if $0 < b-a \leqq 1,$

and therefore M also satisfies (M_0).

4.6. <u>Theorem</u>. <u>Let</u> M <u>be</u> <u>a</u> <u>memory</u> <u>satisfying</u> (M). <u>Then</u> <u>the</u> <u>con-</u> <u>clusions</u> <u>of</u> <u>Theorem</u> <u>4.4</u> <u>hold</u> <u>and</u> <u>there</u> <u>exists</u> <u>a</u> <u>number</u> $\sigma > 0$ <u>such</u> <u>that</u>

(4.14)
$$\|(P(a)v)(t)\| \leqq e^{\sigma[t-a+1]} \sup_{s \leqq a}\|v(s)\|, \qquad v \in \underset{\sim}{K}(E)$$
$$\|(Q(a)r)(t)\| \leqq \|M_{\underset{\sim}{C}}\|^{-1}(e^{\sigma}-1)\int_a^t e^{\sigma[t-s]}\|r(s)\|ds, \quad r \in \underset{\sim}{L}(E)$$
$$t \geqq a,$$

<u>where</u> [ ] <u>denotes</u> <u>the</u> "<u>greatest-integer</u>" <u>function</u>.

<u>Proof</u>. 1. We claim that the conclusion holds with

(4.15) $\sigma = \log(\sum_{i=0}^{n} \|M_{\underset{\sim}{C}}\|^i (1 - 2n^{-1}\|M_{\underset{\sim}{C}}\|)^{-i}),$

where n is an arbitrary fixed integer greater than $2\|M_{\underset{\sim}{C}}\|$.

We use the following temporary notation: if $y \in \underset{\sim}{K}(E)$ and $t \in R$, we set $|y|_t = \sup_{s \leqq t}\|y(s)\| \leqq \infty$.

2. Let $a',b' \in R$, $0 < b'-a' \leqq 1$, and $v \in \underset{\sim}{K}(E)$, $r \in \underset{\sim}{L}(E)$ be given. Then $u = P(a')v + Q(a')r$ is the unique solution of $(4.1)_{[a']}$ that agrees with v on $]-\infty,a']$. We let $v_0 \in \underset{\sim}{K}(E)$ be defined to agree with v on $]-\infty,a']$ and with a constant on $[a',\infty[$, and set f = $= r - Mv_0$, $w = u - v_0$. Using (4.13) and $|v_0|_{b'} = |v|_{a'}$, we find
$$\int_{a'}^{b'}\|f(s)\|ds \leqq \int_{a'}^{b'}(\|(Mv_0)(s)\| + \|r(s)\|)ds \leqq \|M_{\underset{\sim}{C}}\| |v|_{a'} + \int_{a'}^{b'}\|r(s)\|ds.$$
By the argument of the proof of Theorem 4.4 and by Lemma 4.2 and Scholium 4.3, w agrees on $]-\infty,b']$ with a function $w_0 \in \underset{\sim}{C}(E)$ that satisfies $|w_0| \leqq c\int_{a'}^{b'}\|f(s)\|ds$, where we may choose $c = \|M_{\underset{\sim}{C}}\|^{-1}(e^{\sigma}-1)$ on account of (4.13), (4.15). Therefore

$$|P(a')v + Q(a')r|_{b'} = |u|_{b'} \le |v_0 1_{b'} + |w_0| \quad \wedge$$

$$\le |v|_{a'} + ||M_c H^J - (e^a - 1)(||M_c|| |v|_{al} + j_a j|r(s)| |ds) =$$

$$= e^a |v| \quad , + i[M_r||^{-1}(e^{CT}-1)r^{Br'} j|r(s)| |ds.$$

Since v, r were arbitrary and $P(a^f)$, $Q(a^f)$ are linear,

(4.16) $1P(a')v|_t \% e^Q 1v1_- , , \quad |Q(a')r| \cdot \wedge ||M-\_r|^{-1}(e^o-1) j^{b'} \quad ||r(s)| |ds,$

$$0 < b'-a' \wedge 1, \quad v \; \pounds \; \underset{\sim}{K}(E), \quad r \in \underset{\sim}{L}(E).$$

3. Let a,t e R, t > a, and v e $\underset{\sim}{K}(E)$, r $\in$ $\underset{\sim}{L}(E)$ be given. We

apply (4.12) and (4.16) successively to $a' = t-1,...,t-[t-a],a$

and $b^1 = t,...,t-[t-a]+1,t-[t-a]$, and find

$$|P(a)v| \; 1 \; \wedge^{[t-a+1]} e^a M \; , \quad _{a]}$$

$$|Q(a)rl. \quad \wedge \quad ||M_c|r^1(e^a-1)(\; J \; _e{}^{CT} < \wedge^X > f'' \wedge H r (s) \; ||ds + e^{\wedge} \wedge f''^1 \wedge \; ||r_{(s)}| |ds);$$

and (4.14) is an immediate consequence.

5• _Short memories_.

In this section we sketch a typical situation that can be reduced

to a special case of the existence and uniqueness theorems presented

earlier.

We again assume that the Banach space E is given. For every

a e R and every function f defined on a subset of R containing

[a,»[, f_r , denotes the restriction of f to [a,»[. For each a e R,

K_r -i(E), C_r -.(E), L_r -.(E), M_r -,(E) denote the spaces consisting of the

restrictions to [a,»[of the elements of $\underset{\sim}{K}(E)$, $\underset{\sim}{C}(E)^\wedge \underset{\sim}{L}(E)$, $\underset{\sim}{M}(E)$,

respectively. The second and the fourth are Banach spaces with the

obvious norms.

$\underset{\sim}{E}$ shall denote the Banach space $\underset{\sim}{C}([-1,0],E)$ of all continuous

functions v': [-1,0] •* E, with the norm $0 v' i = \sup_{s \in [-1,0]} ||v^T(s)||$. Let

a $\in$ R be given. For each t $\wedge$ a we define the _slicing operator_

$n(t)$: K. $_n(E)$ -+ E by

$(\Pi(t)u')(s) = u'(t+s),$ $\qquad -1 \leqq s \leqq 0,$ $\quad u' \in \underset{\sim}{K}_{[a-1]}(E),$

so that $\Pi(t)u'$ is the "slice" of u' between $t-1$ and t,

transplanted to $[-1,0]$.

A <u>short memory</u> is a linear mapping $M': \underset{\sim}{K}_{[-1]}(E) \to \underset{\sim}{L}_{[0]}(E)$

such that: <u>for each interval</u> $[a,b] \subset [0,\infty[$, <u>if</u> $u',v' \in \underset{\sim}{K}_{[-1]}(E)$

<u>agree on</u> $[a-1,b]$, <u>then</u> $M'u'$, $M'v'$ <u>agree on</u> $[a,b]$. This condition

allows us to define, for each $a \geqq 0$, a "cut-down" memory

$M'_{[a]}: \underset{\sim}{K}_{[a-1]}(E) \to \underset{\sim}{L}_{[a]}(E)$ as follows: if $u' \in \underset{\sim}{K}_{[a-1]}(E)$, then

$u' = u''_{[a-1]}$ for some $u'' \in \underset{\sim}{K}_{[-1]}(E)$, and we set $M'_{[a]}u' = (M'u'')_{[a]}$;

it follows at once from the definition of a short memory that this

construction does not depend on the choice of u'', and satisfies

$(5.1) \qquad M'_{[a']}u'_{[a'-1]} = (M'_{[a]}u')_{[a']}, \qquad a' \geqq a \geqq 0, \quad u' \in \underset{\sim}{K}_{[a-1]}(E).$

For each short memory M' and each interval $[a,b] \subset [0,\infty[$

we may define

$$k'_0(M';a,b) = \sup\left\{\int_a^b \|(M'u')(s)\|ds: u' \in \underset{\sim}{C}_{[-1]}(E), \ |u'| \leqq 1, \ \text{supp } u' \subset [a,b]\right\},$$

and state conditions on M' analogous to (M_0) and (M):

(M'_0): <u>For each</u> $t > 0$ <u>there exists</u> t', $0 \leqq t' < t$, <u>such that</u>

$k'_0(t',t) < \infty$, <u>and for each</u> $t \geqq 0$ <u>there exists</u> $t'' > t$ <u>such that</u>

$k'_0(t,t'') < \infty$.

(M'): <u>The restriction of</u> M' <u>to</u> $\underset{\sim}{C}_{[-1]}(E)$ <u>is a bounded linear</u>

<u>mapping</u> $M'_{\underset{\sim}{C}}: \underset{\sim}{C}_{[-1]}(E) \to \underset{\sim}{M}_{[0]}(E)$.

We assume the short memory M' and the function $r' \in \underset{\sim}{L}_{[0]}(E)$

given, and consider the equation

$(5.2)_{[a]} \qquad \dot{u}'_{[a]} + M'_{[a]}u' = r'_{[a]}$

for each $a \geqq 0$: a <u>solution</u> of $(5.2)_{[a]}$ is defined to be a function

$u' \in \underset{\sim}{K}_{[a-1]}(E)$ such that

$(5.3)_{[a]} \qquad u'(t) = u'(a) - \int_a^t ((M'_{[a]}u')(s) - r'(s))ds, \qquad t \geqq a.$

We now come to the existence and uniqueness theorems for the initial-value problem for equation $(5.2)_{r_{[a]}}$.

5.1. $\overline{\text{Theorem.}}$ $\overline{\text{Let}}$ $M^{!}$ $\overline{be}$ $\overline{a}$ $\overline{\text{short memory satisfying}}$ $(M*)$. $\overline{\text{For}}$ $\overline{\text{every}}$ $a *? 0$ $\overline{\text{there are linear mappings}}$ $P^{f}(a) : \overline{E} \to \overline{K_r^{[a-1]}}(E)$ $\overline{\text{and}}$ $Q^{!}(a) : \overline{L_f^{[0]}}(E) \to \overline{K_f^{[a-1]}}(E)$ $\overline{\text{such that for every}}$ $v^{f} \in \overline{E}$ $\overline{\text{and}}$ $r^{f} \in \overline{L^{[0]}}(E)$:

(1): $u^{!} = P^t(a)v^{1} + Q^{!}(a)r^{!}$ $\overline{\text{JLS the unique solution of}}$ $(5.2)_{[a]}$ $\overline{\text{that satisfies}}$ $\Pi(a)u^T = v^{f}$;

(2): $\overline{\text{If}}$ $u^{f} \in \overline{K^{[a-1]}}(E)$ $\overline{\text{satisfies}}$ $\Pi(^a)u^T = v^{f}$ $\overline{\text{and}}$

$$(5.4) \qquad u'(t) = u'(a) - J_a((Mj_{[a]}u')(s) - r'(s))ds, \qquad a < t \leq b$$

$\overline{\text{for a given}}$ $b > a$, $\overline{\text{then}}$ u^{1} $\overline{\text{agrees with}}$ $P^{f}(a)v^{f} + Q^{f}(a)r^{!}$ $\overline{\text{£n}}$ $[a-1,b]$.

$\underline{\text{Proof.}}$ Let $a \geq 0$ be fixed. Define $M: \underset{\sim}{K}(E) \to \underset{\sim}{L}(E)$ as follows: for each $u \in \underset{\sim}{K}(E)$, Mu agrees with 0 on $]-\infty,a]$ and with $M_{r_{[a]}}^{!} U_{[a-1]}^{1}$ on $[a,\infty[$. it is immediate from the definitions that M is a memory and that it satisfies (M_0).

Let $v^{!} \in \underset{\sim}{E}$ and $r^{!} \in \underset{\sim}{L_{[0]}}(E)$ be given. Define $v \in \underset{\sim}{K}(E)$ to agree with a constant on $]-\infty,a-1]$ and with a (possibly different) constant on $[a,\infty[^$ and to satisfy $v(a+s) = v^{!}(s)$, $s \in [-1,0]$; and define $r \in \underset{\sim}{L}(E)$ to agree with 0 on $]-\infty,0]$ and with $r^{!}$ on $[0,\infty[$.

For each $u^{!} \in K_f_{-}(E)$, let $u \in K(E)$ be the function that agrees with a constant on $]-\infty,a-1]$ and with u^{f} on $[a-1,\infty[$. It is then a matter of direct verification that u^{f} satisfies $\Pi(a)u^{f} = v^{!}$ if and only if u agrees with v on $]-\infty,a]$; that u^{1} satisfies (5.4) for a given b if and only if u satisfies (4.11) for the same b; and hence, by inspection of $(5.3)_{r_{[a]}}$ and $(4.2)_{[a]}$, that u^{1} is a solution of $(5.2)_{r_{[a]}}$ if and only if u is a solution of $(4.1)_{[a]}$.

The conclusion now follows from Theorem 4.4. $P'(a)$ and $Q'(a)$ are defined by $P'(a)v' = (P(a)v)_{[a-1]}$, $Q'(a)r' = (Q(a)r)_{[a-1]}$, where v and r are defined as above and depend linearly on v', r', respectively.

The linear-mapping-valued functions P' and Q' also satsfy functional equations, derived from Theorem 5.1 as (4.12) follows from Theorem 4.4. The key observation is that if $a' \geqq a \geqq 0$ and u' is a solution of $(5.2)_{[a]}$, then $u'_{[a'-1]}$ is a solution of $(5.2)_{[a']}$. The functional equations are

$$(5.5) \quad \begin{aligned} \Pi(t)P'(a) &= \Pi(t)P'(a')\Pi(a')P(a) \\ \Pi(t)Q'(a) &= \Pi(t)(P'(a')\Pi(a')Q'(a) + Q'(a')) \end{aligned} \qquad t \geqq a' \geqq a \geqq 0.$$

We introduce the <u>transition operators</u> $U(t,t_0) : \underset{\sim}{E} \to \underset{\sim}{E}$ defined for all $t \geqq t_0 \geqq 0$ by

$$(5.6) \qquad U(t,t_0) = \Pi(t)P'(t_0).$$

The significance of these operators and the justification of the name we have chosen are explained by the following result.

5.2. <u>Corollary</u>. <u>Let</u> M' <u>be a short memory satisfying</u> (M_0'). <u>Then the transition operators satisfy</u>

$$(5.7) \quad U(t_0,t_0) = I, \qquad U(t_2,t_1)U(t_1,t_0) = U(t_2,t_0), \quad t_2 \geqq t_1 \geqq t_0 \geqq 0;$$

<u>and if</u> $a \geqq 0$ <u>and</u> u' <u>is a solution of</u> $(5.2)_{[a]}$ <u>with</u> $r' = 0$ (<u>the homogeneous equation</u>), <u>then</u>

$$(5.8) \qquad \Pi(t)u' = U(t,t_0)\Pi(t_0)u', \qquad t \geqq t_0 \geqq a.$$

<u>Proof</u>. $U(t_0,t_0) = I$ follows from (5.6) and Theorem 5.1,(1). The rest of (5.7) follows from (5.6) and (5.5). If u' is a solution of the homogeneous equation $(5.2)_{[a]}$, then $u' = P'(a)\Pi(a)u'$, and (5.8) follows from (5.6) and (5.7).

We now examine the quantitative results that may be obtained when the short memory $M^!$ satisfies (M^f).

5.3. <u>Theorem. Let</u> $M^!$ <u>be a short memory satisfying</u> (M^f). <u>Then the conclusions ff Theorem</u> $^\wedge JL_4$ <u>hold, and there exists a number</u> $a^1 > 0$ <u>such that</u>

$$(5.9) \quad \|(P'(a)v')(t)\| \le e^{a![t-a+1]}VQ, \quad v< G \underset{\sim}{E}$$

$$\left| (\phi<(\phi'r)()\phi\right| \wedge_M \wedge |r^1(^a1)jV^{[t-s]}_a\| ») (\} |ds, \quad r' \in \underset{\sim[0]}{L}(E)$$

$$t \ge a \ge 0.$$

<u>Proof</u>. We use the constructions of the proof of Theorem 5.1. Since M^1 satisfies (M^1), the memory M satisfies (M), and $\|M_C\| \% \|M'\|$. We choose the positive integer $n > 2\|M'_C\|$, and define a^1 by (4.15) with $\|M_-\|$ replaced by $\|M'\|$. If Q is given by (4.15) with the same n, but without replacement, we have $a \wedge Q^1$, $\|M_-\|^1(e^\sigma - 1) \wedge \|M\bullet\|^1(e^{\sigma'} - 1)$ (observe that a depends on a, but a^1 does not). We may therefore apply Theorem 4.6 to obtain the conclusion, noting that $\sup\|v(s)\| = flv^\wedge$.

5.4. <u>Corollary. Let</u> M^1 <u>bf a short memory satisfying</u> (M^T). <u>Then the transition operators satisfy</u> (5.7) <u>and</u> (5.8), <u>and</u>

$$(5.10) \quad \|U(t,t_o)\| \% e^{\wedge[t't^\wedge 1]^\wedge} \quad * *t_o*0.$$

<u>Proof</u>. Corollary 5.2, (5.6), and (5.9).

6. <u>Compact transition operators</u>.

Our notations and terminology are those of Section 5. It is often important and useful to know whether certain transition operators are compact. We give a strong affirmative result in this direction for finite-dimensional E.

We need some additional notation. We denote by $E-$ the Banach

space $\underset{\sim}{L}^1([-1,0],E)$ of all (equivalence classes of) Bochner-integrable functions $g: [-1,0] \to E$ with the norm $\int_{-1}^{0} \|g(s)\| ds$. It is expedient to use a separate notation for the slicing operator $\Pi_1(t): \underset{\sim[a]}{L}(E) \to \underset{\sim 1}{E}$ defined for each $a \geqq 0$ and $t \geqq a+1$ by

$$(\Pi_1(t)f')(s) = f'(t+s), \qquad -1 \leqq s \leqq 0, \quad f' \in \underset{\sim[a]}{L}(E),$$

to distinguish it from $\Pi(t)$, which acts on a space of continuous functions. We further need the "Volterra operator" $V: \underset{\sim 1}{E} \to \underset{\sim}{E}$ and the "evaluation operator" $W: \underset{\sim}{E} \to \underset{\sim}{E}$ defined, respectively, by

$$(Vg)(s) = \int_{-1}^{s} g(s')ds', \qquad\qquad -1 \leqq s \leqq 0, \quad g \in \underset{\sim 1}{E}$$

$$(Wv')(s) = v'(0), \qquad\qquad -1 \leqq s \leqq 0, \quad v' \in \underset{\sim}{E}.$$

Both are bounded linear mappings, with $\|V\| = \|W\| = 1$. We recall a useful property of V.

6.1. <u>Lemma</u>. <u>If</u> E <u>is finite-dimensional</u>, V <u>maps each relatively weakly compact set in</u> $E_{\sim 1}$ <u>into a relatively compact set in</u> E.

<u>Proof</u>. Let $\underset{\sim}{Q}$ be a relatively weakly compact set in $E_{\sim 1}$. It is relatively weakly sequentially compact (Eberlein's Theorem). We apply [1; IV.8.10 and IV.8.11] - with the obvious adaptation to a space of functions with values in an arbitrary finite-dimensional Banach space E - and conclude: for each number $\varepsilon > 0$ there exists a number $\delta > 0$ such that $\int_{B} \|g\| dm < \varepsilon$ for all $g \in \underset{\sim}{Q}$ and all measurable sets $B \subset [-1,0]$ with $m(B) < \delta$ (here m denotes Lebesgue measure). It follows that $V(\underset{\sim}{Q})$ is equicontinuous - indeed equi-absolutely continuous. Since $\underset{\sim}{Q}$ is bounded in $E_{\sim 1}$, $V(\underset{\sim}{Q})$ is bounded in $\underset{\sim}{E}$; hence $V(\underset{\sim}{Q})$ is relatively compact in $\underset{\sim}{E}$ (Arzelà-Ascoli Theorem).

6.2. <u>Theorem</u>. <u>Assume that</u> E <u>is finite-dimensional, and let</u> M' <u>be a short memory satisfying</u> (M'). <u>Then the transition operator</u> $U(t_1, t_0)$ <u>is compact for all</u> $t_0 \geqq 0$, $t_1 \geqq t_0 + 1$.

<u>Proof</u>. 1. We show first that $U(a+1,a)$ is compact for each $a \geq 0$. Let $a \geq 0$ be fixed throughout this part of the proof.

Let $v' \in \underset{\sim}{E}$ be given, and define $w' \in \underset{\sim[-1]}{C}(E)$ to agree with a constant on $[-1,a-1]$ with $P'(a)v'$ on $[a-1,a+1]$, and with a constant on $[a+1,\infty[$. Then $\|w'\| = \max_{t \in [a-1,a+1]} \|(P'(a)v')(t)\| \leq$

$\leq \max\{\|v'\|, e^{\sigma'}\|v'\|\} = e^{\sigma'}\|v'\|$, by Theorem 5.3. Therefore

$$\int_{-1}^{0} \|(\Pi_1(a+1)M'_{[a]}P'(a)v')(s)\|ds = \int_{a}^{a+1} \|(M'_{[a]}P'(a)v')(t)\|dt =$$

$$= \int_{a}^{a+1} \|(M'w')(t)\|dt \leq \|M'w'\|_{\underset{\sim}{M}} \leq \|M'_{\underset{\sim}{C}}\|\|w'\| \leq \|M'_{\underset{\sim}{C}}\|e^{\sigma'}\|v'\|.$$

We conclude that $T = \Pi_1(a+1)M'_{[a]}P'(a) : \underset{\sim}{E} \to \underset{\sim 1}{E}$ is a bounded linear mapping. Since E is finite-dimensional, $\underset{\sim 1}{E}$ is weakly complete and T is weakly compact [1; IV.8.6 and VI.7.6].

Let $v' \in \underset{\sim}{E}$ again be given. By Theorem 5.3, $u' = P'(a)v'$ is the solution of $(5.2)_{[a]}$, hence of $(5.3)_{[a]}$, with $r' = 0$ and $\Pi(a)u' = v'$. Using Corollary 5.2, we find

$$(U(a+1,a)v')(s) = (U(a+1,a)\Pi(a)u')(s) = (\Pi(a+1)u')(s) = u'(a+1+s) =$$

$$= u'(a) - \int_{a}^{a+1+s}(M'_{[a]}u')(s')ds' = v'(0) - \int_{-1}^{s}(\Pi_1(a+1)M'_{[a]}P'(a)v')(s')ds' =$$

$$= (Wv')(s) - (VTv')(s), \qquad -1 \leq s \leq 0.$$

Since $v' \in \underset{\sim}{E}$ was arbitrary, we conclude that $U(a+1,a) = W - VT$. Now E is finite-dimensional, and T is weakly compact; therefore W has finite rank, and VT is compact by Lemma 6.1. We conclude that $U(a+1,a)$ is compact.

2. Let now $t_0 \geq 0$ and $t_1 \geq t_0+1$ be given. We have $U(t_1,t_0) = U(t_1,t_1-1)U(t_1-1,t_0)$, and these operators are bounded (Corollaries 5.2 and 5.4); by Part 1 of this proof, $U(t_1,t_1-1)$ is compact; hence $U(t_1,t_0)$ is compact.

<u>Remark 1</u>. A careful perusal of the proofs of Lemma 6.1 and Theorem 6.2 shows that, under the assumptions of the latter, $U(t_1,t_0)$ is

more than compact: the image under $U(t_1, t_0)$ of a bounded set in

$\underset{\sim}{E}$ is not merely equicontinuous, but equi-absolutely continuous.

Remark 2. It is almost obvious that $U(t_1, t_0)$ cannot be compact for any non-trivial E, any short memory, and any t_0, t_1 if $t_0 \leqq t_1 < t_0 + 1$; and that it cannot be compact for any infinite-dimensional E, any short memory, and any t_0, $t_1 = t_0 + 1$. Further, if E in infinite-dimensional, M' = 0, and $t_1 \geqq t_0 + 1$, then $U(t_1, t_0)$ is always equal to the non-compact operator W. Thus Theorem 6.2 is best possible for short memories satisfying (M').

Remark 3. In Theorem 6.2 the condition (M') may be replaced, with obvious amendments in the proof, by the weaker assumption that $\sup \left\{ \int_a^{a+1} \| (M'u')(s) \| ds : u' \in \underset{\sim}{C}_{[-1]}(E), \|u'\| \leqq 1 \right\} < \infty$ for each $a \geqq 0$, or, equivalently, that $M' : \underset{\sim}{K}_{[-1]}(E) \to \underset{\sim}{L}_{[0]}(E)$ is continuous for the obvious Fréchet-space topologies of domain and codomain. However, this case may also be reduced to the case in which (M') does hold: once t_0, t_1 are given, it is enough to apply Theorem 6.2 to the short memory M" defined by requiring that M"u' agree with M'u' on $[0, t_1]$ and with 0 on $[t_1, \infty[$, since this M" satisfies (M').

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