TOPOLOGIES ASSOCIATED WITH OPERATORS ON NORMED FUNCTION SPACES

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Report 73-3

February 1973

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Abstract

Under consideration is the following question. If T is a linear operator from a normed function space 3 into a Banach space X, then may an appropriate topology 3 be defined so that conditions on T may be interpreted in terms of the topology 3? For the present we consider 3 to be the class of Banach function spaces. We define appropriate "semi-norm topologies" on the unit ball $x^!_1$ of the dual of X. Conditions on the topologies give information concerning both the underlying measure and the associated operator. On the other hand, for example, compactness of T is related to the compactness of 3. Indications are given for the study of more general structures in the topological setting.

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1. Introduction

Of general interest is the following situation. If T is a linear operator from a normed function space $\mathfrak{F}$ into a Banach space X, then may a topology $\mathfrak{I}$ be naturally associated with T so that conditions on T might be related to topological considerations of $\mathfrak{I}$? For example, may one relate the compactness of the operator T with the topological compactness of $\mathfrak{I}$. The general theory is developed via some (integral) representation of T in terms of some measure τ. Thus extending the situation further, the topologist may inquire as to what kind of topology $\mathfrak{I}$ will the measures associated with a particular class of operators produce. On the other hand topological properties of $\mathfrak{I}$ may give rise to important classes of associated operators. What is the significance of these classes?

Numerous papers have appeared dealing with this idea. One of the most recent (see [5]) considered the function space $\mathfrak{F}$ to be the collection $C_0(\Omega, E)$ of continuous E-valued functions defined on a locally compact space Ω and vanishing at infinity where E is a Banach space. Another candidate for $\mathfrak{F}$ was the collection $M_E(\Sigma)$ of E-valued functions totally measurable relative to the ring Σ. The main result there was that if E is reflexive then T is weakly compact if and only if τ is strongly bounded. Since the operators under consideration have some representation via a measure τ, we wish to enquire further if this measure may be used to define semi-norms, which in turn define topologies, so that the previous mentioned result may be further illuminated through topology.

An immediate generalization of the above classes of spaces are the Lebesgue spaces $\mathcal{L}^p(\mu)$ for $1 \leq p \leq \infty$. In [3], it is shown that a continuous linear operator T from $\mathcal{L}^p(\mu)$ into a Banach space X with T absolutely continuous with respect to μ, is compact if and only if an appropriate topology on the unit ball X_1^* of the dual space X^* is compact. Moreover compactness of T was shown to make τ countably additive. However a more general setting for these results and questions is appropriate.

Shortly, we will formulate these questions in a more precise and succinct way. However for the time being we direct our attention to the particular abstract function spaces that we wish to presently consider. We will then consider the interpretation of our results to some particular important subclasses and at the end we will consider what may be studied in a more general setting.

About twenty years ago the first papers on Banach function spaces or normed Kothe spaces (or even sometimes called spaces with a length function) were first published. In general these are Banach spaces of either scalar or vector-valued functions defined on a point set Ω and measurable with respect to a given measure μ on Ω (see [10] and [16]). As an abstract class of spaces they include (and are natural generalizations of) the Lebesgue spaces $\mathcal{L}^p$, $1 \leq p \leq \infty$, and the well-known (but possibly less familiar) Orlicz spaces, which themselves are generalizations of the L_p-spaces. On the other side every normed function space is a normed Riesz space (that is, a normed vector lattice). Consequently as one might expect the more abstract theories of either Riesz spaces, ordered topological linear spaces, or normed linear spaces will serve as cornerstones for the present developments.[1] Of course this is not meant to exclude the influence that the theory of measure and integration has on these function spaces or will have on the topologies to be considered. Our interest in this work will be essentially the topological aspects of the theory and the influence that the other theories will have on these topologies. For example, the interaction of measure theory and topology. By this we hope to engage the interested in many of the significant questions that arise and to begin the study of the more abstract setting, again from the topological point of view with the assistance of the more analytical details and concrete realizations that have been established. For the present, we must develop the necessary structural and analytical aspects of Banach function spaces.

2. Basic Definitions and Results

Let μ be a non-negative countably additive measure defined on a σ-field Σ of subsets of the non-empty point set Ω. It is assumed that the Caratheodory extension procedure has already been applied to μ, so that Σ cannot be enlarged by another application of this procedure. It is also assumed that μ is σ-finite, that is, the set Ω is the union of at most a countable number of sets with finite μ-measure. Thus (Ω, Σ, μ) is said to be a σ-finite measure space as is normally understood. For convenience, we will always deal with the equivalence classes of functions on Ω modulo functions of μ-measure zero rather than the individual functions. In a similar vein, subsets E of Ω whose characteristic function χ_E differ only on a set of μ-measure zero will also be identified.

Let M^+ be the collection of all non-negative measurable functions on Ω (equipped with pointwise order). As usual a function $f \in M^+$ may assume $+\infty$ at some (or even at all) points of Ω. A mapping ρ from M^+ into the extended real number system is called a function semi-norm if ρ satisfies the following properties:

(i) if $f \in M^+$ then $0 \le \rho(f) \le \infty$ and $\rho(f) = 0$ if $f = 0$ (almost everywhere);

(ii) $\rho(af) = a\rho(f)$ for all non-negative finite constants a and for all $f \in M^+$;

(iii) $\rho(f+g) \le \rho(f) + \rho(g)$ for all $f, g \in M^+$;

(iv) if $f, g \in M^+$, $f \le g$ then $\rho(f) \le \rho(g)$.

If ρ satisfies also

(v) $\rho(f) = 0$ only if $f = 0$ (almost everywhere)

then ρ is called a <u>function norm</u>.[3] The domain of the function semi-norm ρ may be extended to the collection M of all μ-measurable extended complex valued functions[4] on Ω by defining $\rho(f) = \rho(|f|)$ for any $f \in M$. We will assume that there is an $f \in M$ such that $\rho(f) < \infty$.

The function space $L_\rho = L_\rho(\Omega, \Sigma, \mu) = \{f \in M : \rho(f) < \infty\}$ is a normed linear space of (equivalence classes of) measurable scalar-valued functions on the σ-finite measure space (Ω, Σ, μ) with function norm ρ defined on M^+ (and therefore M) and norm $\|f\| = \rho(|f|)$ for all $f \in M$.[5] Such spaces are called <u>normed Kothe spaces</u>.[6]

In general the function space L_ρ is not complete. However, conditions on ρ to insure the completeness of L_ρ are well-known.[7] The elimination of completeness does not bring that much additional insight. So we will assume throughout that L_ρ is complete and in such case the spaces L_ρ have been referred to as <u>Banach function spaces</u> or <u>complete normed Kothe spaces</u>.[8]

We will also need to make use of another function norm ρ' defined for all measurable f by

$$\rho'(f) = \sup\{\int |fg| d\mu : \rho(g) \leq 1\}.$$

Throughout our work $\int d\mu$ will denote integration (with respect to μ) over the whole set Ω, otherwise over a subset $A \in \Sigma$, $\int_A d\mu$ will be used. It follows that ρ' is a function norm[9] with the (sequential) Fatou property (even if ρ doesn't have it). Consequently it is called the <u>associate norm</u> of ρ and the corresponding Banach function space[10] $L_{\rho'} = \{f \in M : \rho'(f) < \infty\}$ is called the <u>associate space</u> of L_ρ.

The following terminology and notation is necessary for developing the analytical details. Some results from the theory of vector measures is needed to obtain the desired topological theory. The reader is referred to the text [7] by N. Dinculeanu which carefully and systematically develops this theory. The book [27] by A. C. Zaanen has a comprehensive account of Banach function spaces. Either of these texts may be referenced for terminology not herein defined.

We will assume that Ω is now a topological space. Let Σ_0 and Σ_0', respectively, be all sets $A \in \Sigma$ for which $\rho(\chi_A) < \infty$ and $\rho'(\chi_A) < \infty$, respectively. Clearly Σ_0 and Σ_0' are rings (clans, in the sense of [7]) and algebras of sets if Ω belongs to them. Let M^0 be the closed subspace of L_ρ which is the closure of the <u>span</u> of those bounded functions in L_ρ whose <u>support</u> (that is, the smallest closed subset, supp f, contained in Ω such that $x \notin \text{supp } f$ implies that $f(x) = 0$) lies in Σ_0. It is clear that $M^0 = \{f \in L_\rho : f$ is a Σ_0-step function$\}$. To appropriately talk about step functions we need the concept of a partition in Σ_0. A <u>partition</u> in Σ_0 is a finite pairwise disjoint sub-collection of Σ_0 of non μ-null members which are of finite measure.

The letters X,Y,... will designate Banach spaces. For a Banach space X, we designate its <u>unit ball</u> by X_1. The dual of X (that is, all bounded linear functionals on X) will be designated by x^*.

Throughout γ will designate a (finitely) additive set function from Σ_0 into X. Let the norm $n_{\rho'}$ be given by

$$n_{\rho'}(\gamma) = \sup\{\rho'(x^*\gamma) : x^* \in X_1^*\}.$$

To eliminate unnecessary confusion, we will write $x^*\gamma$ when composition of two functions is required. By $\rho_f(\mu)$ we mean all (finitely) additive set functions γ mapping Σ_0 into X such that $n_{\rho_f}(\gamma) < \infty$ and which vanish on sets of μ-measure zero. By $P_{\rho'}(\mu)$ we mean all additive set functions γ from Σ into X that vanish on μ-null sets, whose support is contained in that of an element of $L_P \setminus M^\rho$ and such that for each $x^* \in X^*$, $x^*\gamma$ is purely finitely additive. Recall that a positive finitely additive set function λ defined on a ring $\mathcal{R}$ for which $\lambda(R) < \infty$ for $R \in \mathcal{R}$ is said to be purely finitely additive on $\mathcal{R}$ whenever any countably additive set function u on $\mathcal{R}$ satisfying $0 \leq u(R) \leq \lambda(R)$ for all $R \in \mathcal{R}$ is identically zero.[23] If $|x^*\gamma|$ is the variation of $x^*\gamma$ then $x^*\gamma$ purely finitely additive means that if $0 \leq u \leq |x^*\gamma|$ where u is countably additive then $u \equiv 0$. For $\gamma \in P_\rho(\mu)$,

$$\|\gamma\|(\Omega) = \sup\{|x^*\gamma|(\Omega) : x^* \in X_1^*\},$$

that is, γ is of finite weak semi-variation in the sense of [7].

Let us set

$$V(\mu) = V_{\rho_f}(\mu) \oplus P_{\rho'}(\mu).$$

That is, if $y \in n_{\rho'}(\mu)$ then y is uniquely decomposed as $y_1 + y_2$, $y_1 \in V(M)$, $y_2 \in p_{\rho'}(\mu)$ and

$$\|y\|_{\rho'} = \sup\{\rho'(x^* G_1) + |x^* G_2|(\Omega) : x^* \in X^*\}.$$

The following theorem, found in [23], gives the significance of the sum $U_{\rho'}(\mu)$. For Banach spaces X and Y we let L(X,Y) be the set of all bounded linear operators from X into Y.

THEOREM 2.1. If $T \in L(L_\rho, X)$ then there is a unique $T \in U_{\rho'}(\mu)$ such that

$$T(f) = \int f dT$$

and

$$\|T\| = \|T\|_{\rho'}$$

where the integral is similar to that in [9] (see IV. 10 of [9]).

Thus this theorem states that $L(L_\rho, X)$ is isomorphic -- isometric to $u_{\rho'}(\mu)$. Furthermore, let us assume that $T \in \mu_{\rho'}(\mu)$ is the unique correspondent for $T \in L(L_\rho, X)$. If T is decomposed into $T_1 + T_2$ and ρ is continuous at zero then $T_2 = 0$ and T_1 is both μ-continuous and countably additive on Σ (see [23], 3.2).

The function norm ρ is said to be continuous at zero if for every $\epsilon > 0$ there is a $\delta > 0$ such that $\mu(E) < \delta$ implies $\rho(X_E) < \epsilon$.

In the following all integrals pertaining to (finitely) additive measures will be understood to be in the sense of [9] (namely, IV.10). Consequently we will make no further mention of this.

We may now define a collection of semi-norms to generate the required topologies. If $A \in V_{\rho_0}$ and if $y \in \mu_{\rho'}(\mu)$, we may define a semi-norm $p_{\gamma,A}$ on X^* by

$$p_{\gamma,A}(x^*) = \sup\{|\int_A fd(x^*\gamma_1)| : f\in M_1^{\rho}\}.$$

For $A\in\Sigma_0$ let us denote by γ_A the restriction of γ to $\Sigma | A = \{B\cap A : B\in\Sigma\}$. It is clear that

$$\|\gamma_A\|_{\rho'} = \sup\{p_{\gamma,A}(x^*) : x^*\in X_1^*\}.$$ [12]

The set function γ above is called ρ' <u>countably additive</u> if for every sequence $\{A_n\}_{n\in N}$ in Σ_0, <u>decreasing monotonically</u> to $\emptyset$ (that is, $\bigcap_{n=1}^{\infty} A_n = \emptyset$), one has the sequence $\{\|\gamma_{A_n}\|_{\rho'}\}_{n\in N}$ converging to zero.[13]

The <u>boundary</u> of $\mathfrak{U}_{\rho'}(\mu)$ is the set

$$\{\gamma\in\mathfrak{U}_{\rho'}(\mu) : \text{for each } A\in\Sigma_0 \text{ there is } x^*\in X_1^* \text{ with } \|\gamma_A\|_{\rho'} = p_{\gamma,A}(x^*)\}.$$

If $\mathfrak{B}$ is a collection of semi-norms on X^* then by the <u>topology generated by</u> $\mathfrak{B}$ we mean the coarsest topology on X^* which makes all semi-norms $\beta\in\mathfrak{B}$ continuous (as such it is always locally convex). In particular if $\gamma\in\mathfrak{U}_{\rho'}(\mu)$ and if $A\in\Sigma_0$ by $\mathfrak{P}(\gamma,A)$ we mean the topology on X^* generated by the single semi-norm $p_{\gamma,A}$. If $\mathfrak{U}$ is any subcollection of $\mathfrak{U}_{\rho'}(\mu)$ then by $\mathfrak{P}(\mathfrak{U}')$ we mean the topology generated by $\mathfrak{U}'$. Of particular interest will be the subcollection $\mathfrak{U}' = \{\gamma\}$ whose generated topology we will designate by $\mathfrak{P}(\gamma)$ and the subcollection $\mathfrak{U}' = \mathfrak{U} = \mathfrak{U}_{\rho'}(\mu)$ whose generated topology we will simply designate by $\mathfrak{P}(\mathfrak{U})$ or simply $\mathfrak{P}$. Of course it is clear that $\mathfrak{P}(\gamma,A) \subset \mathfrak{P}(\gamma) \subset \mathfrak{P}(\mathfrak{U})$. Of especial interest for our work is the restriction of these topologies to the linear subspace X_1^* of X^*. Consequently for the rest of this discussion, the above designations will refer to the respective topologies restricted to X_1^* unless specific mention is made otherwise.

3. Main Results

Some statement about the relationships of these topologies to the generating set functions γ (and therefore semi-norms) is appropriate here. If $\gamma \in \mathcal{U}_{\rho'}(\mu)$ and if the topology $\mathcal{P}(\gamma,A)$, $A \in \Sigma_0$, on X_1^* is compact, then the ρ'-norm of γ_A is precisely given by the evaluation of the semi-norm $\rho_{\gamma,A}$ at some $x^* \in X_1^*$, that is γ lies in the boundary of $\mathcal{U}_{\rho'}(\mu)$. Moreover if X_1^* is compact in the topology $\mathcal{P}(\mathcal{U})$ then the above is true for each $\gamma \in \mathcal{U}_{\rho'}(\mu)$. This we state formally in the following easily shown lemma.

<u>LEMMA 3.1.</u> <u>Let</u> $\gamma \in \mathcal{U}_{\rho'}(\mu)$. <u>If for each</u> $A \in \Sigma_0$, X_1^* <u>is compact in the topology</u> $\mathcal{P}(\gamma,A)$ <u>then</u> γ <u>is in the boundary of</u> $\mathcal{U}_{\rho'}(\mu)$. <u>In particular, if</u> X_1^* <u>is compact in the topology</u> $\mathcal{P}(\mathcal{U})$ <u>then</u> $\mathcal{U}_{\rho'}(\mu)$ <u>coincides with its boundary</u>.

<u>Proof</u>. Let $\{x_n^*\}_{n \in N}$ be a sequence in X_1^* for which

$$\|\gamma_A\|_{\rho'} = \lim p_{\gamma,A}(x_n^*).$$

Since X_1^* is compact in the topology $\mathcal{P}(\gamma,A)$, without loss of generality, we may assume for $x^* \in X_1^*$ that $\{x_n^*\}_{n \in N}$ converges to x^*.

Then $\|\gamma_A\|_{\rho'} = p_{\gamma,A}(x^*)$ and the rest of the lemma follows.

This lemma leads us to the countable additivity of the dual norm[14] ρ'. But first we need to define the concept of a set function enjoying the Fatou property.

If μ and η are scalar valued set functions defined on a ring $\mathcal{E}$ where η is subadditive, then μ has the <u>Fatou property</u> (or μ <u>has</u> property (F_1)) if for every sequence of sets $\{E_n\}_{n\in N}\in\mathcal{E}$, $E_n \subset E\in\mathcal{E}$, for which the sequence $\{\eta(E-E_n)\}_{n\in N}$ converges to zero, one also has

$$\liminf \mu(E_n) \geq \mu(E).$$

The set function μ is said to be <u>strongly bounded</u> if the sequence $\{\mu(E_n)\}_{n\in N}$ converges to zero for every sequence $\{E_n\}_{n\in N}$ of pairwise disjoint sets.

The following theorem shows how these concepts are used to yield the ρ'-countable additivity of $\gamma\in\mathcal{U}_{\rho'}(\mu)$. The proof is obtained by applying Theorem 7 in Orlicz's paper [21] to our specific situation.

<u>THEOREM 3.2. If μ is finite on Σ_0 and if there is a scalar valued set function λ defined on Σ_0 such that</u>

(1) <u>λ is strongly bounded;</u>

(2) <u>for each $x^*\in X_1^*$, $p_{\gamma,A}(x^*)$ satisfies (F_1);</u>

(3) <u>$p_{\gamma,A}(x^*) \leq \lambda(A)$ for all $x^*\in X_1^*$ and $A\in\Sigma_0$</u>

<u>then γ is ρ'-countably additive.</u>

On the other hand, under an appropriate topology on X_1^*, γ being ρ'-countably additive is equivalent to a sequence $\{p_{\gamma,A}(x^*)\}_{A\in\mathcal{U}}$

converging to zero for all $x^* \in X_1^*$ where $\mathfrak{u}$ is a sequence of sets in Σ_o. This will follow from the next theorem.

If $\mathfrak{a}$ is a subcollection of Σ_o, we will say that the semi-norms $p_{\gamma,A}$ for $A \in \mathfrak{a}$ are ρ'-norm attainable if for each $A \in \mathfrak{a}$ there is an $x^* \in X_1^*$ such that $p_{\gamma,A}(x^*) = \|\gamma_A\|_{\rho'}$. The point $x^* \in X_1^*$ is called the ρ'-attained point for A. It is clear that if γ is in the boundary of $\mathfrak{u}_{\rho'}(\mu)$ then for each $A \in \Sigma_o$ the semi-norm $p_{\gamma,A}$ is ρ'-norm attainable.

THEOREM 3.3. Let $\mathfrak{a} = \{A_n\}_{n\in N}$ be a sequence in Σ_o decreasing monotonically to $\emptyset$ and assume that the topological space $(X_1^*, \mathcal{P}(\gamma, A_1))$ is compact. Then there is a sequence $\{x_n^*\}_{n\in N} \in X_1^*$ such that x_n^* is the ρ'-attained point for A_n. Furthermore if x^* is a limit point of $\{x_n^*\}_{n\in N}$ then the sequence $\{\|\gamma_{A_n}\|_{\rho'}\}_{n\in N}$ converges to zero whenever $\{p_{\gamma,A_n}(x^*)\}$ does.

Proof. By Lemma 3.1 for each $A_n \in \Sigma_o$ there is an $x_n^* \in X_1^*$ such that x_n^* is the ρ'-attained point for A_n. If γ is not ρ'-countably additive then for some $\epsilon > 0$, $\|\gamma_{A_n}\|_{\rho'} > \epsilon$. Since for n sufficiently large $p_{\gamma,A_1}(x^*-x_n^*) < \epsilon/4$, we have $p_{\gamma,A_n}(x^*-x_n^*) < \epsilon/4$. Consequently, $p_{\gamma,A_n}(x^*) > \epsilon/2$ which contradicts the convergence to zero of the sequence $\{p_{\gamma,A_n}(x^*)\}_{n\in N}$.

COROLLARY 3.4. Let $\{A_n\}_{n\in N}$ be a sequence in Σ_o decreasing monotonically to $\emptyset$ for which the sequence $\{\sup|\int_{A_n} fd(x^*\gamma)|\}_{n\in N}$ converges to zero (where $x^* \in X_1^*$ and where the supremum is taken over all $f \in M^{\rho}$). If $(X_1^*, \mathcal{P}(\gamma))$ is compact then γ is ρ'-countably additive.

For $f \in L_0$, one defines the "averaged" step function of f to be

$$f_{\mathcal{E}} = \sum_{E \in \mathcal{E}} \left(\int_E \frac{|f|\,d\mu}{\mu(E)}\right) \chi_E$$

where $\mathcal{E}$ is a partition in Σ_0. The function norm ρ is weakly leveling if, for each partition $\mathcal{E}$ in Σ_0, $\rho(f_{\mathcal{E}}) \le \rho(f)$.

All well known Banach function spaces[15] such as the Orlicz spaces (and in particular the Lebesgue spaces) have weakly leveling function norms. In [11] this concept was referred to as ρ having property (J). We prefer the present terminology since the condition is weaker than the concept of leveling as discussed in [10].

COROLLARY 3.5. *If $(X_1^*, \rho(\gamma))$ is compact, then the following statements are equivalent.*

(1) *The set function γ is ρ'-countably additive;*

(2) *For every sequence $\{A_n\}_{n \in N} \in \Sigma_0$ decreasing monotonically to $\emptyset$ and for every $x^* \in X_1^*$, the sequence $\{p_{\gamma, A_n}(x^*)\}$ converges to zero;*

(3) *For every sequence $\{A_n\}_{n \in N} \in \Sigma_0$ decreasing monotonically to $\emptyset$, the sequence $\{\int_{A_n} f d\gamma\}_{n \in N}$ converges to zero in the norm topology (uniformly for $f \in M_1^{\rho}$).*

Proof. In 3.4 we have shown that (2) implies (1), and the converse is obvious. The other equivalence follows from the fact that

$$\|\gamma_{A_n}\|_{\rho'} = \sup\{ |\int_{A_n} fd(x^*\gamma)| : f \in M_1^{\rho},\ x^* \in X_1^* \}.$$

An interesting interpretation of $p^!$-countable additive is the following. Suppose we define the set function m_{f,x^*} on E_o for $f \in M^p$ and $x^* \in X_1^*$ by

$$m_{f,x^*}(A) = \int_A f d(x^* y) .$$

Assuming $(X_1^*, P(y))$ to be compact, the mentioned interpretation is given as a "uniform[24] countable additivity" on E_o of the family $\{m_{f,x^*} : f \in M^p\}$.

Let us now consider the topology P on X_1^* which is generated by all the semi-norms $p_{y,A}$ for all $y \in U_p(\mu)$ and all $A \in S^o$. We will see later that this topology is not Hausdorff. But first we investigate conditions equivalent to Hausdorff.

Subcollections $U^!$ of $U_p(\mu)$ of set functions mapping S^o into X give rise to interesting linear subspaces of X. We define the $U^!$-hull in X to be the set $X_{U^!}$ of all finite sums of the form $\Sigma a y(A)$ as A ranges over Σ^o, a ranges over the scalar field, and y ranges over $U^!$. Again of particular interest will be the case when $U^! = \{y\}$ and $U^! = U = U_p(\mu)$. These we will simply refer to as the y-hull in X and the U-hull in X, respectively.

The importance of the $U^!$-hull in X stems from its influence on the relationship between the $P(U^!)$ and weak* topologies on X_1^*. In particular, as the following lemma demonstrates, if the closure of the $U^!$-hull in X is X then the weak* topology on X_1^* is coarser than the $P(U^!)$ topology. Consequently the $P(U^!)$ topology must be Hausdorff.

<u>LEMMA 3.6. Let</u> $\mathcal{U}'$ <u>be some subcollection of</u> $\mathcal{U}_{0'}(\mu)$. <u>If</u> $c\ell\, X_{\mathcal{U}'} = X$ <u>then the weak*-topology on</u> X_1^* <u>is coarser than its</u> $\rho(\mathcal{U}')$ <u>topology</u>.

<u>Proof</u>. Let us suppose that $\{x_\alpha^*\}_{\alpha\in I}$ is a net in X_1^* convergent in the $\rho(\mathcal{U}')$ topology to the point $x^*\in X_1^*$. We wish to show that it is also convergent in the weak*-topology to x^*. If y is any point in X then there is a finite set of scalars a_i, set functions $\gamma_i\in\mathcal{U}'$ and sets $A_i\in\Sigma_0$, $i = 1,\dots,n$, such that $\|y-\Sigma\alpha_i\,\gamma_i(A_i)\| < \epsilon/2$. Since the net converges in the $\rho(\mathcal{U}')$ topology, we have for $\alpha \geq \alpha_0$,

$$\Sigma|\alpha_i|\rho(\chi_{A_i})\,p_{\gamma_i,A_i}(x_\alpha^*-x^*) < \frac{\epsilon}{2}.$$

Thus

$$|\langle\Sigma\alpha_i\,\gamma_i(A_i),x_\alpha^*-x^*\rangle| \leq \Sigma|\langle\int_{A_i}\alpha_i\chi_{A_i}d\gamma_i,x_\alpha^*-x^*\rangle| \leq$$

$$\leq \Sigma|\alpha_i|\rho(\chi_{A_i})\rho'[(x_\alpha^*-x^*)\gamma_i(A_i)] < \frac{\epsilon}{2}.$$

Consequently

$$|\langle y,x_\alpha^*-x^*\rangle| \leq |\langle y-\Sigma\alpha_i\,\gamma_i(A_i),x_\alpha^*-x^*| + |\langle\Sigma\alpha_i\,\gamma_i(A_i),x_\alpha^*-x^*\rangle| < 2\epsilon.$$

This completes the proof of the Lemma.

We may now formulate Hausdorffness of (X_1^*,ρ) in terms of the weak*-topology on X_1^*.

<u>PROPOSITION 3.7. The following conditions are equivalent.</u>

(1) <u>The topological space</u> (X_1^*,ρ) <u>is Hausdorff</u>;

(2) The closure of the U-hull X_1 is X_1;

(3) The weak topology P is finer than the weak*-topology.

In particular if (X_1^*,P) is Hausdorff then (X_1^*,P) is compact if and only if P coincides with the weak*-topology.

Proof. Since the locally convex weak*-topology is always Hausdorff, the implication (3) implies (1) is obvious. If (2) is assumed to be false then there must be some $x^* \in X_1^*$, $x^* \neq 0$ such that $\langle y(A), x^* \rangle = 0$ for all $A \in S_o$, $y \in U_\rho(\mu)$. But this means $p_A(x^*) = 0$ which contradicts (1). Now statement (2) implies (3) from Lemma 3.6 and the final implication follows directly from (3). This completes our proof.

For the topology P(Y), we can also formulate conditions for when $(X_1^*, P(Y))$ is Hausdorff.

PROPOSITION 2.8. If the topological space (X_1^*,P) is Hausdorff then the following statements are equivalent.

(1) The topological space $(X_1^*,P(y))$ is Hausdorff.

(2) If $\int_A f\, d(x^*Y) = 0$ for all $f \in M^\circ$ and $A \in S_o$ then $\int_A f\, d(x^*y) = 0$ for all $f \in M^\circ$, $A \in S_o$ and $y \in U_\rho(\mu)$;

(3) The topology P(y) is finer than the weak*-topology.

Proof. The implications (1) implies (2) and (3) implies (1) are both clear. In showing (2) implies (1), let us assume that (1) does not hold, that is, there is some $x^* \in X_1^*$, $x^* \neq 0$, yet $p_{Y,A}(x^*) = 0$ for all $A \in S_o$. But (2) says that $p_{y,A}(x^*) = 0$ for all $y \in U_\rho(\mu)$ and $A \in S_o$ which contradicts P being Hausdorff.

A few comments are now in order. If (Ω, Σ, μ) is a finite measure space, and if we let

$$\rho(f) = \left(\int |f|^p d\mu\right)^{\frac{1}{p}}$$

then $\rho'(x^*\gamma)$ is the q-semi-variation of $x^*\gamma$ for $\gamma \in \mathcal{U}_{\rho'}(\mu)$. If $\mu = 0$ then $\gamma = 0$ and (2) of Proposition 3.7 shows that $(X_1^*, \mathcal{P})$ is not Hausdorff. On the other hand if μ is <u>purely atomic</u>[22] we may split Ω into atoms A_0 such that $\mu(A_0) > 0$. If t is a fixed point of Ω, we may define γ_t on Σ to be 0 if $t \notin A$ and $\gamma_t(A) = x_0$ if $t \in A$ where x_0 is some element of X. Then $\gamma_t \in \mathcal{U}_{\rho'}(\mu)$ and Proposition 3.7 shows that $(X_1^*, \mathcal{P})$ is Hausdorff.

For $A \in \Sigma_0$, let us denote by $L_\rho(A)$ those functions in L_ρ which vanish on the complement of A. For $T \in L(L_\rho, X)$ we will denote by T_A the restriction of T to $L_\rho(A)$ and by τ the correspondent of T given by (2.1).

<u>THEOREM 3.9.</u> <u>For</u> $T \in L(L_\rho, X)$ <u>and for</u> τ <u>the correspondent element in</u> $\mathcal{U}_{\rho'}(\mu)$, <u>the following statements hold.</u>

(1) <u>If</u> T <u>is a compact operator, then</u> $(X_1^*, \mathcal{P}(\tau))$ <u>is a compact topological space;</u>

(2) <u>If the topological space</u> $(X_1^*, \mathcal{P}(\tau))$ <u>is compact then the operators</u> T_A, $A \in \Sigma_0$, <u>are compact.</u>

<u>In particular if</u> $\rho(\chi_\Omega) < \infty$ <u>then</u> T <u>is compact if and only if the topology</u> $\mathcal{P}(\tau)$ <u>is compact on</u> X_1^*.

<u>Proof.</u> Assuming that T is a compact operator, let $\{x_\alpha^*\}_{\alpha \in I}$ be a net in X_1^* converging in the weak*-topology to $x^* \in X_1^*$. We show that convergence is retained with the $\mathcal{P}(\tau)$ topology. In the

norm topology on X^*, one has the net $\{T^*(x^*_\alpha)\}_{\alpha\in I}$ converging to $T^*(x^*)$. But

$$p_{\tau,A}(x^*_\alpha - x^*) = \sup\{|\langle\int_A f d\tau, x^*_\alpha - x^*\rangle| : f\in M^\rho_1\}$$

$$= \sup\{|\langle T(f\chi_A), x^*_\alpha - x^*\rangle| : f\in M^\rho_1\}$$

$$\leq \sup\{\rho(f\chi_A)\|T^*(x^*_\alpha - x^*)\| : f, M^\rho_1\} \leq \|T^*(x^*_\alpha - x^*)\|.$$

Thus we have convergence in the $\rho(\tau)$ topology.

To show statement (2) let $\{x^*_\alpha\}$ again be a net in X^*_1 converging to x^* in the $\rho(\tau)$ topology on X^*_1 and let $f\in L_\rho$ with $\rho(f) \leq 1$. If $A\in\Sigma_o$ then $f\chi_A$, having support in Σ_o, is in M^ρ and

$$|\langle f\chi_A, T^*(x^*_\alpha - x^*)\rangle| = |\langle\int_A f d\tau, x^*_\alpha - x^*\rangle| \leq p_{\tau,A}(x^*_\alpha - x^*).$$

Consequently the net $\{T^*_A(x^*_\alpha)\}_{\alpha\in I}$ converges in the norm of X^* to $T^*_A(x^*)$ which is the compactness of T_A (see [9]). The rest of the theorem follows immediately.

In [13], some characterizations of the compactness of T in terms of τ have been given. It would be interesting to characterize the weakly compact operators on L_ρ since L^{**}_ρ has a known representation. To this end one may apply the Kakutani representation for abstract M spaces (see Theorem 2.7 in [23]).

The semi-norms $p_{\gamma,A}$ have been defined and have been shown to be worthwhile. Let us now consider the semi-norm $p_{\gamma,\Omega}$ defined for $x^*\in X^*_1$ by

$$p_{\gamma,\Omega}(x^*) = \sup\{|\int fd(x^*\gamma_1)| : f\in M^\rho_1\}.$$

Since $\rho'(x^* \gamma_1)$ is finite, it is clear that $p_{\gamma,\Omega}(x^*)$ is also. Let $\mathcal{P}(\gamma,\Omega)$ be the topology generated by this semi-norm $p_{\gamma,\Omega}$.

If x^* is an element in X_1^*, we may define the operator $\langle T,x^*\rangle$ on M^ρ by

$$\langle T,x^*\rangle(f) = \langle T(f),x^*\rangle.$$

As above, it is the restriction of this operator to $L_\rho(A)$ which will be of interest. This we will designate by $\langle T,x^*\rangle_A$. It is clear that for any $x^* \in X_1^*$, one has

$$p_{\tau,A}(x^*) = \|\langle T,x^*\rangle_A\|.$$

Utilizing this terminology we may now formulate a condition for τ to be ρ'-countably additive.

COROLLARY 3.10. *Assume that $T \in L(L_\rho,X)$ is compact and that for every sequence $\{A_n\}_{n\in N}$ in Σ_0 monotonically decreasing to $\emptyset$, the sequence $\{\|\langle T,x^*\rangle_{A_n}\|\}_{n\in N}$ converges to zero. Then $(X_1^*,\mathcal{P}(\tau,\Omega))$ is a compact space and τ is ρ'-countably additive.*

Proof. Since both topologies $\mathcal{P}(\tau,A_1)$ and $\mathcal{P}(\tau,\Omega)$ are subcollections of $\mathcal{P}(\tau)$ if the operator T is compact then it is clear from the theorem that these topologies are also compact. To show τ is ρ'-countably additive we must demonstrate that for sequences $\{A_n\}_{n\in N}$ in Σ_0 monotonically decreasing to $\emptyset$, also the sequence $\{\|\tau_{A_n}\|_{\rho'}\}_{n\in N}$ converges to zero, where $\|\tau_{A_n}\|_{\rho'} = \sup\{p_{\tau,A_n}(x^*) : x^* \in X_1^*\}$. To this end we will apply Theorem 3.3. That is, we will assume that x^* is a $\mathcal{P}(\tau,A_1)$ limit point of a sequence

$\{x_n^*\}_{n\in N}$ in X_1^* for which

$$p_{\tau,A_n}(x_n^*) = \|\tau_{A_n}\|_{\rho},$$

and we will show that $\{p_{\tau,A_n}(x^*)\}_{n\in N}$ converges to zero. For $\epsilon > 0$, there is $N > 0$ such that $p_{\tau,A_1}(x_n^*-x^*) < \epsilon/4$ for all $n \geq N$ and such that for $r > N$

$$\left| \|\langle T,x_r^*\rangle\|_{A_n} - \|\langle T,x_r^*\rangle\|_{A_m} \right| < \frac{\epsilon}{4}$$

for all $n,m > N$. Thus for all $n,m > N$ we have

$$|p_{\tau,A_n}(x^*) - p_{\tau,A_m}(x^*)| \leq p_{\tau,A_n}(x^*-x_r^*)$$

$$+ |p_{\tau,A_n}(x_r^*) - p_{\tau,A_m}(x_r^*)| + p_{\tau,A_m}(x_r^*-x^*)$$

$$\leq 2p_{\tau,A_1}(x^*-x_r^*) + \left| \|\langle T,x_r^*\rangle_{A_n}\| - \|\langle T,x_r^*\rangle_{A_m}\| \right| < \epsilon.$$

This completes the proof.

In the second part of Theorem 3.3 we considered conditions which would insure the σ^τ-countably additivity. We note that a similar thing may be done here for $\|\langle T,x^*\rangle_A\|$.

We now assume that Ω is a locally compact space and that $\mathcal{G}$ is a semi-tribe (δ-ring) of Baire sets (see [7]). We will say that a set function $\gamma \in U_\sigma^f(\mu)$ is regular at $A \subset \Omega$ if for every $\epsilon > 0$ there is a compact set K and an open set O both in $\mathcal{G}$ such that $K \subset A \subset O$ and if $S \subset O\setminus K$, $S \in \mathcal{G}$, then $\|\gamma_S\| < \epsilon$. As in [8] it is easy to see that γ is regular on every compact G_δ-set and the collection of subsets on which γ is regular is a ring of sets. Thus γ is regular on $\mathcal{G}$.

PROPOSITION 3.11. If $\gamma \in \mathcal{U}_{\rho'}(\mu)$ and if $\rho(\chi_K) < \infty$ for every compact set $K \in \Sigma$, then the following statements are equivalent.

(1) The set function γ is ρ'-countably additive on $\mathcal{B}$;

(2) For every sequence $\{A_n\}_{n\in N}$ of Σ_o monotonically decreasing to $\emptyset$, there is a sequence $\{B_n\}_{n\in N} \in \mathcal{B}$ of open Baire sets such that $A_n \subset B_n$ and the sequence $\{\|\int f_n d\gamma\|\}_{n\in N}$ converges uniformly to 0 for every sequence $\{f_n\}_{n\in N}$ of M_1^ρ for which $f_n \equiv 0$ on the complement of B_n.

Proof. Since γ is regular on $\mathcal{B}$, in showing that (1) implies (2) let us obtain a sequence $\mathcal{B}_o$ of open sets in $\mathcal{B}$ such that $\{\|\gamma_B\|\}_{B\in\mathcal{B}_o}$ converges to zero. Let us note that since $c\ell\ B$ is compact, $B_n \in \Sigma_o$. If $\{f_n\}_{n\in N}$ is a sequence as defined in (2) and is related to $\mathcal{B}_o$, we have for $x^* \in X_1^*$,

$$|\int f_n d(x^*\gamma)| \leq \rho(f_n)\|\gamma_{B_n}\|_{\rho'}.$$

Thus statement (2) follows.

If γ is not ρ'-countably additive on $\mathcal{B}$ then we may assume that for the sequence $\mathcal{C} \subset \mathcal{B}$ monotonically decreasing to $\emptyset$, $\|\gamma_A\|_{\rho'} > \epsilon$ for some $\epsilon > 0$ and for all $A \in \mathcal{C}$. Let $\mathcal{B}_o$ be a sequence in $\mathcal{B}$ as utilized in (2). Then for $A \in \mathcal{C}$, since

$$\|\gamma_A\|_{\rho'} = \sup\{\|\int_A f d\gamma\| : f \in M_1^\rho\}$$

one has ilj$^f{}_A X_A{}^d$>':! I_ G/2 for a11 AcQ where {' $f_A\}_{A\in Q}$ is a sequence in M^. Since f'X, vanishes on the complement of Be R where A c B, we have a contradiction to (2). Thus y is p'-countably additive on (B.

We have made the assumption that ft be locally compact. As it is, this is not much of a restriction. If $L_\rho = L_\rho$(Q, £,fi) is a Banach function space, then one may find a measure space (0^f JRJY) where ft^1 is a locally compact space, IB is the o~ field generated by the compact subsets, y is finite on the compact subsets and L_ρ is isometric and (lattice) isomorphic to L_ρ (QP ,fo,Y)• Moreover if (£i,Z,/i) is o-finite or if there is some $f_o \in L_\rho$ such that $f_o > 0$ almost everywhere, then IB is the a-field generated by all clopen subsets of the compact space Q' and y (SV) is finite. The reader is referred to [23] for more in this direction. Finally let us note that if p has the weak leveling property then |i(E) < oo implies p(X_F) ' °° (see [23]).

4. Operators on Bounded Functions of L_ρ

In conjunction with our remarks ending the previous section, we will assume that there is some $f_o \in L_\rho$ (where L_ρ is now a real Banach function space) such that f > 0 almost everywhere. Let B^ρ be the algebra of essentially bounded functions in L^ρ and $cl^\infty B^\rho$ will be its closure in L^∞(ft,Z,/i). (These definitions are given in the footnotes). For any topological space Q, we will let C(Q) represent all continuous real-valued functions on ft.

Since B_ρ is a vector lattice, the closed subspace $c\ell\, B_\rho$ in L_∞ is an abstract M-space (in the sense of Kakutani, see [13]). Also since $f_n = \min\{f, n\}$ belongs to B_ρ and since $f_n > 0$ almost everywhere, it follows from Theorem 2.1 of [24] that there is a compact Stone space Ω_1 such that $c\ell_\infty B_\rho$ is isometrically isomorphic $C(\Omega_1)$ (the topology being that obtained from the sup norm). The adjoint space $C(\Omega_1)^*$ is an abstract L-space (also in the sense of Kakutani, see [18]). It is isometrically equivalent to $\mathcal{L}^1(\Omega_2, \mathcal{B}, \upsilon)$ where υ is a finite regular measure on the Borel sets $\mathcal{B}$ of some Ω_2 (see [23], Theorem II.1.1). From Theorem I.3.2 Ω_1 can be homeomorphically identified with a closed subset of Ω_2.

In our particular case we are interested in bounded linear operators $T \in L(L_\rho, X)$ for which T restricted to $c\ell\, B_\rho$ is also a bounded linear operator.

THEOREM 4.1. *Let T be a continuous linear operator from $c\ell_\infty B_\rho$ into X.*

(1) *For every sequence $\{A_n\}_{n\in N}$ in Σ_0 of pairwise disjoint sets, for every sequence $\{\alpha_n\}_{n\in N}$ of scalars with $|\alpha_n| \leq 1$ and for $x^* \in X_1^*$, the series $\Sigma\langle x^*, \alpha_n \tau(A_n)\rangle$ is convergent where τ is the measure corresponding to T (in the representation Theorem 2.1).*

(2) *If T is a compact (respectively, weakly compact) operator from $c\ell_\infty B_\rho$ into X, if $\{\alpha_n\}_{n\in N}$ and τ are as in (1) and if $\{B_n\}_{n\in F}$ is a finite sequence in Σ_0 then the subset X_F of X consisting of all elements of the form $\Sigma\{\alpha_n \tau(B_n) : n \in F\}$ is conditionally compact (respectively, weakly compact). In addition if*

$\{A_n\}_{n\in N}$ <u>in</u> Σ_o <u>is monotonically decreasing to</u> $\emptyset$, <u>then the sequence</u> $\{\tau(A_n)\}_{n\in N}$ <u>converges to zero in the norm of</u> X.

<u>Proof</u>. Since $c\ell_\infty B_\rho$ may be identified with $C(\Omega_1)$, we may consider T as a bounded linear operator from $C(\Omega_1)$ into X. Then there is a finitely additive set function γ from $\mathcal{B}$, the σ-field of Borel subsets of Ω_1, into X such that $T(f) = \int \bar{f} d\gamma$ for $f \in c\ell_\infty B_\rho$ where $\bar{f}$ is the function in $C(\Omega_1)$ corresponding to f. Since T is bounded, the semi-variation of γ is finite. In [4] it is shown that this is equivalent to the series $\Sigma \langle x^*, \alpha_n \gamma(B_n)\rangle$ being convergent for all sequences $\{B_n\}_{n\in N}$ in $\mathcal{B}$ of pairwise disjoint subsets. On the other hand for $A \in \Sigma_o$, its correspondent $\bar{A}$ in $\mathcal{B}$ is clopen and

$$T(\chi_A) = \int \chi_{\bar{A}} d\gamma = \gamma(\bar{A}).$$

Thus $\tau(A) = \gamma(\bar{A})$ which shows statement (1). It follows that if T is a compact (respectively, weakly compact) operator on $C(\Omega_1)$ then the subset X_F of X as given in statement (2) is conditionally compact (respectively, weakly compact). In this case it is known that the γ utilized in the proof of (1) above must be countably additive in the norm of X. If $\bar{A} \in \mathcal{B}$ denotes the corresponding clopen set of $A \in \Sigma_o$, then for $x^* \in X_1^*$,

$$\gamma_{x^*}(A) = \int \chi_A d\gamma_{x^*} = \langle T(\chi_A), x^*\rangle = \langle \tau(A), x^*\rangle.$$

Now for $x^* \in X_1^*$, γ_{x^*} are uniformly countably additive.[25] Consequently the second part of statement (2) holds.

For the locally compact Hausdorff space Ω, let $C_0 = C_0(\Omega, X)$ be the set of continuous functions on Ω mapping into the Banach space X and vanishing at infinity. The uniform norm is placed on C_0. Let Σ_1 be a ring of subsets of Ω and for the Banach space Y, let γ be a finitely additive measure from Σ_1 into $L(X,Y)$ with finite semi-variation $\tilde{\gamma}$ [16].

For $y^* \in Y^*$, a set function γ_{y^*} from Σ_1 and X^* may be defined by

$$\langle \gamma_{y^*}(A), x \rangle = \langle y^*, \gamma(A)x \rangle$$

for $x \in X$ and $A \in \Sigma_1$. If $|\gamma_{y^*}|$ denotes the total variation [17] of γ_{y^*} then for $A \in \Sigma_1$ the semi-norm $g_{\gamma,A}$ on Y^* may be defined as

$$g_{\gamma,A}(y^*) = |\gamma_{y^*}|(A).$$

The topology generated by the collection $\{g_{\gamma,A} : A \in \Sigma_1\}$ will be designated by $Q(\gamma)$. It is shown in [11] that every bounded linear operator $T \in L(C_0(\Omega,X),Y)$ corresponds to a unique weakly regular, finitely additive vector measure γ from the Borel σ-algebra of subsets of Ω into $L(X,Y^{**})$. The weak regularity of T means that $\langle \gamma(\cdot)x, y^* \rangle$ is a regular Borel measure for each $x \in X$ and $y^* \in Y^*$. The operator T is compact if and only if $(Y_1^*, Q(T))$ is a compact topological space. We can now develop this for operators T from $cl\, B^0$ into X.

We will say that the unit ball X_1^* is weak*-sequentially compact relative to T if every sequence $\{x_n^*\} \subset X_1^*$ has a subsequence $\{x_{n_p}^*\}$ and a point $x^* \in X_1^*$ for which the sequence $\{\sup\{|\langle T(f), x_{n_p}^* - x^* \rangle| : f \in L_{\infty,\mu}, \|f\| \le 1\}\}_{n \in N}$ converges to zero.

PROPOSITION 4.2. *The bounded linear operator* T *from* $cl_{\infty}B_{\rho}$ *into* X *is compact if and only if* X_1^* *is weak*-sequentially compact relative to* T.

Proof. Considering T again as a bounded linear operator from $C(\Omega_1)$ into X then T is a compact operator if and only if $(X_1^*, \mathcal{G}(\tau))$ is a compact space. But Ω_1 is a compact space also. Consequently T is compact if and only if for every sequence S in X_1^* there is a subsequence $\{x_n^*\}_{n\in N} \subset S$ and a point $x^*\in X_1^*$ such that[18] $\{\bar{\tau}_{x^*-x_n^*}(\Omega)\}_{n\in N}$ converges to zero. If $\bar{f}\in C(\Omega_1)$ corresponds to the function $f\in cl\ B_{\rho}$ then for all f

$$|\langle T(f), x_n^* - x^*\rangle| = |\int \bar{f} d\tau_{x^*-x_n^*}|.$$

Now $cl_{\infty}B_{\rho}$ is a dense subset of $C(\Omega_1)$ using the essential sup norm relative to μ. Consequently

$$\bar{\tau}_{x^*-x_n^*}(\Omega_1) = \sup\{|\langle T(f), x_n^* - x^*\rangle| \ :\ f\in L_{\rho},\ \|f\|_{\infty,\mu} \le 1\}.$$

This completes the proof.

5. Operators into L_{ρ}

Let us now consider operators T which map the Banach space X into L_{ρ}. We will assume that the dual norm ρ' has the weak leveling property.[19] It is easy to see that if ρ has the weak leveling property then also does ρ'.

Now γ will be a finitely additive set function from $\Sigma_0' = \{E\in\Sigma : \rho'(\chi_E) < \infty\}$ into X^* where $\langle\gamma(\cdot),x\rangle$, for $x\in X$, is countably additive and μ-continuous for each $x\in X$. Let $\mathcal{V}_0$ be the linear space of all such finitely additive set functions γ. We may define a norm V_ρ on $\mathcal{V}_0$ by taking

$$V_\rho(\gamma) = \sup\{\rho(\sum_{\mathcal{E}} \frac{\tau(E)x}{\mu(E)} \chi_E) : \mathcal{E} \text{ partition in } \Sigma_0',\ \|x\| \leq 1\}.$$

This norm is often called the ρ-variation of γ and with it $\mathcal{V}_\rho$ is complete (see [11] for the details).

If $\gamma\in\mathcal{V}_\rho$ then γ is finitely additive and not, in general, countably additive. In addition it vanishes on μ-null sets. Let us also note that if $\mathcal{E}$ is a partition then the simple function $f = \Sigma\{\alpha\chi_E : E\in\mathcal{E}\}$ taking its constant values on the members of $\mathcal{E}$ is in both L_ρ and $L_{\rho'}$. For the sake of simplicity we will allow for $E\in\mathcal{E}$, $\mu(E) = 0$ in which case $\frac{0}{0}$ will be interpreted as 0. For a subset $A \subset \Omega$, $\mathcal{E}_A$ will denote a finite family of disjoint sets $E\in\mathcal{E}$ with $E \subset A$.

Motivated by the definition of the ρ-variation of γ, we may define a family of semi-norms on X. If $\gamma\in\mathcal{V}_\rho$ then define r_γ on X to be

$$r_\gamma(x) = \sup\{\sum_{\mathcal{E}} \frac{\langle\gamma(E),x\rangle}{\mu(E)} \chi_E : E\in\mathcal{E}\}$$

and for $A \subset \Omega$ define $r_{\gamma,A}$ by

$$r_{\gamma,A}(x) = \sup\{\sum_{\mathcal{E}} \frac{\langle\gamma(E),x\rangle}{\mu(E)} \chi_E : E\in\mathcal{E}_A\}.$$

As was done previously we may use these semi-norms to generate topologies on X. We designate them in an analogous fashion as $\mathfrak{T}(\gamma, \mathcal{A})$, $\mathfrak{T}(\gamma)$ and $\mathfrak{T}$. Utilizing these, we may then define for any $A \subset \Omega$ the V_p-norm of γ relative to A or $V_p[\gamma]$-relative to A as

$$V_p[\gamma](A) = \sup\{r_{\gamma,A}(x) : x \in X_1^*\}.$$

We shall say that γ is V_0-countably additive if for every sequence $\{A_n\}_{n \in N} \subset \mathcal{E}$ the sequence $\{V_0[\gamma](A_n)\}_{n \in N}$ converges to zero. If for each $A \in \mathcal{E}$ there is some $x \in X_1$ with $V_0[\gamma](A) = r^{\gamma,A}(x)$ then we shall say, as was done previously, that $r^{\gamma,A}$ is V_0-attainable. Again as we have done previously we may define the boundary of V_p to be

$$\text{bdry } V_p = \{\gamma \in V_0 : \text{for each } A \in \mathcal{E}_0,\ r_{\gamma,A} \text{ is } V_0\text{-attainable}\},$$

Adjusting our proof to Lemma 3.1 we may now state

PROPOSITION 5.1. *If $(X_n, \mathfrak{T})$ is a compact space then the boundary of V_p is U_0.*

With appropriate adjustments in the proof of Theorem 3.3 we may use the above to obtain

THEOREM 5.2. *For a sequence $\mathcal{G} = \{A_n\}_{n \in N} \subset \mathcal{E}$ monotonically decreasing to $\emptyset$ let us assume that $(X_n, \mathfrak{B}(\gamma, A_n))$ is a compact space. There is a sequence $S \subset X_1$ such that for each A_n, r_{γ,A_n} is V_0-attainable. Furthermore if x is a limit point of S then the sequence $\{V_0[\gamma](A_n)\}_{n \in N}$ converges to zero whenever*

$\{r_{\gamma,A_n}(x)\}_{n\in N}$ does. *Thus if* $(X_1,\mathfrak{R}(\gamma))$ *is a compact space and if* $\{r_{\gamma,A}(x)\}_{A\in\mathcal{S}}$ *converges to zero for all such sequences* $\mathcal{S}$ *in* Σ *then* γ *is* V_ρ*-countably additive.*

Assuming reflexivity on X, we may give equivalences for $(X_1,\mathfrak{R})$ to be Hausdorff. The space being Hausdorff will readily follow if the weak topology on X_1 is coarser than $\mathfrak{R}$. But more interesting appears this analogy to the $\mathcal{U}$-hull in X. We will define the *averaged* $\mathcal{V}_\rho$*-hull in* X^* to be the set $avX^*_{\mathcal{V}}$ of all finite sums of the form $\Sigma a\frac{\gamma(E)}{\mu(E)}$ where $E\in\mathcal{E}$, $\mathcal{E}$ is some finite partition of Σ, $\gamma\in\mathcal{V}_\rho$, $\mathcal{V}=\mathcal{V}_\rho$ and a is a scalar.

PROPOSITION 5.3. *If* X *is a reflexive space then the following conditions are equivalent.*

(1) *The space* $(X_1,\mathfrak{R})$ *is Hausdorff;*

(2) *The closure of the averaged* $\mathcal{V}_\rho$*-hull in* X^* *in the norm topology is* X^*;

(3) *The weak topology on* X_1 *is coarser than the* $\mathfrak{R}$*-topology.*

Proof. To show (1) *implies* (2) assuming X is reflexive, let us also assume there is an $x\in X$, $x\neq 0$, such that $\langle\gamma(E),x\rangle = 0$ for all E in a partition $\mathcal{E}$ and $\gamma\in\mathcal{V}_\rho$. As in Proposition 3.7 this yields a contradiction to (1). In showing that (2) *implies* (3), assume the net $\{x_\alpha\}_{\alpha\in I}$ in X_1 converges in the $\mathfrak{R}$-topology to $x\in X_1$ and let $x^*\in X^*$. Then there are finite sets of scalars a_i, set functions γ_i, and a partition $\mathcal{E}$ such that for $E_i\in\mathcal{E}$

$$\|x^*-\Sigma a_i\frac{\gamma_i(E_i)}{\mu(E_i)}\| < \epsilon/2.$$

On the other hand for all α greater than some fixed $\alpha_0 \in I$, we have

$$|\langle \Sigma a_i \frac{\gamma_i(E_i)}{\mu(E_i)}, x_\alpha - x\rangle| \leq \Sigma |\alpha_i| \frac{|\langle \gamma_i(E_i) x_\alpha - x\rangle}{\mu(E_i)} < \frac{\epsilon}{2} .$$

The rest of the proof follows as in Proposition 3.1.

Now it is clear, that as we have proceeded in Section 3, one could give conditions for which the space $(X_1, \Re(\tau))$ would be Hausdorff. Then utilizing results by Orlicz in [21] conditions for γ to be V_ρ-countably additive could be given.

In Theorem II.6 of [11], there is exhibited an isomorphism between $L(X, L_\rho)$ and $\mho_\rho$ (under the assumption that ρ' satisfies the weak leveling property). To this end let τ be the correspondent in $\mho_\rho$ of the operator $T \in L(X, L_\rho)$. More precisely this correspondence is given by

$$\langle \tau(E), x\rangle = \int Txd\mu$$

and

$$Tx = \frac{d}{d\mu}\langle \tau(\cdot), x\rangle$$

where the integral is as defined in [11].

<u>THEOREM 5.4.</u> <u>Suppose that ρ has the weak leveling property.</u>

(1) <u>If X is reflexive and if T is a compact operator from X into L_ρ is compact then $(X_1, \Re(\tau))$ is a compact space.</u>

(2) <u>If $(X_1, \Re(\tau))$ is a compact space then T must be a compact operator.</u>

Proof. Whenever X is reflexive, every net in X_1 has a weakly convergent subnet. Thus in showing (1) we may assume that $\{x_n\}_{n\in N} \subset X_1$ is a net converging weakly to x. Now T^* is compact and T^{**} may be identified with T. Thus the net $\{T(x_\alpha)\}_{\alpha\in I}$ converges in the norm of L_ρ to $T(x)$. Utilizing the representation for τ, we now have

$$r_\tau(x_\alpha - x) = \sup\{\rho[\sum_{E\in\mathcal{E}} \frac{\langle \tau(E), x_\alpha - x\rangle}{\mu(E)} \chi_E] : \text{partition } \mathcal{E} \subset \Sigma\} \leq$$

$$\leq \sup\{\rho[\sum_{E\in\mathcal{E}} \frac{\chi_E}{\mu(E)} \int_E |T(x_\alpha - x)| d\mu : \text{partition } \mathcal{E} \subset \Sigma\}$$

$$\leq \sup \rho[T(x_\alpha - x)_{\mathcal{E}}]$$

where $T(x_\alpha - x)_{\mathcal{E}}$ is the averaged step function of $T(x_\alpha - x)$. Since ρ has the weak leveling property it follows that

$$r_\tau(x_\alpha - x) \leq \rho[T(x_\alpha - x)].$$

Since the net $\{\rho[T(x_\alpha - x)]\}_{\alpha\in I}$ converges to zero it follows that $(X_1, \Re(\tau))$ is compact.

In Lemma II.5 of [11], it is shown that

$$\text{(I)} \quad k\rho[T(x)] \leq \sup\{\rho[\sum_{E\in\Sigma} \frac{|\langle \tau(E), x\rangle|}{\mu(E)} \chi_E] : \mathcal{E} \text{ partition in } \Sigma\}$$

where k is some constant, $0 < k \leq 1$. Thus to show (2) since $(X_1, \Re(\tau))$ is compact we may assume the net $\{x_\alpha\}_{\alpha\in I}$ in X_1 converges in the $\Re(\tau)$ topology to x. Applying inequality (I) to $x_\alpha - x$, then the right side will converge to zero. Thus $\{Tx_\alpha\}_{\alpha\in I}$ converges to Tx in the L_ρ-norm and T is compact.

6. Operators with Range in L(X,Y)

Our attention is now directed to the vector valued case for spaces of the type L_ρ. We will consider the same questions as previously posed in Section 3. In particular for the Banach space X let $L_\rho{}^*X$ be the Banach function space of strongly measurable X-valued functions f defined on Q, with $\rho(f) = \rho(|f|)$ where $|f|$ is the X-norm of f. Corresponding to M_ρ we have the linear subspace $M_\rho{}^*X$ of $L_\rho{}^\# X$ defined as

$$M_\rho{}^\cdot X = \overline{sp}\{f^*x : f \in L^\infty, x \in X\}.$$

Our questions then are formulated in terms of obtaining meaningful topologies related to bounded linear operators T from $M_\rho{}^*X$ into the Banach space Y. The restriction of T to elements of $M_\rho{}^*X$ which vanish on the complement of $A \in \Sigma_0$ will be designated by T_A.

In [23], a characterization of the subspace $L(M_\rho{}^*X,Y)$ of $L(L_\rho{}^\cdot X,Y)$ is obtained. For this we define the norm $\|T\|_\rho$ of T to be

$$\|T\|_\rho = \sup\{\sum_{i=1}^n |\langle y^*, T(f_i x_i)\rangle| : y^* \in Y^*, \rho(\sum_{i=1}^n f_i \cdot x_i) \leq 1\}.$$

As before if Z is also a normed linear space, let us define $G_{\rho'}(L(Z,Y),\mu)$ to be the family of all additive set functions γ mapping Σ_0 into L(Z,Y) which vanish on μ-null sets and for which $N_{\rho'}(\gamma) < \infty$ where

$$N_{\rho'}(\gamma) = \sup\{n_{\rho'}(\gamma z) : z \in Z_1\}.$$

Let $B_{\rho'}(L(Z,Y),\mu)$ be the family of all additive set functions γ mapping Σ into L(Z,Y) which vanish on μ-null sets and such that for each $y^* \in Y^*$, $z \in Z$, $\langle y), y^{\# z}\rangle$ is purely finitely additive with its support contained in the support of a function of $L_{\rho} M_{\rho}$ and for which $\|y\|(\Omega) < \infty$ where

$$\|y\|(\Omega) = \sup\{\|y^{z}\|(\Omega) : z \in Z\}$$

($\|y^z\|$ is just the variation of y^{*z}). Finally we let

U f(L(Z,Y),a) =G f (L(Z,Y) ,(i)©» f (L(Z,Y) , fi) and Y^U . (L(Z,Y) , fi) ,

$Y = Y_1 + Y_2$ implies $R_{\rho}(Y) < \infty$ where

$$R_{\rho}(y) = \sup\{\|y^{*z}\|_{\rho} : z \in Z_1\}.$$

Of particular interest for us will be the case when $Z = X$. Thus for $Y \in U_{\rho}(L) = U_{\rho}(L(X,Y), \mu)$ we may define the semi-norm $s_{Y,A}$ on Y^* to be

$$s_{Y,A}(y^*) = \sup\{|\int_A f\,d\langle x, y^* Y\rangle| : x \in X_1, f \in M_{\rho}]_1\}$$

where $A \in S_0$. By $g(y, A)$ we will mean the topology on Y^* generated by the one semi-norm $s_{Y,A}$. Meanings similar to that used for the topologies $P(Y)$ and P will be attached to $\wedge(Y)$ and $\wedge$.

PROPOSITION 6.1. Let $T \in L(M_{\rho}, X, Y)$ with $\|T\|_{\rho} < \infty$. There is a unique $T \in U_{\rho}(L(X,Y), \mu)$ and a bounded linear operator $T' \in L(L_{\rho}, L(X,Y))$ such that for $f \in L_{\rho}$ and $x \in X$

$$T'(f)\cdot x = T(f\cdot x) = \int f\,d(T\cdot x).$$

If T is a compact operator then Y^*_1 will be a compact space with the topology $g(T)$. Conversely if $(Y^*_1, g(T))$ is a compact space then all T_A, for $A \in \Sigma$, will be compact operators.

Proof. The first statement is demonstrated in [23]. As in Section 3, let us assume that the net $\{y_\alpha^*\}_{\alpha \in I} \in Y^*$ converges in the weak*-topology to y^*. Since

$$|\langle \int_A f\,d(T.x, y_\alpha^* - y^*\rangle| = |\langle T(f \cdot \chi_A(x)), y_\alpha^* - y^*\rangle|,$$

for $x \in X$, it follows that

$$s_{T,A}(y_\alpha^* - y^*) \leq \|T\| \, (y_\alpha^* - y^*) \ldots$$

This shows that $(Y^*, \mathcal{S}(T))$ is compact whenever T is a compact operator. For the converse utilize the inequality

$$|\langle f \cdot \chi_A(x), T^*(y_\alpha^* - y^*)\rangle| \leq s_{T,A}(y_\alpha^* - y^*).$$

Consequently if $\{y_\alpha^*\}_{\alpha \in I}$ in Y^* converges in the $\mathcal{S}(T)$ topology to y^* then T_A must be compact for each $A \in \Sigma_o$. This completes the proof.

In [23], it is shown that $T'(f) = \int_\Omega f\,d\tau$ and that

$$\|T'\| = R_{\rho'}(\tau) \leq \|T\| \leq \|T\|^\infty.$$

We should also note that the semi-norms s_λ will yield information for both T and T' in much the same way as we have already done in Section 3. The statements and their proofs would be similar to the corresponding ones in Section 3. Consequently they are omitted here.

Some of the basic results dealing with the scalar case of function spaces L_ρ may be transferred to the vector valued spaces $L_\rho \cdot X$. This not only pertains to our results here but also to many results as found in [23]. Thus the importance of tensor products in function spaces L_ρ is largely due to this. To this end, we need to consider results pertinent to greatest and least cross-norms for L_ρ and X (see [23]).

For the Banach spaces X and Y we will consider the bilinear forms $x \otimes y$ defined on $X' \times Y'$ (X' and Y' are the algebraic duals of X and Y respectively) by

$$x \otimes y(x', y') = x'(x) \cdot y'(y).$$

If t with representation

$$t = \sum_{i=1}^{n} x_i \otimes y_i \in X \otimes Y$$

is a formal tensor product, then the <u>greatest</u> <u>cross</u> <u>norm</u> g is defined by

$$g(t) = \inf\{ \sum_{i=1}^{n} \|x_i\| \cdot \|y_i\| \ : \text{all representations of } t\}.$$

By $X \otimes_g Y$ is meant the completion, in this norm, of the tensor product $X \otimes Y$. In a similar manner one defines the <u>least</u> <u>cross</u> <u>norm</u> ℓ by

$$\ell(t) = \sup\{ | \sum_{i=1}^{n} x^*(x_i) \cdot y^*(y_i) | \ : \ x^* \in X_1^*, y^* \in Y_1^* \}$$

and $X \otimes_\ell Y$ to be the completion, in this norm, of $X \otimes Y$.

Our interest will be in the completion $L_\rho \otimes_\ell X$ where we will assume that $M_\rho = L_\rho$. This last assumption can occur under some very general conditions. For example in the case of Orlicz spaces, one may consider the so-called Δ_2-condition (see [22]).

For a function f mapping Ω into the Banach space X we may define the norm w_ρ as

$$w_\rho(f) = \sup\{\rho(x^* \circ f) : x^* \in X_1^*\}.$$ [21]

Such a function f is said to be <u>weakly measurable</u> if the scalar function $x^* \circ f$ is measurable for each $x^* \in X^*$. By $W_{\rho,X}$ is meant the closed linear subspace

$$W_{\rho,X} = c\ell\ sp\{f : \Omega \to X : f \text{ is weakly measurable, } w_\rho(f) < \infty\}.$$

From the definition it is clear, for example, that if $f_i \in L_\rho$ and $x_i \in X$, then for the case of $L_\rho \cdot X = M_\rho \cdot X$ one has $w_\rho(\sum_{i=1}^{n} f_i \cdot x_i)$ well-defined. Also for $A \in \Sigma$ we may define w'_ρ to be

$$w'_\rho(A) = \sup\{w_\rho(\Sigma \chi_E \cdot x : E \in \mathcal{E} \subset 2^A,\ \mathcal{E} \text{ a partition,}$$
$$E \in \Sigma_0 \cap \Sigma_F,\ \rho(\Sigma \chi_E \cdot x) \leq 1\},$$

where $\Sigma_F = \{A \in \Sigma : \mu(A) < \infty\}$.

The function norm ρ is said to be <u>countably additive</u> if $w_\rho(\Omega) < \infty$ and if for every sequence $\{A_n\}_{n \in N} \subset \Sigma$, monotonically decreasing to $\emptyset$, the sequence $\{w'_\rho(A_n)\}_{n \in N}$ converges to zero.

We are interested in defining a topology on X_1^* such that if X_1^* is compact in this topology then ρ countable additivity will be equivalent to the statement that some elements of $L_\rho \otimes_\ell X$ converge uniformly in the ℓ-norm to zero. This, in turn, will be equivalent to a family of compact operators of $L(L_\rho^*, X)$ converging uniformly to zero.

In particular let $\mathcal{E}_A$ denote a finite family of sets $E \in \Sigma_0 \cap \Sigma_F$, $E \subset A$, which are pairwise disjoint. Let t in the formal tensor product $L_\rho \otimes X$ be defined as

$$t = \Sigma\{\chi_A \otimes x \ : \ A \in \mathcal{E}_\Omega, \ x \in X\}.$$

In particular, if this finite sum is taken over sets $A \in \mathcal{E}_B$ then we will designate t as t_B. We will also consider

$$\overline{\ell}(t_B) = \sup\{\ell(t_B) \ : \ A \in \mathcal{E}_B, \ \rho(\Sigma\chi_A \cdot x) \leq 1\}.$$

If $\chi^t = \Sigma\chi_A x$ for t as above then χ^t is <u>strongly</u>[26] (and therefore <u>weakly</u>) <u>measurable</u> and

$$w_\rho(\chi^t) \leq \Sigma o\,(\chi_A)\,\|x\| < \infty.$$

For each such t, the linear operator T^t of L_ρ^* into X may be defined as follows. Let us recall (as mentioned in Theorem 2.1) that for each $\varphi \in L_\rho^*$ there is a unique $\gamma_\varphi \in \mu_{\rho'}(\mu)$ such that

$$\varphi(f) = \int_\Omega f d\gamma_\varphi, \quad f \in L_\rho.$$

Thus for $\varphi \in L_\rho^*$ we define T^t from L_ρ^* into X to be

$$<T^t(\gamma_\varphi), x^*> = \int_\Omega <\chi^t, x^*> d\gamma_\varphi, \quad x^* \in X^*.$$

Thus it is easy to see that

$$\text{(A)} \qquad \ell(t) = \|T^t\| = \sup\{\,|<T^t(\gamma_\varphi), x^*>| \ : \ \rho'(\gamma_\varphi) \leq 1, \ x^* \in X_1^*\}$$

$$= \sup\{\rho(x^*(\chi^t)) \ : \ x^* \in X_1^*\} = w_\rho(\chi^t).$$

We may now define the semi-norm o_A for $A \in \Sigma_0$ on X_1^* by

$$o_A(x^*) = \sup\{\rho(\Sigma \chi_B \cdot x^*(x)) : B \in \mathcal{C}_A, \rho(\Sigma \chi_B \cdot x) \leq 1\}.$$

By $\Theta(A)$ and Θ we mean the topologies generated (respectively) by o_A for fixed $A \in \Sigma_0$ and by o_A for all $A \in \Sigma_0$.

It is clear from the definitions that

(B) $\quad \sup\{o_A(x^*) : x^* \in X_1^*\} = \sup\{w_\rho(\Sigma \chi_B \cdot x) : B \in \mathcal{C}_A\} = \overline{w}'_\rho(A).$

<u>THEOREM</u> <u>6</u>.<u>1</u>. <u>If</u> $(X_1^*, \Theta(\Omega))$ <u>is</u> <u>compact</u> <u>then</u> <u>the</u> <u>following</u> <u>statements</u> <u>are</u> <u>equivalent</u>.

1. <u>The</u> <u>function</u> <u>norm</u> ρ <u>is</u> <u>countably</u> <u>additive</u>;
2. <u>For</u> <u>every</u> <u>sequence</u> $\mathcal{S} \subset \Sigma$ <u>monotonically</u> <u>decreasing</u> <u>to</u> $\emptyset$, <u>the</u> <u>sequence</u> $\{\overline{\ell}(t_A) : A \in \mathcal{S}\}$ <u>converges</u> <u>to</u> <u>zero</u>;
3. <u>For</u> <u>every</u> <u>sequence</u> $\mathcal{S} \subset \Sigma$ <u>monotonically</u> <u>decreasing</u> <u>to</u> $\emptyset$, <u>the</u> <u>sequence</u> $\{\sup \ell(T^{t_A}) : A \in \mathcal{S}\}$ <u>converges</u> <u>to</u> <u>zero</u>;
4. <u>For</u> <u>every</u> <u>sequence</u> $\mathcal{S} \subset \Sigma$ <u>monotonically</u> <u>decreasing</u> <u>to</u> $\emptyset$, <u>the</u> <u>sequence</u> $\{o_A(x^*) : A \in \mathcal{S}\}$ <u>converges</u> <u>to</u> <u>zero</u> <u>for</u> <u>each</u> $x^* \in X_1^*$.

<u>Proof</u>. Using an argument similar to that of Theorem 3.3 and the observation (B) above we may show that (1) and (4) are equivalent. We also bring forth the fact that $\ell(t_A) = \|T^{t_A}\|$ and that $w_\rho(\chi^t) = \ell(t_A)$ (from (A)) to prove the rest of the theorem. That is, the countable additivity of ρ is translated into the sequence $\{\overline{w}'_\rho(A) = \ell(t_A) : A \in \mathcal{S}\}$ converging to zero for $\mathcal{S} \subset \Sigma$ as described in (2), (3) or (4). This completes the proof.

The above theorem has an interesting interpretation. _If X_1^* is compact in the topology $\mathfrak{S}(\Omega)$ then the countable additivity of ρ is equivalent to a family of compact operators in $L(L_\rho^*, X)$ converging to zero._ Surely the operators T^t defined above are compact. Again we omit the results that may be obtained analogous to those in Section 3 by considering the topologies $\mathfrak{S}$ and $\mathfrak{S}(A)$.

For the purposes of our discussion below let $\mathfrak{n}$ designate the norm topology and $\mathfrak{w}^*$ the weak*-topology. We also designate by $\mathfrak{a}_{\rho'}(K)$ that subset of $\mathfrak{a}_{\rho'} = \mathfrak{a}_{\rho'}(L(X,Y),\mu)$ consisting of all $\tau\in\mathfrak{a}_{\rho'}$ whose corresponding operator T from L_ρ into $L(X,Y)$ is compact. For the sake of simplicity we will assume that μ is positive and finite on Ω and that ρ still has the weak leveling property (and $L_\rho = M_\rho$).

For $\gamma\in\mathfrak{a}_{\rho'}(K)$ and for $\mu(A) > 0$, $A\in\Sigma_0$ we define the seminorm $k_{\gamma,A}$ on Y_1^* by

$$k_{\gamma,A}(y^*) = \sup\{|\int_A fd\langle y^*,\gamma\cdot x\rangle| \ : \ x\in X_1, \ f\in M_1^\rho)\}.$$

By $\mathfrak{K}(A,X)$ we mean the topology on Y_1^* generated by such $k_{\gamma,A}$ where A is fixed in Σ_0 and γ varies through $\mathfrak{a}_{\rho'}(K)$. The space Y is called an Ω-_space_ if for some fixed $A\in\Sigma_0$ with $\mu(A) > 0$ and for all X, $(Y_1^*,\mathfrak{K}(A,X))$ is a compact space.

THEOREM 6.2. If $(Y_1^*,\mathfrak{K}(A,X))$ is a Hausdorff space then the $\mathfrak{n}$-topology on Y_1^* is finer than the $\mathfrak{K}(A,X)$-topology which in turn is finer than the $\mathfrak{w}^*$-topology. Moreover there are Banach spaces X_1 and X_2 such that $\mathfrak{w}^* = \mathfrak{K}(A,X_1)$ and $\mathfrak{n} = \mathfrak{K}(A,X_2)$.

In particular if Y is an 0-space, then Y must be finite dimensional.

Proof. Using a proof similar to that found in Proposition 3.7, it is clear that if $(Y_1^*, K(A,X))$ is Hausdorff then to* c K(A,X) . Clearly if Y is an *Or* space then to* = X(A,X) . To see that K(A,X) is a subfamily of ft follows from

$$k_{\tau,A}(y^*) \leq \sup\{\|\int_\Omega f\chi_A d(\tau\cdot x)\|\cdot\|y^*\| : x\in X_1,\ f\in M_1^\rho\}$$

$$\leq \sup\{T(f\chi_A)\cdot\|y^*\| : f\in M_1^\rho\} \leq \|T\|\cdot\|y^*\|,$$

where T is the compact operator corresponding to *red* $_b$ (K) .

If Y is not an Q-space, we may find an X_1 such that $M(A,X_1)$ = UD^. Let X_1 be the scalar field. If the net {y^} c: Y_1 converges in the to*-topology to Y^* then for the compact operator T corresponding to reQ $_{\rho'}$(K) we have

$$k_{\tau,A}(y_\alpha^*-y^*) = \sup\{|\langle T(f\chi_A)x,\ y_\alpha^*-y^*\rangle| : x\in X_1,\ f\in M_1^\rho\}$$

$$\leq \|T^*(y_\alpha^*-y^*)\|.$$

Thus {y*j must converge to y^* also in the $X(A,X_1)$-topology.

We now exhibit an X_9 for which $M(A,X_9)$ = ft. In fact let $X_2 = Y$ and define y from 2^Ω into $L(X,Y)$ by

$$Y(A) = |U(A)\ I$$

where I is the identity operator of $L(X,Y)$. It is clear that y is finitely additive and attains the value zero on $\backslash x$ zero sets.

Also $y \in G_{\rho'}(K)$. Obviously the collection

$\{\frac{\mu(E)}{\rho(\chi_E)} = p_1(E) : E \in \Sigma_0, p_1(E) \leq 1\}$ is conditionally compact since μ is bounded on Σ_0. This implies that

$\{\frac{\gamma(E)}{\rho(\chi_E)} = p_2(E) : E \in \Sigma_0, p_2(E) \leq 1\}$ is conditionally compact. In [23], it is shown that the latter is a sufficient condition to demonstrate that $y \in G_{\rho'}(K)$. Thus

$$k_{\gamma,A}(y^*) = \sup \left| \int f \chi_A \langle Ky^*, d\mu \rangle \right| .$$

Since ρ has the weak leveling property, $\rho(\chi_A) > 0$ and in fact $\mu(A) = \rho(\chi_A)\rho'(\chi_A)$. Let $f = \frac{\chi_A}{\rho(\chi_A)}$. Then $k_{\gamma,A}(y^*) \leq \|y^*\| \rho'(\chi_A)$. Consequently $K(A,X) = h$. Finally if Y is an $\mathcal{L}$-space then $U)^* = H$ and Y must be finite-dimensional. This completes our proof.

7. Concluding Remarks

We would propose that other classes of operators be considered in this topological setting. For example, the weakly compact operators, nuclear operators or others as mentioned in [9] may be interpreted appropriately. It would be interesting to see if all operators can be so interpreted, even the non-linear ones. On the other hand, one may want to consider appropriate conditions (for example, paracompactness) on the topologies herein defined and to determine the significance of the operators obtained.

In another direction, one may want to consider more general structures than the normed linear space structure we have studied.

For example, the generalization obtained by replacing the norm with local convexity seems to be fruitful. Here one would need to replace the "variations" used to define our topologies by an appropriate concept. Pertinent to this appears to be the paper [12].

Since semi-norms define pseudometrics which in turn define uniformities, it is clear that uniform space theory is appropriate to inter-relate the theories of operators, topology, and measure, as we have begun to show. How the latter two interact may be seen in [31] and other recent papers by Frolik. Some other papers assisting in this development would be [29] and [30].

As we have mentioned in the Introduction one would also want to consider the above questions for the more general classes of normed vector lattices or ordered topological linear spaces. They both include the normed Köthe spaces and Banach function spaces.

In summary, we have been considering linear operators T in (1) $L(L_\rho, X)$, in (2) $L(X, L_\rho)$, in (3) $L(M_\rho \cdot X, Y)$ and in (4) $L(c\ell_\infty B_\rho, X)$. Appropriate topologies, in each case, have been defined to relate conditions on T to conditions on the topologies.

In the above cases, a statement about compactness of T can readily be interpreted using the underlying representative measure τ. In particular for (1), if $\rho(\chi_\Omega) < \infty$ then T is compact if and only if $(X_1^*, \mathcal{P}(\tau))$ is a compact space (see 3.9). For (2), we assume ρ has the weak leveling property. If X is reflexive and if T is compact then $(X_1, \mathcal{R}(\tau))$ must be a compact space. Conversely if $(X_1, \mathcal{R}(\tau))$ is a compact space then T must be a compact operator (see 5.4). For (3), we need to assume that norm $\|T\|^{\sim}$ of T is finite. Then T compact implies that $(Y_1^*, \mathcal{S}(\tau))$ is a compact space which in turn implies that the operators T_A, for $A \in \Sigma_o$, are compact operators (see 6.1). For (1) a similar statement was made in (3.9) when $\rho(\chi_\Omega)$ was not necessarily finite. In [14] it is shown that if $T \in L(C_o(\Omega_1, X), Y)$ then T is a compact operator if and only if $(Y_1^*, \mathcal{G}(\tau))$ is a compact space where Ω_1 is a compact Stone space. However, the interest here lies in $T \in (cl_\infty B_\rho, X)$. As such T is compact if and only if X_1^* is weak*-sequentially compact relative to T (see 4.2). Moreover we have related compactness (respectively, weak compactness) of T to the conditional compactness (respectively, weak compactness) of the appropriate subset X_F of X. Also here it will turn out that if T is compact then its corresponding set function τ is countably additive.

A concept that proved significant throughout the discussions was for a set function $\gamma \in \mathcal{U}_{\rho'}(\mu)$ to be ρ' countably additive. The concept arises naturally as a conclusion to conditions which are practical for scalar valued set functions as shown in the work [21] of Orlicz. This continuity type of condition on γ has an interesting interpretation under the assumption that $(X_1^*, \mathcal{P}(\gamma))$ is compact. Precisely it is that the set functions $m_{f,x^*}(A) = \int_A fd(x^*\gamma)$ satisfy a uniform continuity condition. If Ω is a locally compact Hausdorff space, then the compactness of $\mathcal{P}(\gamma)$ may be relaxed to give an interesting characterization of ρ' countable additivity more in terms of the topology (in particular its open Baire sets) on Ω (see 3.11). The countable additivity of the function norm ρ, as we have defined it in terms of the continuity of the norm w'_ρ also has an interesting interpretation for operators. As in 6.1 if it is assumed that $(X_1^*, \mathcal{O}(\Omega))$ is compact then this may be characterized as the family of operators $T^t \in L(L_\rho^*, X)$ for $t \in L_\rho \otimes X$ being compact.

The assumption that Ω be locally compact is rather interesting. In fact for any Banach function space $L_\rho(\Omega, \Sigma, \mu)$, one may find a measure space $(\Omega', \mathcal{B}, \gamma)$ such that L_ρ is isometric and lattice isomorphic to $L_\rho(\Omega', \mathcal{B}, \gamma)$ where $\mathcal{B}$ is the σ-field generated by the compact subsets of the locally compact space Ω' and where γ is finite on compact sets. When (Ω, Σ, μ) is σ-finite then Ω' may be taken to be a compact (extremally disconnected) Hausdorff space and γ develops as a regular Borel measure on the σ-field $\mathcal{B}$ generated by the clopen subsets of Ω' (see [23]). Reminiscent of the maximum modulus principle in complex analysis, the boundary of the collection $\mathcal{U}_{\rho'}(\mu)$ of set functions was defined. Compactness of $(X_1^*, \mathcal{P}(\gamma, A))$ for all $A \in \Sigma_0$ puts γ in the boundary, and compactness of $(X_1^*, \mathcal{P}(\mathcal{U}))$ makes $\mathcal{U}_{\rho'}(\mu)$ coincident with its boundary. An interesting interpretation is that if γ is in the boundary then for each $A \in \Sigma_0$, the semi-norm $p_{\gamma, A}$ is ρ'-norm attainable. Under an appropriate compact topology on X_1^*, sequences $\{A_n\}_{n \in N} \in \Sigma_0$ monotonically decreasing to $\emptyset$ give rise to sequences $\{x_n^*\}_{n \in N} \in X_1^*$ such that x_n^* is the ρ'-attained point for A_n (see 3.3) with an appropriate convergence statement holding.

The weak topologies defined herein were in general not Hausdorff. However if the $\mathfrak{U}'$-hull $X_{\mathfrak{U}'}$, for $\mathfrak{U}' \subset \mathfrak{U}_{\rho'}(\mu)$, has its norm closure as X itself, then $(X_1^*, \wp)$ must be Hausdorff. Consequently if $(X_1^*, \wp)$ is Hausdorff, that is if there are enough semi-norms to distinguish points on X_1^*, then $(X_1^*, \wp)$ will be compact if and only if $\wp$ coincides with the weak* topology on X_1^* (see 3.7). However the compactness of $T \in L(L_\rho, X)$ is related to the compactness of the smaller $\wp(\tau)$ weak topology. It is interesting to note that if $(X_1^*, \wp)$ is Hausdorff then $(X_1^*, \wp(\gamma))$ being Hausdorff is equivalent to a type of absolute continuity for appropriate set functions (see 3.8). If (Ω, Σ, μ) is a finite measure space and if ρ is the $\mathcal{L}^p(\mu)$ norm then $(X_1^*, \wp)$ is not Hausdorff. However if μ is purely atomic, (3.7) shows that it will be Hausdorff. For the other topologies, herein defined, analogous situations occured. These give additional information interrelating the operators, the underlying measures and the topologies.

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Footnotes

1. For a very coherent compendium of the theory of Banach function spaces,, answering most of the relevant questions as known at the time, the reader is referred to the series of papers by W. A. J. Luxemburg and A. C. ^aanen in [17].

2. The a-finite case is sufficiently complicated. More generally, the study of this case is an essential prerequisite for the non-a-finite case (see [17]). Many of the results will follow under even weaker conditions (see [18]).

3. For any real number p, 1 <^ p < oo we may define a function norm o for feM+ by p (f) = (jf^p dji)^p. If p = 1 we have simply the integral of f over Q. A function norm p oo may also be defined as follows. For any a, 0 £ a < oo, set A = fx€O : f (x) > a}. If fi(A) > 0 for all a define p (f) = +oo ; if, (i(A) = 0 for some value of a, define o (f) = a where a = inffa : ^(A) = °}« 1^ie number P (f) is called the essential upper bound of f (for p (f) < oo there is f-|M , almost equal to f, such that f.. is bounded on X with its least upper bound equal to p (f)). The triangle inequality for these norms p , 1 <^ p <^ oo , follows from the Holder inequality discussed in the footnote below. Later on in Section 4 we will have occasion to make use of the Banach function space L which designates the space L for o - p

4. These are functions f = f + if2 where f and f2 are functions whose values lie in the extended real number system. Besides the usual rules for operating with +oo and -oo it is understood that (+oo) + (Too) = (+oo) - (+oo) = 0 and 0(+oo) = 0. Associativity of addition no longer holds unless we are in the non-negative extended real numbers. However the presence of finite valued functions will guarantee the associativity in our function space.

5. Actually one should consider the collection M* = (feM : p(f) < 00} and its decomposition into equivalence classes [f] modulo the collection of /i-null functions. Then each class [f] contains a finite-valued function on Q, namely, f•\n -, where E = [x : |f(x)| =00}, guaranteeing associativity of addition (see previous footnote). If f = g then p(f) = P (g) and one may define p([f]) = p(f). Using f for [f] one may now return to the above definition of L_o. Actually L_o is a linear subspace of the linear space of all (i-almost everywhere finite functions in M which in itself is not a. linear space (recall that addition is not associative in M) .

6. A few remarks are pertinent here regarding the definition. It follows immediately that if f is ji-measurable and if geL_ρ, with |f| £ |g| on *Q,* then p(f) <L p (g) < 00 and feL_P. If feL_P and if E = fx : |f(x) I =00} then the characteristic function *x* satisfies *p (n\ )* <= p(f) < 00 for all neN. Consequently p (y) = 0 which by (v) of the axioms for p implies f = 0 almost everywhere, that is, JLX(E) = 0 or more succintly f is finite almost everywhere on *Cl.* In another vein, the axioms do not exclude the existence of a positive measure set Ae £ such that not only $p(X_A)$ = oo but also $o(x_B)$ = oo for all B c: A and B of positive measure. Such sets A are called, in the literature, <u>unfriendly sets</u>. Using the above argument, it follows that if A is an unfriendly set then any feL_ρ is identically zero on A. Consequently to investigate L_ρ-spaces, it is worthwhile to remove the unfriendly sets A. Throughout we will assume that this has been done. It is shown in [17] that there is a largest unfriendly set A_{max} and once removed the remaining set again designated by fi contains no unfriendly sets and ju(O) is still positive. (See [19] for additional remarks). It seems more appropriate to call such unfriendly sets A above ~~purely infinite~~.

7. A sufficient condition to insure completeness is that the function norm p satisfy the <u>weak Fatou property,</u> that is, whenever the sequence $(f_n\}_{n\in N}$ and feM*, $[f_n)_{n\in N}$ monotonically increasing and pointwise convergent to f, sup $p(f_n)$ < 00 implies that p(f) < 00. This is weaker than the more recognizable <u>(sequential) Fatou</u> property (all £^-spaces, 0 <^ p < 00, satisfy it and it may be used to prove completeness of these spaces);

(continued on next page)

that is, where the pointwise convergence of the above sequence implies that the sequence $\{\rho(f_n)\}_{n\in N}$ is monotonically increasing and convergent to $\rho(f)$. Completeness of L_ρ is equivalent to the norm ρ having the Riesz-Fischer property, that is, for any sequence $\{f_n\}_{n\in N} \in L_\rho \cap M^+$, $\Sigma\rho(f_n) < \infty$ implies $\Sigma f_n \in L_\rho$ (or $\rho(\Sigma f_n) < \infty$). Some examples from [27] are appropriate. Let μ be discrete measure on $\Omega = N$ and for any $f \in M^+$, let $\rho(\mu) = \sup f(n) + \alpha \limsup f(n)$, for α a positive constant. Then the function norm ρ has the weak Fatou property but not the Fatou property. If ρ is adjusted in this example so that for any $f \in M^+$, $\rho(f) = \sup f(n)$ if $\{f(n)\}$ converges to 0, and $\rho(f) = +\infty$ otherwise, then ρ is a function norm having the Riesz-Fischer property but not the weak Fatou property.

8. We have already mentioned the early works for the development of the general theory in [10] and [16]. However definite connections exist with the theory of Kothe spaces as evidenced in the work of Dieudonne in [6]. Besides the later work of Luxemburg and Zaanen already cited, also the work of J. J. Schaffer in [25] and [26] should be noted. The book [20] by Massera and Schaffer has a summary of these results in a more general setting.

9. That ρ' is actually a norm and not a semi-norm follows from the fact that unfriendly sets have been removed (see previous footnote on unfriendly sets).

10. Using our above stated conventions for infinite arithmetic it follows that $\int |fg| d\mu \leq \rho(f)\rho'(g)$ for any f and $g \in M$. In particular if $f \in L_\rho$ and $g \in L_{\rho'}$ then a Holder inequality (as in the case $L_\rho = \mathfrak{L}^p$, $1 \leqq p \leqq \infty$, and $L_{\rho'} = \mathfrak{L}^q$, $\frac{1}{p} + \frac{1}{q} = 1$) holds as

$$\left|\int fg d\mu\right| \leq \int |fg| d\mu \leq \rho(f)\rho'(g) < \infty .$$

Moreover if $\rho'(g) < \infty$ then $\rho'(g) = \sup\{|\int fg d\mu| : \rho(f) \leq 1\}$. From this it follows that if G is defined for all $f \in L_\rho$ by

$$G(f) = \int fg d\mu$$

then G is a bounded linear functional on L_ρ and $\|G\| = \rho'(g)$.

(continued on next page)

Consequently $L_{\rho'}$ is isometrically and algebraically embedded in the first Banach dual space L_ρ^* of L_ρ, that is, $L_{\rho'}$ is a closed linear subspace of L_ρ^*. Thus the elements $G \in L_\rho^*$ belonging to $L_{\rho'}$ are characterized by the existence of a function $g \in L_{\rho'}$ which is identified with G and such that $G(f) = \int fg d\mu$ for all $f \in L_\rho$. In [18], a characterization of these G is given without requiring knowledge in advance that G is represented by a function g. In particular $G \in L_{\rho'}$ if and only if for every sequence $\{f_n\}_{n \in N} \in L_\rho$ converging pointwise to zero it follows that $\{G(f_n)\}_{n \in N}$ converges to zero.

In the recent work [23], a representation of $(L_\rho)^*$ is given. For this purpose a modification of the norm ρ' for set functions is made. In brief let $A_{\rho'}(\mu) = A_{\rho'}(\Omega, \Sigma_0, \mu)$ be the class of all finitely additive scalar-valued functions γ on Σ_0 which vanish on μ-null sets and for which $\rho'(\gamma) < \infty$ where now

$$\rho'(\gamma) = \sup\{|\int_\Omega f d\gamma| : \rho(f) \leq 1, f \in M^\rho\}.$$

The integrals given relative to finitely additive set functions are in the sense of [9] (see Chapter III). This definition of ρ' does give a norm for which $A_{\rho'}(\mu)$ is a normed linear space. If, however, $d\gamma = g d\mu$ then $\rho'(\gamma) = \rho'(g)$ (see [18]) then the two definitions agree. Let $B_{\rho'}(\mu)$ represent the set of purely finitely additive scalar-valued set functions γ, vanishing on μ-null sets and having their support in the support of some $f \in L_\rho \setminus M^\rho$ such that γ has finite variation. Then as in [23] let $\mathcal{C}_{\rho'}(\mu) = A_{\rho'}(\mu) \oplus B_{\rho'}(\mu)$ and $(L_\rho)^*$ is isometrically-isomorphic to $\mathcal{C}_{\rho'}(\mu)$.

11. The theorem in [23] gives a decomposition of T into components T_1 and T_2. The T_1 component is the restriction of T to the space M^ρ and $\rho'(\tau_1) = \|T_1^*\|$. The component T_2 is simply $T - T_1$ and $\|\tau_2\|(\Omega) = \|T_2^*\|$.

12. It is worthwhile to remark here that if η is a finitely additive set function in $\mu_{\rho'}[\mu]$ and if $A \in \Sigma_0$ then we may write

$$p_{\eta,A}(x^*) = \rho'(x^* \gamma_{1A}) + |x^* \gamma_2|(A).$$

This follows from the definition of ρ' and from the fact that $x^* \gamma_2$ is purely finitely additive with support contained in the support of a function in $L_\rho \setminus M^\rho$. Thus for $A \in \Sigma_0$, we have $|x^* \gamma_2| = 0$.

13. The fact that ρ' is countably additive should yield some information on the T_1 component (as discussed above) of T. In fact in [11], it is pointed out that if γ is positive and purely finitely additive measure then there is a sequence $\{B_n\}_{n \in N}$ of sets in Σ_0 such that for all countably additive measures η the sequence $\{\eta(B_n)\}_{n \in N}$ converges to zero and $\gamma(B_n) = \gamma(\Omega)$.

14. This in turn leads to information about the component T_1.

15. For our purposes let us remark that in [11], it was shown that if ρ is weakly leveling then $\mu(E) < \infty$ implies $\rho(\chi_E) < \infty$. Since $x^* \gamma_2$ is purely finitely additive, if $\mu(\Omega) < \infty$ and if ρ is weakly leveling then γ being ρ'-countably additive implies that $\gamma_2 = 0$.

16. The <u>semi-variation</u> $\tilde{\gamma}$ of a vector-valued set function γ from Σ_1 into $L(X,Y)$ is defined as follows. It is assumed that $\gamma(\emptyset) = 0$. For every set $A \subset \Omega$ let

$$\tilde{\gamma}(A) = \sup\{|\Sigma\gamma(A_i)x_i| : \text{finite partitions } \{A_i\} \text{ and}$$
$$\text{finite collections } \{x_i\} \in X_1\}.$$

Note that the semi-variation $\tilde{\gamma}$ of γ depends essentially on the spaces X and Y and as such should be emphasized as $\tilde{\gamma}_{X,Y}$.

17. If γ is a set function from Σ into X, the (total) variation $\bar{\gamma}$ of γ is defined (it is assumed that $\gamma(\emptyset) = 0$) for $A \subset \Omega$ as

$$\bar{\gamma}(A) = \sup\{\Sigma|\gamma(A_i)| : \text{finite partitions } \{A_i\} \text{ of } A\}.$$

It is shown in [7], page 54, that the spaces X and Y may be so chosen so that the semi-variation $\tilde{\gamma}$ of γ (relative to X and Y) is the variation $\bar{\gamma}$ of γ.

18. Since $\tau_{x^*-x_n^*}$ is scalar-valued, its variation coincides with its semi-variation (see [7]).

19. Thus partitions in Σ_o' and in Σ_o are identical (see [11]).

20. It is pointed out in [23] that this definition is necessary to restate a version of the representation theorem for $T \in L(L_\rho, X)$, that is $L(L_\rho, X) \cong \mathcal{U}_{\rho'}(\mu)$, when $X = L(Z,Y)$. This is useful in characterizing some tensor product spaces $L_\rho \otimes_\gamma X$ as discussed later in the reference.

21. This norm is defined by Pettis in the paper, "On integration in vector spaces", Trans. Amer. Math. Soc. 44(1938), 277-304, for the L_p-case.

22. A set $A \in \Sigma$ is an atom (with respect to μ) if $\mu(A) > 0$ and if for every set $B \in \Sigma$, $B \subset A$, either $\mu(B) = 0$ or $\mu(B) = \mu(A)$. Then μ is atomic if there is at least one atom in Σ; μ is purely atomic if Ω is a finite union of atoms.

23. In the case that η is not positive then η is defined to be purely finitely additive if its variation is.

24. That is, if $\{A_n\}_{n \in N} \in \Sigma$ is monotonically decreasing to $\emptyset$ then for all $\epsilon > 0$ there is an $N_\epsilon > 0$ such that for all $n > N_\epsilon$, $|m_{f,x^*}(A)| < \epsilon$ for all $x^* \in X_1^*$ and $f \in M_1^\rho$.

25. This is in the sense of footnote 24 except that the uniform aspect is only over $x^* \in X_1^*$.

26. The function f from Ω into X is <u>strongly measurable</u> if it is the limit in μ-measure of a sequence of μ-simple functions. It is <u>weakly measurable</u> if $x^* \circ f$ is measurable in the usual scalar sense for $x^* \in X^*$.