A Note on Infinite Systems of
Linear Inequalities in $R^n$

by

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ABSTRACT

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A necessary and sufficient condition for infinite systems of linear inequalities to have solutions is given, using the Kuhn-Fourier Theorem, which deals with finite systems.

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Consider a system of inequalities A given by:

$$a_i \cdot x \geq b_i, \quad i \in I$$

where $a_i$, $x \in R^n$ and $b_i \in R$, and I is arbitrary. We shall say that an inequality $c \cdot x \geq d$ can be <u>linearly deduced</u> (abbreviated: <u>deduced</u>) from A iff there is a finite subset $J = \{j_1, \ldots, j_k\}$ of I and non-negative reals $m_1, \ldots, m_k$ such that

(i) $\sum_{i \in J} m_i a_i = c$ and

(ii) $\sum_{i \in J} m_i b_i \geq d$.

(Compare with the notion of legal linear combination in [2], p. 8.)

Clearly, if x satisfies the system A, x must satisfy any inequality deduced from A. Notice that if B is a set of inequalities, each of which can be deduced from A, then any inequality deducible from B is also deducible from A. (See [2], Chapter 1.)

For I finite the Kuhn-Fourier Theorem [2, p. 10] gives a useful necessary and sufficient condition for the system A to have a solution: A has no solution iff the inequality $0 \cdot x \geq d$ can be deduced from A for some positive d.

In this note, we prove the following result for infinite systems.

<u>Theorem</u>: Suppose that every finite subsystem of **A** has a solution.

Then **A** has no solution iff there is a $v \in R^n$ such that $v \cdot x \geq N$ can be deduced from A for every N.

<u>Proof</u>: The "if" part is immediate: If there were an x satisfying A, then x would satisfy $v \cdot x \geq N$ for every N, which is impossible.

To prove the "only if" part we first show:

<u>Lemma</u>: If A has no solution but every finite subsystem of A has a solution, then there is a sequence of $w_i \in R^n$ with $|w_i| = 1$ such that, for all i, $w_i \cdot x \geq i$ can be deduced from A.

<u>Proof of Lemma</u>: Let $e_i$ be the vector in $R^n$ whose ith coordinate is 1 with all other coordinates 0. If, for some N, every finite subsystem of A had a solution x such that $|e_i \cdot x| \leq N$ $(i = 1,\ldots,n)$ then A itself would have a solution. (For $j \in I$, let $C_j$ be the set of x whose coordinates are between -N and N and $a_j \cdot x \geq b_j$. The $C_j$ are compact and, by assumption, every finite set of $C_j$ has nonempty intersection, so $\bigcap_{j \in I} C_j \neq \emptyset$.)

Therefore we know that, for every N, there is a finite subsystem $F \subset A$, $F = \{a_i \cdot x \geq b_i | i \in J\}$, J finite, such that the system $F \cup \{e_i \cdot x \geq -N | i = 1,\ldots,n\} \cup \{-e_i \cdot x \geq -N | i = 1,\ldots,n\}$ has no solution. By the Kuhn-Fourier Theorem, there are non-negative $m_i,\ldots,m_k$, $r_i,\ldots,r_n$, $s_i,\ldots,s_n$ such that

$$\text{(iii)} \quad \sum_{i \in J} m_i a_i + \sum r_i e_i + \sum s_i(-e_i) = 0 \quad \text{and}$$

(iv) $\sum_{i\in J} m_i b_i + \sum r_i(-N) + \sum s_i(-N) > 0.$

Let $u_N = \sum_{i\in J} m_i a_i = \sum (s_i - r_i)e_i$. We can use the $m_i$ to deduce from F (hence, from A)

$$u_N \cdot x = (\sum_{i\in J} m_i a_i)\cdot x \geq \sum_{i\in J} m_i b_i > N(\Sigma(r_i + s_i)) \geq N|u_N|.$$

We now obtain the sequence needed by setting $w_N = \frac{u_N}{|u_N|}$. Q.E.D.

Next we obtain a $v$ such that $v\cdot x \geq N$ can be deduced, for all $N$, from the inequalities $\{w_i \cdot x \geq i | i = 1,2,\ldots\}$ given by the Lemma. Since each of the w-inequalities can be deduced from A, this will complete the proof. Since the $w_i \in R^n$, $\{w_i | i = 1,2,3,\ldots\}$ contains a finite linearly independent set $\{x_1,\ldots,x_t\}$ such that every $w_i$ can be written as a linear combination of the $x_i$. It is a trivial exercise in linear algebra to show that there exists a bound $R > 0$, such that for every vector

$$w = \sum_{i=1}^{t} \alpha_i x_i$$

in the span of the $\{x_i,\ldots,x_t\}$ with $|w| = 1$, we have $|\alpha_i| < R$ for $i = 1,\ldots,t$.

Now let $v = \sum_{i=1}^{t} x_i$. For any $w_N$, we have $w_N = \sum_{i=1}^{t} \alpha_i x_i$, where $|\alpha_i| < R$, (with the $\alpha_i$ depending on $N$). But

$$v\cdot x = \frac{1}{R} w_N \cdot x + (\sum_{i=1}^{t} (1 - \alpha_i/R) x_i)\cdot x \geq \frac{N}{R}$$

can be deduced from A.

The Kuhn-Fourier Theorem is valid in any ordered field, but our theorem is not (recall our use of compactness arguments in the preceding proof). The following infinite system of linear inequalities in Q (= the rationals) provides a counterexample to our theorem for the rational field:

$$\text{(v)} \qquad x \geq q, \quad \text{for each } q < \sqrt{2}$$
$$-x \geq q, \quad \text{for each } q < -\sqrt{2}\,.$$

In fact, if our theorem held for Q, the vector $v \in Q$ would exist with the properties stated, and (v) would not be solvable in R, by the argument for the trivial direction of our theorem. However, $x = \sqrt{2}$ solves (v) in R.

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## References

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