Smoothness and Duality in $L_p(E,\mu)$

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Research Report 73-7

March, 1973

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3/21/73

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Abstract. Let (T,Σ,μ) be a measure space, E a Banach space, and $L_p(E,\mu)$ the Lebesgue-Bochner function spaces, $1 < p < \infty$. It is shown that $L_p(E,\mu)$ is smooth if and only if E is smooth. From this result a Radon-Nikodym theorem for conjugates of smooth Banach spaces is established, and thus a general geometric condition on E sufficient to ensure that $L_p(E,\mu)^* \cong L_q(E^*,\mu)$ for all p, $1 < p < \infty$. Alternate proofs of certain known results concerning the duals of $L_p(E,\mu)$ spaces are provided.

After the manuscript was finalized, Diestel and Faires mailed a copy of their paper, "Vector Measures". Theorem 4.3 in their paper provides a sufficient condition in order that a conjugate Banach space may possess the R. N. property. However their theorem is a corollary of Theorem 4.2 presented in this paper.

1. Introduction.

The problem of concretely representing the dual of the Lebesgue-Bochner function spaces $L_p(E,|JO,$ $1 < p < oo$, has been discussed by several authors, Bochner and Taylor [2], E>aY [5], Dieudonné [7], Dinculeanu [8], Fortet and Mourier [10], Gretsky and Uhl [11], and Mourier [14]. In spite of the fact that a representation of the dual as a certain Banach space of E -valued measures is known, Dinculeanu [8], the fundamental problem, obtaining necessary and sufficient geometric conditions on E which ensure that L _(E,MO is congruent to L _(E ,ji) under the canonical mapping, is still open for arbitrary measure spaces (T,2,|Jb). The purpose of the present paper is to provide a sufficient condition on E so that

$$\text{(a)} \qquad L_p(E,MO^* \cong L_g(E^*,n)$$

for any arbitrary measure |i.

The result appears to be all the more interesting in that this sufficient condition can be described in terms of a purely geometric property of E. The theorems in [5], [7], [8], [10], [11], and [14], providing conditions under which the assertion (a) is true, can be deduced as corollaries of the representation theorem obtained here, as is shown in the last section of this note.

2. Definitions and Notation.

In the following, E denotes a real Banach space and E^* the dual of E. The unit ball of E is U = {XGE| ||X|| £ 1} and its boundary $S = \{x \in E \mid \|x\| = 1\}$ is the unit sphere of E. The unit ball and unit sphere of E^* are defined analogously, and are denoted by U^* and S^*, respectively.

A Banach space E is said to be smooth at $x \in S$ if and only if there exists a unique hyperplane of support at x, that is, there exists only one continuous linear functional $G_x \in E^*$ with $\|G_x\| = 1$ such that $G_x(x) = 1$. Such a linear functional $G_x \in E^*$ is called the support functional of U at x, and $G_x^{-1}(\{1\})$ is called the hyperplane of support of U at x. A Banach space E is said to be a smooth Banach space if it is smooth at every $x \in S$.

The norm $\|\cdot\| : E \longrightarrow \mathbb{R}^+$ is said to be Gâteaux differentiable at $x \in E \sim \{0\}$ if and only if there exists a functional $G_x \in E^*$, with

$$\lim_{t \to 0} \left| \frac{\|x+th\| - \|x\|}{t} - G_x(h) \right| = 0$$

for every $h \in E$. The functional G_x is called the Gâteaux derivative of the norm at $x \in E$. The norm $\|\cdot\| : E \longrightarrow \mathbb{R}^+$ is said to be differentiable (Fréchet differentiable) at $x \in E \sim \{0\}$ if and only if there exists a functional $G_x \in E^*$ such that

$$\lim_{\|h\| \to 0} \frac{\left| \|x+h\| - \|x\| - G_x(h) \right|}{\|h\|} = 0.$$

The norm $\|\cdot\| : E \longrightarrow \mathbb{R}^+$ is said to be of class C^1 or continuously differentiable if and only if the mapping $G : E \sim \{0\} \longrightarrow E^* \sim \{0\}$, given by $G(x) = G_x$, is continuous.

Remark: It should be mentioned that the norm in E is:

(i) Gâteaux differentiable at $x \in E$ if and only if

$$\lim_{t \to 0} \frac{\|x+th\| - \|x\|}{t} = G_x(h)$$

exists for all $h \in E$.

(ii) Smooth at $x \in E$ if and only if it is Gâteaux differentiable at $x \in E$.

(iii) Fréchet differentiable if and only if it is of class C^1.

Statement (i) is Mazur's theorem, and can be found in Mazur [13], while (ii) and (iii) can be found in Day [6].

If E is a smooth Banach space, then the mapping $\nu : E \longrightarrow E^*$, defined by

$$\nu(x) = \begin{cases} \|x\| G_x, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

for each $x \in E$, is called the extended spherical image map. From the results in Cudia [4], it follows that if E is a smooth Banach space, then the extended spherical image map $\nu : (E, \|\cdot\|) \longrightarrow (E^*, \omega^*)$ is continuous. If the norm $\|\cdot\| : E \longrightarrow \mathbb{R}^+$ is Fréchet differentiable, then the extended spherical image map $\nu : (E, \|\cdot\|) \longrightarrow (E^*, \|\cdot\|)$ is continuous.

Let E be a real Banach space and (T, Σ, μ) a measure space, that is, T is a non-empty set, Σ is a σ-algebra of subsets of T, and $\mu : \Sigma \longrightarrow \overline{\mathbb{R}}^+$ is a countably additive measure. (Here μ is assumed to be non-trivial, that is, μ is not supported by finitely many atoms and the range of μ contains at least one non-zero real number.) If f : T—v E, then f is said to be ^-measurable if $f\sim^{1}$(G)eZ for every open set G c E, and there exists a set Ne2, with p<(N) = 0, and a countable set H c E, such that f (T ~ N) c $\overline{H}$. if 1 £ p £ oo , the Lebesque-Bochner function spaces L_P (E,M<) are defined as follows:

L_P (E,M») = {f|f : T—* E is measurable, and J_T ||f (t) $||^P$dp,(t) < CD }

for 1 <£ p < oo , and

L_{00} (E,|i) = {f|f : T—vE is measurable, and ess $\sup_{teT}$||f (t) || < ao)

(as usual, identifying functions which agree p,-a.e.). When T is the set of positive integers and |i is the counting measure, L_p(E,n.) is usually denoted by I_p(E) .

Let m : 2 —* E be a countably additive measure, and 1 <£ q < oo , if AeS, then the Q-variation of m is defined by

$$\overline{m}_q(A) = \sup(\wedge \frac{n\ HmfA\wedge \ ||^q}{\mu(A_i)q-1})^{1/q}$$

where the supremum is taken over all finite families $\{A_i\}?_{i=1}$ of disjoint sets from 2 contained in A.

The dual of L_P(E,M<), 1 < p < oo , is isometrically isomorphic to the Banach space of all countably additive measures m : 2 — V E^* for which $\overline{m}_q$(T) < oo . ([8, p. 261, Corollary 1].)

It is known, [9] that for a a-finite measure |i, if E is reflexive or if E is separable, then L_p(E,|jb) is isometrically isomorphic to $L_q(E^*,\mu)$ for $1 < p < \infty$. The correspondence is given as follows: For each $F \in L_p(E,\mu)^*$, there exists a unique $g \in L_q(E^*,\mu)$ such that

$$F(f) = \int_T \langle g(t), f(t)\rangle d\mu(t)$$

for all $f \in L_p(E,\mu)$ and such that

$$\|F\| = \|g\| = \left(\int_T \|g(t)\|^q d\mu(t)\right)^{1/q}.$$

The fact that this result does not hold in general can be seen from the example in [11], where it is shown that $L_2(\ell_1)$ is not isometrically isomorphic to $L_2(\ell_\infty)$.

3. <u>Smoothness of</u> $L_p(E,\mu)$.

<u>Theorem 3.1</u>. Let (T,Σ,μ) be a measure space, and let E be a real Banach space. If $1 < p < \infty$, then $L_p(E,\mu)$ is smooth if and only if E is smooth.

<u>Proof</u>: Let $f,h \in L_p(E,\mu)$, with

$$\|f\| = \left(\int_T \|f(t)\|^p d\mu(t)\right)^{1/p} = 1.$$

In order to show that $L_p(E,\mu)$ is smooth at f, it suffices (by Mazur's theorem) to show that

$$\lim_{\lambda \to 0} \frac{\|f+\lambda h\| - \|f\|}{\lambda}$$

exists for all $h \in L_p(E,\mu)$; and in order to show this, it suffices (by the chain rule) to show that

$$\lim_{\lambda \to 0} \frac{\|f+\lambda h\|^p - \|f\|^p}{\lambda} = \lim_{\lambda \to 0} \frac{1}{\lambda} \int_T (\|f(t)+\lambda h(t)\|^p - \|f(t)\|^p)\, d\mu(t)$$

exists for all $h \in L_p(E,\mu)$. Let $T_0 = \{t \in T | f(t) = 0\}$, since $p > 1$, it follows that

$$\lim_{\lambda \to 0} \frac{1}{\lambda} \int_{T_0} (\|f(t)+\lambda h(t)\|^p - \|f(t)\|^p)\, d\mu(t)$$

$$= \left(\int_{T_0} \|h(t)\|^p d\mu(t)\right) \lim_{\lambda \to 0} (\operatorname{sgn} \lambda)\, |\lambda|^{p-1} = 0 \qquad (*)$$

for all $h \in L_p(E,\mu)$.

If $t \in T \sim T_0$, since E is smooth at $f(t)$, then

$$\lim_{\lambda \to 0} \frac{\|f(t)+\lambda h(t)\|^p - \|f(t)\|^p}{\lambda} = p\|f(t)\|^{p-1} G_{f(t)}(h(t)),$$

where $G_x \in E^*$ is the Gâteaux derivative of the norm $\|\cdot\| : E \to \mathbb{R}^+$ at $x \neq 0$. Define $\psi_t : \mathbb{R}^+ \to \mathbb{R}$ by

$$\psi_t(\lambda) = \|f(t) + \lambda h(t)\|^p$$

for all $\lambda \in \mathbb{R}^+$. Then

$$\psi_t'(\lambda) = p\|f(t) + \lambda h(t)\|^{p-1} G_{f(t)+\lambda h(t)}(h(t)),$$

and from the mean-value theorem, Cartan [3],

$$|\psi_t(\lambda) - \psi_t(0)| \le |\lambda| \sup_{0 \le \varphi \le \lambda} |\psi_t'(\varphi)|.$$

Thus,

$$\|f(t)+\lambda h(t)\|^p - \|f(t)\|^p \leq |\lambda| \sup_{0\leq\varphi\leq\lambda} \{p\|f(t)+\varphi h(t)\|^{p-1}|G_{f(t)+\varphi h(t)}(h(t))|\}$$

$$\leq |\lambda| \sup_{0\leq\varphi\leq\lambda} \{p\|f(t)+\varphi h(t)\|^{p-1}\|h(t)\|\}$$

$$\leq p|\lambda| \|h(t)\|(\|f(t)\| + |\lambda| \|h(t)\|)^{p-1}.$$

Therefore, for $0 < \lambda \leq 1$,

$$\left|\frac{\|f(t)+\lambda h(t)\|^p - \|f(t)\|^p}{\lambda}\right| \leq p(\|f(t)\| + \|h(t)\|)^{p-1}\|h(t)\|$$

for all $t \in T \sim T_o$. The mappings $t \longmapsto \|f(t)\|$ and $t \longmapsto \|h(t)\|$ are in $L_p(\mathbb{R},\mu)$, and hence the mapping $t \longmapsto (\|f(t)\| + \|h(t)\|)^{p-1}\|h(t)\|$ is in $L_1(\mathbb{R},\mu)$. From Lebesgue's dominated convergence theorem,

$$\lim_{\lambda\to 0^+} \int_{T\sim T_o} \frac{\|f(t)+\lambda h(t)\|^p - \|f(t)\|^p}{\lambda}\, d\mu(t)$$

$$= \int_{T\sim T_o} \lim_{\lambda\to 0^+}\left(\frac{\|f(t)+\lambda h(t)\|^p - \|f(t)\|^p}{\lambda}\right) d\mu(t),$$

and therefore,

$$\lim_{\lambda\to 0^+} \int_{T\sim T_o} \frac{\|f(t)+\lambda h(t)\|^p - \|f(t)\|^p}{\lambda}\, d\mu(t)$$

$$= p\int_{T\sim T_o} \|f(t)\|^{p-1} G_{f(t)}(h(t))\, d\mu(t). \qquad (**)$$

A similar argument shows that

$$\lim_{\lambda \to 0^-} \int_{T \sim T_o} \frac{\|f(t)+\lambda h(t)\|^p - \|f(t)\|^p}{\lambda} \, d\mu(t)$$

$$= p \int_{T \sim T_o} \|f(t)\|^{p-1} G_{f(t)}(h(t)) \, d\mu(t). \qquad (***)$$

Combining (*), (**), and (***),

$$\lim_{\lambda \to 0} \frac{\|f+\lambda h\|^p - \|f\|^p}{\lambda} = \lim_{\lambda \to 0} \frac{1}{\lambda} \cdot \int_{T_o} (\|f(t)+\lambda h(t)\|^p - \|f(t)\|^p) \, d\mu(t)$$

$$+ \lim_{\lambda \to 0} \frac{1}{\lambda} \cdot \int_{T \sim T_o} (\|f(t)+\lambda h(t)\|^p - \|f(t)\|^p) \, d\mu(t)$$

$$= p \int_{T \sim T_o} \|f(t)\|^{p-1} G_{f(t)}(h(t)) \, d\mu(t),$$

exists for all $h \in L_p(E,\mu)$. Therefore, Mazur's theorem implies that $L_p(E,\mu)$ is smooth at f.

Conversely, if $L_p(E,\mu)$ is smooth, since E is isometrically isomorphic to a closed subspace of $L_p(E,\mu)$, then E is smooth.

Q.E.D.

4. Representation Theorem.

Suppose that E is a smooth Banach space and $1 < p < \infty$. Let $f \in L_p(E,\mu)$, with $\|f\| = (\int_T \|f(t)\|^p d\mu(t))^{1/p} = 1$, and define $\hat{G}_f : L_p(E,\mu) \longrightarrow \mathbb{R}$ by

$$\hat{G}_f(h) = \int_{T \sim T_o} \|f(t)\|^{p-1} G_{f(t)}(h(t)) \, d\mu(t)$$

for each $h \in L_p(E,\mu)$, where $T_o = \{t \in T \mid f(t) = 0\}$. Then,

$$v(x) = \begin{cases} \frac{\|x\| \, G_x}{\|x\|}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is continuous, then the mapping $\beta : T \to E^*$ given by

$$\beta(t) = \begin{cases} \|f(t)\|^{p-1} G_{f(t)}, & t \in T \sim T_o \\ & t \in T_o \end{cases}$$

is μ-measurable.

Suppose now that the mapping $\beta : T \to E^*$ given above is μ-measurable, then

$$\int_T \|\beta(t)\|^q d\mu(t) = \int_{T \sim T_o} \|f(t)\|^{q(p-1)} \|G_{f(t)}\|^q d\mu(t)$$

$$= \int_{T \sim T_o} \|f(t)\|^p d\mu(t) = \int_T \|f(t)\|^p d\mu(t) = \|f\|^p < \infty,$$

that is, $\beta \in L_q(E^*, \mu)$. Therefore, for each $f \in L_p(E, \mu)$, $\|f\| = 1$,

$$\hat{G}_f(h) = \int_T \langle \beta(t), h(t) \rangle d\mu(t)$$

for all $h \in L_p(E, \mu)$, where $\beta \in L_q(E^*, \mu)$. Hence, all the support functionals in $L_p(E, \mu)$ belong to the closed subspace $M \subset (L_p(E, \mu))^*$ which is isometrically isomorphic to $L_q(E^*, \mu)$.

Now let A be the set of all continuous linear functionals on $L_p(E, \mu)$ which attain their supremum on the unit sphere of $L_p(E, \mu)$; so that

$$\Lambda \subset M \subset L_p(E,\mu)^*.$$

Since Λ is norm dense in $L_p(E,\mu)^*$, Bishop and Phelps [1], and since M is a closed subspace of $L_p(E,\mu)^*$, $M = L_p(E,\mu)^*$. Therefore, $L_q(E^*,\mu)$ is isometrically isomorphic to $L_p(E,\mu)^*$, and the linear isometry $\alpha : L_p(E,\mu)^* \to L_q(E^*,\mu)$ is given as follows: For each $F \in L_p(E,\mu)^*$, $\alpha(F) = g$, where g is the unique element of $L_q(E^*,\mu)$ such that

$$F(f) = \int_T \langle g(t), f(t)\rangle \, d\mu(t)$$

for all $f \in L_p(E,\mu)$ and $\|F\| = \|g\| = \left(\int_T \|g(t)\|^q \, d\mu(t)\right)^{1/q}$. Hence the following theorems are true:

<u>Theorem 4.1</u>. Let E be a smooth Banach space, (T,Σ,μ) a measure space, and let $1 < p < \infty$. If, for each $f \in L_p(E,\mu)$, $\|f\| = 1$, the mapping $\beta : T \to E^*$ given by

$$\beta(t) = \begin{cases} \|f(t)\|^{p-1} G_{f(t)}, & f(t) \neq 0 \\ 0, & f(t) = 0 \end{cases}$$

is μ-measurable, then $L_p(E,\mu)^* = L_q(E^*,\mu)$.

<u>Theorem 4.2</u>. Let E be a smooth Banach space, (T,Σ,μ) a measure space, and let $1 < p < \infty$. If, for each $f \in L_p(E,\mu)$, $\|f\| = 1$, the mapping $\beta : T \to E^*$ given by

$$\beta(t) = \begin{cases} \|f(t)\|^{p-1} G_{f(t)}, & f(t) \neq 0 \\ 0, & f(t) = 0 \end{cases}$$

is μ-measurable, and F is a Banach space which is isomorphic to E, then $L_p(F,\mu)^* \cong L_q(F^*,\mu)$, and hence F has the Radon-Nikodym property [11].

In the case when the norm in E is Fréchet differentiable, the extended spherical image map $\nu : (E,\|\cdot\|) \longrightarrow (E^*,\|\cdot\|)$ given by

$$\nu(x) = \begin{cases} \|x\|G_x, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is continuous. Thus, $\beta : T \longrightarrow E^*$ as defined above, being the composition of a μ-measurable function and a continuous function, is μ-measurable. In this case then, the following theorem is true:

Theorem 4.3. Let E be a real Banach space, (T,Σ,μ) a measure space, and $1 < p < \infty$. If the norm $\|\cdot\| : E \longrightarrow \mathbb{R}^+$ is Fréchet differentiable, then $L_p(E,\mu)^* \cong L_q(E^*,\mu)$.

The following two corollaries are immediate consequences of Theorem 4.3:

Corollary 4.4. Let E be a reflexive Banach space and (T,Σ,μ) an arbitrary measure space, then $L_p(E,\mu)^* \cong L_q(E^*,\mu)$ for all p, $1 < p < \infty$.

Proof: Since every reflexive Banach space has an equivalent Frechet differentiable norm (Troyanski [16]), the result follows immediately from Theorem 4.2 and Theorem 4.3. Q.E.D.

Corollary 4.5. Let E be a Banach space such that E^* is separable, and let (T,Σ,μ) be an arbitrary measure space, then

$L_p(E,\mu)^* \cong L_q(E^*,\mu)$ for all p, $1 < p < \infty$.

Proof: Since E is separable, then E has an equivalent Frechet differentiable norm (Restrepo [15]), and again, the result follows immediately from Theorem 4.2 and Theorem 4.3. Q.E.D.

In conclusion, it might be noted that if E is isomorphic to a Banach space whose norm is Frechet differentiable, Theorem 4.3 implies that E^* has the Radon-Nikodym property [11].

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$$|\hat{G}_f(h)| \leq \int_{T\sim T_o} \|f(t)\|^{p-1}\|h(t)\|d\mu(t)$$

$$\leq \left(\int_T \|f(t)\|^p d\mu(t)\right)^{1/q}\left(\int_T \|h(t)\|^p d\mu(t)\right)^{1/p},$$

that is,

$$|\hat{G}_f(h)| \leq \|f\|^{p/q}\|h\|$$

for all $h \in L_p(E,\mu)$. Hence $\|\hat{G}_f\| \leq \|f\|^{p/q} = \|f\|^{p-1} = 1$. Also,

$$\hat{G}_f(f) = \int_{T\sim T_o} \|f(t)\|^{p-1} G_{f(t)}(f(t))d\mu(t) = \int_{T\sim T_o} \|f(t)\|^p d\mu(t),$$

so that $\hat{G}_f(f) = \|f\|^p = 1$. Thus, $\|\hat{G}_f\| = 1$ and $\hat{G}_f(f) = 1$, and $\hat{G}_f \in L_p(E,\mu)^*$ is the support functional at $f \in L_p(E,\mu)$, $\|f\| = 1$; and

$$\hat{G}_f(h) = \int_{T\sim T_o} \|f(t)\|^{p-1} G_{f(t)}(h(t))d\mu(t)$$

for all $h \in L_p(E,\mu)$.

Define the mapping $\gamma : T \longrightarrow \mathbb{R}$ by

$$\gamma(t) = \begin{cases} \|f(t)\|^{p-1} G_{f(t)}(h(t)), & t \in T \sim T_o \\ 0, & t \in T_o. \end{cases}$$

Then γ is a measurable function, being the limit of a sequence of measurable functions. Also, since the mapping

$$\nu : (E, \|\cdot\|) \longrightarrow (E^*, \omega^*)$$

given by