DICHOTOMIES FOR LINEAR DIFFERENTIAL EQUATIONS WITH DELAYS: THE CARATHÉODORY CASE

by

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1. Introduction.

We consider on $[0,\infty)$ an equation of the form

$$\dot{u} + Mu = r \tag{1.1}$$

in a Banach space E, and the corresponding homogeneous equation

$$\dot{u} + Mu = 0; \tag{1.2}$$

here r is a locally integrable vector-valued function; the "solution" u is defined on $[-1,\infty)$, and M, the "memory" functional, takes a continuous function u into a locally integrable function Mu in such a way that the values of Mu on an interval $[a,b]$ depend on the values of u on $[a-1,b]$ only. The equations are to be satisfied "locally in $\underset{\sim}{L}^1$ ".

The purpose of our investigation, which continues the work in [10] (and also in [2] and [3]) is to relate properties of (1.1) such as "admissibility" ("for every r in some given function space there is a solution in some given function space") and certain forms of conditional stability behaviour ("dichotomies") of the solutions of (1.2) and of its restrictions to intervals of the form $[m,\infty)$. The method consists, as in [10], in reducing this problem to a similar problem about a linear difference equation in a function space; this difference equation can then be studied by means of the theory developed in [1]. We refer to the introduction of [10] for further comments on method and significance, and to the work of Pecelli [7] for some related results obtained under more special assumptions and by a different method.

In [10] a special instance of the "continuous case" was considered: that is, r and Mu were assumed to be continuous, and the equations were to hold everywhere; and $(Mu)(t)$ depended only on the values of u in $[t-1,t]$. In this paper we describe instances of the "Carathéodory case", in which continuity is replaced by local integrability. The reduction of the problem to one about difference equations is much simpler in the Carathéodory case (contrast Theorem 8.1 with [10; Theorem 6.2]); the more basic question of the existence, uniqueness, and growth of solutions, which is almost trivial in the continuous case, becomes, on the other hand, quite complicated under our very general assumptions. We wish to avoid, in particular, any assumption on the representability of M as, say, a Stieltjes integral. We are thus forced to devote quite a bit of space to these questions (Sections 5 and 7); and yet we do not feel that our present formulation of the assumptions on M - summarized in Lemma 7.2 - is definitive. In a forthcoming paper we plan to discuss in detail the autonomous ("constant coefficient") case.

This paper is best read in conjunction with [10], although the formal dependence on that paper consists only in the use of some proofs. On the other hand, our present approach does depend, especially in Section 10, on material in [1] and [9],

2. Spaces.

Throughout this paper, E shall denote a real or complex Banach space. The norm in E, as in all normed spaces other than the scalar fields and the function and sequence spaces described below, is denoted by $\| \ \|$. If X and Y are Banach spaces, $[X\to Y]$ denotes the Banach space of operators (bounded linear mappings) from X to Y, and we set $\check{X} = [X \to X]$.

In this paper, spaces of sequences occur together with spaces of functions on certain intervals of the real line. For the former, we adopt without elaboration the notation described in [1; Sections 2 and 3]. In particular: $\omega = \{0, 1, \ldots\}$, and $\underset{\sim}{s}_{[m]}(X)$ denotes the Fréchet space of all functions on $\omega_{[m]} = \{m, m+1, \ldots\}$ with values in the Banach space X, where $m \in \omega$; and notations such as $\underset{\sim}{l}^{p}_{[m]}(X)$ are to be understood by the obvious analogy. If $f \in \underset{\sim}{s}_{[m]}(X)$ and $m' \geqq m$, then $f_{[m']} \in \underset{\sim}{s}_{[m']}(X)$ is the restriction of f to $\omega_{[m']}$.

The intervals that occur as domains of measurable functions will be $[-1,0]$ and $[m,\infty)$ for real numbers m. We shall in general follow the notation and terminology of [6; Chapter 2] for spaces consisting of such functions, with some special simplifying conventions.

Spaces of functions on $[-1,0]$ will have no label indicating the domain. For instance, $\underset{\sim}{L}^{1}(E)$ is the Banach space of (equivalence classes modulo null sets of) Bochner integrable functions $f\colon [-1,0] \to E$, with the norm $|\!|\!| f |\!|\!|_1 = \int_{-1}^{0} \|f(t)\| dt$. The space $\underset{\sim}{C}(E)$ of continuous functions $f\colon [-1,0] \to E$ with the norm $|\!|\!| f |\!|\!| = \max \|f(t)\|$, which plays a central part in our work, is abbreviated to $\underset{\sim}{E}$, and its norm written without a subscript.

As indicated in these examples, thick hollow bars are used for the norms of function spaces with $[-1,0]$ as domain. This convention permits the following arrangement: suppose that, e.g., $g \in \underset{\sim}{l}^{q}_{[m]}(\underset{\sim}{L}^{p}(E))$, where $1 \leqq p,q \leqq \infty$ and $m \in \omega$; then $\|g\|$ is the element of $\underset{\sim}{l}^{q}_{[m]}(\underset{\sim}{L}^{p})$ - the argument R is omitted, as usual - given by $\|g\|(n) = \|g(n)\|$, $n \in \omega_{[m]}$ (where $\|g(n)\|(t) = \|(g(n))(t)\|$ for all $t \in [-1,0]$; the latter norm is the norm in E); $|\!|\!| g |\!|\!|_p$ is the element of $\underset{\sim}{l}^{q}_{[m]}$ given by $|\!|\!| g |\!|\!|_p(n) = |\!|\!| g(n) |\!|\!|_p = |\!|\!| \|g(n)\| |\!|\!|_p$, $n \in \omega_{[m]}$; thus $|\!|\!| g |\!|\!|_p = |\!|\!| \|g\| |\!|\!|_p$;

and $\|g\|_q = \|\|g\|_p\|_q$ is the norm of g as an element of $l^q_{[m]}(L^p(E))$.

We recall from [6; Chapter 2] that $b\mathfrak{F}$ is the class of all Banach spaces F of (equivalence classes of) measurable functions $\varphi: [-1,0] \to R$ such that

(N): F is stronger than L^1, i.e., F is algebraically contained in L^1 and there exists a number $\alpha_F > 0$ such that $|\varphi|_1 \leqq \alpha_F |\varphi|_F$ for all $\varphi \in F$;

(F): if $\varphi \in F$ and $\psi: [-1,0] \to R$ is measurable and $|\psi| \leqq |\varphi|$, then $\psi \in F$ and $|\psi|_F \leqq |\varphi|_F$.

If $F \in b\mathfrak{F}$, then $F(E)$ denotes the Banach space of (equivalence classes of) measurable functions $f: [-1,0] \to E$ such that $\|f\| \in F$, with the norm $|f|_F = |\|f\||_F$.

In considering spaces of functions defined on intervals of the type $[m,\infty)$, we shall use the following conventions. If $m \leqq m'$ and f is some function defined on $[m,\infty)$, $f_{[m']}$ shall denote its restriction to $[m',\infty)$. The subscript $[m]$ is also used when the fact that $[m,\infty)$ is the domain has to be recorded (these usages are compatible). Thus $L_{[m]}(E)$ denotes the space of all (equivalence classes of) measurable functions $f: [m,\infty) \to E$ that are Bochner integrable on each compact interval; $K_{[m]}(E)$ denotes the space of <u>all</u> continuous functions $f: [m,\infty) \to E$ (cf.[10]); and similarly for the space $M_{[m]}(E)$ of all functions $f \in L_{[m]}(E)$ with $|f|_M = \sup_{t \geqq m} \int_t^{t+1} \|f(s)\| ds < \infty$; for the spaces $L^p_{[m]}(E)$, $1 \leqq p \leqq \infty$; and for the space $C_{[m]}(E)$ of bounded continuous functions $f: [m,\infty) \to E$ with the supremum norm, and the subspace $C_{0[m]}(E)$ of those that tend to 0 at infinity. The norms of all normed spaces of this kind will be indicated, as in [6], by thick bars with the appropriate subscript; the subscript is omitted for the supremum norm.

3. Slicing operations.

Let $m \geqq 0$ be a given real number. For each $t \geqq m$ we define the linear mapping $\Pi(t): \underset{\sim}{L}_{[m-1]}(E) \to \underset{\sim}{L}^1(E)$ by

$$(3.1) \qquad (\Pi(t)f)(s) = f(t+s), \qquad s \in [-1,0], \; f \in \underset{\sim}{L}_{[m-1]}(E).$$

Thus $\Pi(t)$ maps f into the "slice" of f between $t-1$ and t, transplanted to $[-1,0]$ for convenience. (Note that indication of m is omitted; this will not cause any confusion.)

When m is an integer, we define $\varpi f \in \underset{\sim}{s}_{[m]}(\underset{\sim}{L}^1(E))$ for each $f \in \underset{\sim}{L}_{[m-1]}(E)$ by

$$(3.2) \qquad (\varpi f)(n) = \Pi(n)f, \qquad n = m, m+1, \ldots .$$

Thus $\varpi\colon \underset{\sim}{L}_{[m-1]}(E) \to \underset{\sim}{s}_{[m]}(\underset{\sim}{L}^1(E))$ is a linear bijective mapping. This mapping has obvious restrictions to linear mappings of $\underset{\sim}{K}_{[m-1]}(E)$ into $\underset{\sim}{s}_{[m]}(\underset{\sim}{E})$, of $\underset{\sim}{C}_{[m-1]}(E)$ into $\underset{\sim}{l}^{\infty}_{[m]}(\underset{\sim}{E})$, and of $\underset{\sim}{C}_{0[m-1]}(E)$ into $\underset{\sim}{l}^{\infty}_{0[m]}(\underset{\sim}{E})$.

The mapping ϖ has other restrictions that are "natural" isomorphisms between certain normed function spaces: e.g., $\varpi : \underset{\sim}{L}^p_{[m-1]}(E) \to \underset{\sim}{l}^p_{[m]}(\underset{\sim}{L}^p(E))$ is a congruence (linear isometry) for $1 \leqq p \leqq \infty$; $\varpi : \underset{\sim}{M}_{[m-1]}(E) \to \underset{\sim}{l}^{\infty}_{[m]}(\underset{\sim}{L}^1(E))$ is an isomorphism with norm 1, the norm of the inverse being 2; $\varpi : \underset{\sim}{T}_{[m-1]}(E) \to \underset{\sim}{l}^1_{[m]}(\underset{\sim}{L}^{\infty}(E))$ is another isomorphism, with norm 2, the norm of the inverse being 1. We might indeed define new normed spaces of functions on $[m,\infty)$ in this way, but we shall not do this here.

4. Memories.

In this section we shall make precise some of the assumptions on the "memory functional" M that appears in (1.1). We express the linearity of the functional and the fact that the scope of the memory extends at most one unit of time into the past by the following definition.

A memory is a linear mapping $M: \underset{\sim}{K}_{[-1]}(E) \to \underset{\sim}{L}_{[0]}(E)$ such that

(4.1) $\quad \chi_{[a-1,b]}u = 0$ implies $\chi_{[a,b]}(Mu) = 0$ for all $u \in \underset{\sim}{K}_{[-1]}(E)$ and each interval $[a,b] \subset [0,\infty)$.

It is clear that a memory is uniquely determined by its restriction to $\underset{\sim}{C}_{[-1]}(E)$.

Condition (4.1) permits, for each $m \geqq 0$, the "cutting down" of M to a linear mapping $M_{[m]}: \underset{\sim}{K}_{[m-1]}(E) \to \underset{\sim}{L}_{[m]}(E)$: indeed, each $u \in \underset{\sim}{K}_{[m-1]}(E)$ can be written as $u = v_{[m-1]}$ for some $v \in \underset{\sim}{K}_{[-1]}(E)$, and we may set $M_{[m]}u = (Mv)_{[m]}$; since $v'_{[m-1]} = u = v_{[m-1]}$ implies $\chi_{[m-1,t]}(v'-v) = 0$ for each $t \geqq m$, (4.1) implies $(M(v'-v))_{[m]} = 0$; thus the definition does not depend on the choice of v. We have $M_{[0]} = M$; if $m' \geqq m \geqq 0$, these "cut-down" memories satisfy

(4.2) $$M_{[m']}u_{[m'-1]} = (M_{[m]}u)_{[m']} \qquad u \in \underset{\sim}{K}_{[m-1]}(E).$$

A memory is usually assumed to have some continuity or boundedness properties; it is typical to assume (or imply by the assumptions on M) that the restriction of M to $\underset{\sim}{C}_{[-1]}(E)$ is continuous (equivalently, closed) as a mapping from $\underset{\sim}{C}_{[-1]}(E)$ to the Fréchet space $\underset{\sim}{L}_{[0]}(E)$. For our purposes, we shall usually require a uniform condition of this type, namely:

(M): The restriction of the memory M to $\underset{\sim}{C}_{[-1]}(E)$ is a bounded linear mapping $M_{\underset{\sim}{C}}: \underset{\sim}{C}_{[-1]}(E) \to \underset{\sim}{M}_{[0]}(E)$.

Thus $M \mapsto \|M_{\underset{\sim}{C}}\|$ is a norm on the linear space of all memories satisfying (M).

In order to obtain an existence and uniqueness theorem for the initial-value problem as well as certain bounds for the growth of solutions, it seems necessary to impose an additional condition, expressing the "uniform local smallness" of the memory when acting on selected functions u. For this purpose, we assume that the memory M

satisfies (M), and define, for each interval $[a,b] \subset [0,\infty)$,

(4.3)
$$k(M;a,b) = \sup\{\|\int_t^{t'}(Mu)(s)ds\|: a \leqq t < t' \leqq b,\ u \in \underset{\sim}{C}_{[-1]}(E),\ \|u\| \leqq 1\}$$
$$k_0(M;a,b) = \sup\{\|\int_t^{t'}(Mu)(s)ds\|: a \leqq t < t' \leqq b,\ u \in \underset{\sim}{C}_{[-1]}(E),\ \|u\| \leqq 1,\ \chi_{[-1,t]}u = 0\}.$$

(Observe that, on account of (4.1), the value of $k_0(M;a,b)$ is not altered if $t = a$ is required in the definition.)

A (countable) set $S \subset [0,\infty)$ is _uniformly sparse_ if there exists a number N such that no interval $[j,j+1)$, $j \in \omega$, contains more than N points of S; the least number with this property is the _sparseness_ $sp(S)$; i.e., $sp(S) = \sup_{j\in\omega} \operatorname{card}(S \cap [j,j+1)) < \infty$. We need two trivial facts about uniformly sparse sets.

4.1. _Lemma. If S, S' are uniformly sparse sets, then so is $S \cup S'$, and_ $sp(S \cup S') \leqq sp(S) + sp(S')$.

4.2. _Lemma. Let $(a_n)_{n\in\omega}$ be a strictly increasing sequence in $[0,\infty)$ such that $\{a_n: n \in \omega\}$ is uniformly sparse, with sparseness N, say. Then $[a_{n'}] - [a_n] \geqq [(n'-n)/N]$, where $[\,]$ denotes the "greatest integer" function._

Let $\mathfrak{S}$ denote the class of all uniformly sparse sets $S \subset [0,\infty)$ such that $\omega \subset S$. If $S \in \mathfrak{S}$, then $S = \{a_n : n \in \omega\}$ for a well-defined strictly increasing sequence $(a_n)_{n\in\omega}$ with $a_0 = 0$, $a_{n+1} - a_n \leqq 1$, $n \in \omega$. For this S, and for a memory M satisfying (M), we set

$$k(M;S) = \sup_{n\in\omega} k(M;a_n,a_{n+1}) \qquad k_0(M;S) = \sup_{n\in\omega} k_0(M;a_n,a_{n+1}).$$

From (M) and (4.3) we have $k_0(M;S) \leqq k(M;S) \leqq \|M_{\underset{\sim}{C}}\|$; we may therefore define

$$\rho(M) = \inf\{k(M;S): S \in \mathfrak{S}\} \qquad \rho_0(M) = \inf\{k_0(M;S): S \in \mathfrak{S}\}$$

and find

(4.4)
$$\rho_0(M) \leqq \rho(M) \leqq \|M_{\underset{\sim}{C}}\|.$$

4.3. Lemma. p and p_n are seminorms on the linear space of all memories satisfying (M).

Proof. k(-;a,b) is a seminorm for each interval [a,b]; hence so is k(.;S) for each S *e* g. By Lemma 4.1, the set g ordered by inclusion is directed. Since k(M;a,b) increases with the interval [a;b], the net S**k(.;S) of seminorms is decreasing; therefore its limit, which is p, is itself a seminorm. The proof for p^O is the same.

Remark. The values of p(M) and $P_{\pi}(M)$ do not change if we replace the predicate "S e g^M in their definition by "S is a uniformly sparse infinite set with 0 *e* S"

The condition we shall have to impose in general on the memory M, in addition to (M), is $P_Q(M) < 1$.

There are several conditions on M that ensure that $o_{\pi}(M) = 0$ or even p(M) = 0; it follows from Lemma 4.3 that the condition $p^O(M) < 1$ is unaffected if terms satisfying such sufficient conditions are added to M. We shall discuss two of these conditions now; one includes a uniform version of the assumptions usually made in the literature to ensure existence and uniqueness of solutions. A third condition of this kind will be presented in Section 7.

We first consider memories that have a "gap" in their recollection of the immediate past. Specifically, a memory M is said to be uniformly delayed if there exists 6, 0 < 6 ≤ 1, such that

(4.5) $X_{\{-i, t-6]}u = \circ$ implies $X_{[0\ t]}(Mu) = \circ$ for all

u 6 K_f -,-,(E) and each t e (0,o>) .

(This could be combined with (4.1) into: $Xr_{[a-i,b-oj]}$ - K $RI(^U)$ = ^ implies $)(_{r\ ,\ n}(Mu) = 0$.) It is clear - and well known - that an equation (1.1) with a uniformly delayed memory can be solved by step-by-step integration; here we wish to include this case in our general treatment.

4.4. Lemma. If M is a uniformly delayed memory satisfying (M), then $\rho_0(M) = 0$.

Proof. With δ as in (4.5), let h be a positive integer so great that $h\delta \geqq 1$. For each $n \in \omega$, $\chi_{[-1,n/h]}u = 0$ then implies $\chi_{[-1,(n+1)/h - \delta]}u = 0$, which in turn implies, by (4.5), $\int_{n/h}^{(n+1)/h} \|(Mu)(s)\| ds = 0$. Thus $k_0(M;n/h,(n+1)/h) = 0$. But $S = \{n/h: n \in \omega\} \in \mathcal{S}$, and therefore $\rho_0(M) = k_0(M;S) = 0$.

A memory M is said to be uniformly narrow if there exists $\varphi \in \underset{\sim}{M}_{[0]}$, $\varphi \geqq 0$, such that

$$\|Mu\| \leqq |u|\varphi \quad \text{for all} \quad u \in \underset{\sim}{C}_{[-1]}(E). \tag{4.6}$$

4.5. Lemma. If M is a uniformly narrow memory, it satisfies (M) and $\rho(M) = \rho_0(M) = 0$.

Proof. With φ as in the definition, (M) is obviously satisfied, with $\|M_{\underset{\sim}{C}}\| \leqq |\varphi|_{\underset{\sim}{M}}$. Let the positive integer h be given; there exists a strictly increasing sequence $(a_n)_{n\in\omega}$ such that $a_{hn} = n$ for each $n \in \omega$, and $\int_{a_n}^{a_{n+1}} \varphi(s)ds \leqq |\varphi|_{\underset{\sim}{M}}/h$, $n \in \omega$. Then (4.6) and (4.3) imply that $k(M;a_n,a_{n+1}) \leqq |\varphi|_{\underset{\sim}{M}}/h$, $n \in \omega$. Now $S_h = \{a_n: n \in \omega\} \in \mathcal{S}$ (with $sp(S_h) = h$), and therefore $\rho_0(M) \leqq \rho(M) \leqq k(M;S_h) \leqq |\varphi|_{\underset{\sim}{M}}/h$. Since h was arbitrarily great, we conclude that $\rho_0(M) = \rho(M) = 0$.

Remark 1. The condition of uniform narrowness includes those usually imposed in the literature for the "Carathéodory case" (e.g., [4; p.30]), except that we assume the majorant φ in (4.6) to be in $\underset{\sim}{M}_{[0]}$ instead of merely in $\underset{\sim}{L}_{[0]}$ - hence the qualifier "uniform". We do not discuss here to what extent our definition provides - up to this uniformity - a genuine generalization of those conditions.

Remark 2. If $\hat{M} \in \underset{\sim}{M}_{[0]}([\underset{\sim}{E \to E}])$ and M is defined by $(Mu)(t) =$

= $\hat{M}(t)\Pi(t)u$, $t \in [0,\infty)$, $u \in K_{[0]}(E)$, then M is obviously a uniformly narrow memory, with $\varphi = \|\hat{M}\|$. It might be thought that most uniformly narrow memories are of this form, but this is not so. For instance, the condition that $\hat{M}$ be measurable with respect to the norm topology of $[E \to E]$, and hence almost-separable-valued [5; Theorem 3.5.3], excludes even such simple uniformly narrow memories as the M, given by $(Mu)(t) = u(t-A(t))$, where A is a continuously varying delay.

There is one special case of the kind of uniformly narrow memory described in Remark 2 that should be recorded separately.

4.6. Lemma. If $L \in M_{[0]}(E)$, the mapping M: $K_{[-1]}(E) \to L_{[0]}(E)$ defined by

(4.7) $\qquad (Mu)(t) = L(t)u(t) \qquad t \in [0,\infty)$

is a uniformly narrow memory, so that $P_n(M) = p(M) = 0$.

Proof. $\|Mu\| \leq \|u\|\,\|L\|$ for all $u \in K_{[-1]}(E)$.

5. Solutions.

We say that a function $f \in K_{[m]}(E)$ is a primitive (function) if there exists $g \in L_{[m]}(E)$ such that $f(t) - f(m) = \int_m^t g(s)ds$ for all $t \in [m,\infty)$; then g is unique, is denoted by $\dot{f}$, and is called the derivative of f.

Assume that we are given a memory M and, in addition, a function $r \in L_{[0]}(E)$. A solution of the "differential equation with delay"

(5.1) $\qquad \dot{u} + Mu = r$

is a function $u \in K_{[-1]}(E)$ whose restriction $u_{[0]}$ to $[0,\infty)$ is a primitive whose derivative $\dot{u}_{[0]}$ satisfies $\dot{u}_{[0]} + Mu = r$ in $L_{[0]}(E)$. More generally, for each $m \geq 0$, a solution of

$(5.1)_{[m]}$ $\qquad \dot{u}_{[m]} + M_{[m]}u = r_{[m]}$

is a function $u \in K_{[m-1]}(E)$ whose restriction $u_{[m]}$ to $[m,\infty)$ is a primitive whose derivative $\dot{u}_{[m]}$ satisfies $(5.1)_{[m]}$ in $L_{[m]}(E)$.

These definitions of course also apply to the homogeneous equations

(5.2) $$\dot{u} + Mu = 0$$

$(5.2)_{[m]}$ $$\dot{u}_{[m]} + M_{[m]}u = 0.$$

As usual, it is preferable to deal with integral equations equivalent to these differential equations.

5.1. Lemma. Let the memory M and $r \in L_{[m]}(E)$ be given. A function $u \in K_{[m-1]}(E)$ is a solution of $(5.1)_{[m]}$ if and only if it satisfies

(5.3) $$u(t) = u(m) - \int_m^t ((M_{[m]}u)(s) - r(s))ds$$

for all $t \geq m$. If $m' \geq m \geq 0$ and u is a solution of $(5.1)_{[m]}$, then $u_{[m'-1]}$ is a solution of $(5.1)_{[m']}$.

Proof. Definition of "solution" and (4.2).

Our next major aim is an existence-and-uniqueness theorem for solutions of $(5.1)_{[m]}$, with estimates on their growth.

5.2. Lemma. Let the memory M satisfy (M). Let the interval $[a,b] \subset [0,\infty)$ be given and satisfy $b-a \geq 1$ and $k_0(M;a,b) < 1$. For given $v \in E$, $r \in L_{[a]}(E)$ there exists $u \in K_{[a-1]}(E)$ such that

(5.4) $$\Pi(a)u = v$$

(5.5) $$u(t) = u(a) - \int_a^t ((M_{[a]}u)(s) - r(s))ds, \quad a \leq t \leq b.$$

The restriction of u to $[a-1,b]$ is uniquely determined by these properties; and u satisfies

(5.6) $$\|u(t)\| \leq (1 + (1-k_0)^{-1} [illegible]) \|v\| + (1-k_0)^{-1}\int_a^b \|r(s)\|ds, \quad a-1 \leq t \leq b,$$

where $k_0 = k_0(M;a,b)$.

Proof. Let A be the affine subspace of $C_{[a-1]}(E)$ consisting of those functions y that satisfy $\Pi(a)y = v$ and are constant on $[b,\infty)$. Consider the affine mapping $F: A \to A$ defined by

(5.7) $$(Fy)(t) = \begin{cases} y(t) = v(t-a) & a-1 \leq t \leq a \\ y(a) - \int_a^{\min\{b,t\}} ((M_{[a]}y)(s) - r(s))ds & a \leq t. \end{cases}$$

This mapping is well defined. It is contractive: indeed, $y, y' \in \underset{\sim}{A}$ implies $\chi_{[a-1,a]}(y'-y) = 0$ and hence, by (5.7), (4.1), (4.2), (4.3),

$$(5.8) \quad \|(Fy'-Fy)(t)\| = \begin{cases} 0 & a-1 \leqq t \leqq a \\ \left\| \int_a^{\min\{b,t\}} ((M_{[a]}(y'-y))(s)ds \right\| & a \leqq t; \end{cases} \leqq k_0 \| y'-y \|$$

and $k_0 < 1$ by assumption. Therefore F has a unique fixed point, say u_0.

Condition (4.1) implies that $u \in \underset{\sim}{K}_{[a-1]}(E)$ satisfies (5.4), (5.5) if and only if u coincides on $[a-1,b]$ with a fixed point of F; i.e., precisely with u_0; this establishes the existence of u satisfying (5.4), (5.5) and the uniqueness of its restriction to $[a-1,b]$.

Define $w \in \underset{\sim}{A}$ by $\Pi(a)w = v$ and $w(t) = w(a) = v(0)$, $t \geqq a$. Since $b-a \leqq 1$, (5.7) implies

$$\| Fw - w \| \leqq \int_a^b \|(M_{[a]}w)(s)\| ds + \int_a^b \|r(s)\| ds \leqq \|\underset{\sim}{M_C}\| \| w \| + \int_a^b \|r(s)\| ds.$$

Since $u_0 = Fu_0$, it follows from this estimate and (5.8) that

$$\| u_0 - w \| \leqq \| Fu_0 - Fw \| + \| Fw - w \| \leqq k_0 \| u_0 - w \| + \|M_{\underset{\sim}{C}}\| \| w \| + \int_a^b \|r(s)\| ds;$$

finally, since $\| w \| = \| v \|$, we conclude that

$$\| u_0 \| \leqq \| w \| + \| u_0 - w \| \leqq \| v \| + (1-k_0)^{-1}(\|M_{\underset{\sim}{C}}\| \| v \| + \int_a^b \|r(s)\| ds).$$

Since u coincides with u_0 on $[a-1,b]$, (5.6) holds.

5.3. <u>Lemma</u>. <u>Let the memory</u> M <u>satisfy</u> (M) <u>and</u> $\rho_0(M) < 1$. <u>Then there exist positive numbers</u> σ, C, C' <u>with the following properties</u>. <u>For given</u> m <u>and</u> t_0, $t_0 \geqq m \geqq 0$, <u>and given</u> $v \in \underset{\sim}{E}$ <u>and</u> $r \in \underset{\sim}{L}_{[0]}(E)$, <u>there exists</u> $u \in \underset{\sim}{K}_{[m-1]}(E)$ <u>satisfying</u> $\Pi(m)u = v$ <u>and</u> (5.3) <u>for all</u> t, $m \leqq t \leqq t_0$; <u>the restriction of</u> u <u>to</u> $[m-1,t_0]$ <u>is uniquely determined by these properties</u>; <u>and</u>

$$\|u(t_0)\| \leqq Ce^{\sigma(t_0-m)} \| v \| + C' \int_m^{t_0} e^{\sigma(t_0-s)} \|r(s)\| ds.$$

<u>Proof</u>. 1. Since $\rho_0(M) < 1$ there exists a uniformly sparse set $S \subset [0,\infty)$ including the integers and such that $k_0 = k_0(M;S) < 1$.

Set $N = sp(S)$. We claim that the conclusion holds with

(5.9) $\quad \sigma = N \log K, \quad C = K^{2N+2}, \quad C' = (1-k_0)^{-1}K^{2N+1}$, where

$$K = 1 + (1-k_0)^{-1}\|\underset{\sim}{M}_C\|.$$

We know that S is the range of a strictly increasing sequence $(a_n)_{n\in\omega}$ with $a_0 = 0$, $\lim_{n\to\infty} a_n = \infty$. Let now m and t_0 be given, with $t_0 > m \geqq 0$ (the conclusion is trivial for $t_0 = m$); define the integers $n_0 \geqq 0$, $h > 0$ by $a_{n_0} \leqq m < a_{n_0+1}$, $a_{n_0+h-1} < t_0 \leqq a_{n_0+h}$. Set $b_0 = m$, $b_n = a_{n_0+n}$ for $n = 1,\ldots,h-1$, and $b_h = t_0$. Then $0 < b_n - b_{n-1} \leqq 1$ and $k_0(M;b_{n-1},b_n) \leqq k_0(M;S) = k_0 < 1$, $n = 1,\ldots,h$.

2. Let $v \in \underset{\sim}{E}$ and $r \in \underset{\sim}{L}_{[0]}(E)$ be given. We claim that there is a sequence $(u_0,\ldots,u_h)$ in $\underset{\sim}{K}_{[m-1]}(E)$ satisfying

(5.10) $$\Pi(m)u_n = v$$

(5.11) $$\chi_{[m-1,b_i]}u_n = \chi_{[m-1,b_i]}u_i \qquad i = 0,\ldots,n$$

(5.12) $$u_n(t) = u_n(m) - \int_m^t ((M_{[m]}u_n)(s) - r(s))ds \qquad m \leqq t \leqq b_n$$

for each $n = 0,\ldots,h$, and that the restriction of each u_n to $[m-1,b_n]$ is uniquely determined by these conditions.

For $n = 0$, the conditions merely require $\Pi(m)u_0 = v$, u_0 being otherwise arbitrary. Assume that $0 < j \leqq h$ and that $u_0,\ldots,u_{j-1}$ satisfy (5.10), (5.11), (5.12) for $n = 0,\ldots,j-1$, and that these conditions determine the restriction of u_{j-1} to $[m-1,b_{j-1}]$ uniquely. Then $u_j \in \underset{\sim}{K}_{[m-1]}(E)$ satisfies (5.10), (5.11), (5.12) with $n = j$ if and only if

(5.13) $$\chi_{[m-1,b_{j-1}]}u_j = \chi_{[m-1,b_{j-1}]}u_{j-1}$$

(5.14) $$u_j(t) = u_j(b_{j-1}) - \int_{b_{j-1}}^t ((M_{[m]}u_j)(s) - r(s))ds \qquad b_{j-1} \leqq t \leqq b_j;$$

by (4.2), this will be the case if and only if (5.13) holds and $u' = (u_j)_{[b_{j-1}-1]} \in \underset{\sim}{K}_{[b_{j-1}-1]}(E)$ satisfies

$$\Pi(b_{j-1})u' = \Pi(b_{j-1})u_{j-1}$$

$$u'(t) = u'(b_{j-1}) - \int_{b_{j-1}}^t ((M_{[b_{j-1}]}u')(s) - r(s))ds \qquad b_{j-1} \leqq t \leqq b_j$$

Now Lemma 5.2 shows that such a u^1 exists and that its restriction to $[b_{j-1}-1,b_j]$ is uniquely determined; it follows that u_j satisfying (5.10), (5.11), (5.12) for $n = j$ exists (define it to coincide with u_{j-1} on $[m-1,b_{j-1}]$ and with u^1 on $[b_{j-1}-1,b_j]$), and that its restriction to $[m-1,b_j]$ is uniquely determined by these conditions. The existence and claimed properties of the sequence $(u_0,\ldots,u_h)$ have thus been established by induction on j.

Now $u = u_h$ satisfies $\Pi(m)u = v$ and (5.3) for $m \le t \le b_h = t_0$ (by (5.10), (5.12)); conversely, if $u \in K_{r,[m-1]}(E)$ satisfies $\Pi(m)u = v$ and (5.3) for $m \le t \le t_0$, the constant sequence defined by $u_n = u$, $n = 0,\ldots,h$, satisfies (5.10), (5.11), (5.12) for each n; therefore the restriction of $u = u_h$ to $[m-1,t_0] = [m-1,b_h]$ is uniquely determined. We remark that up to this point the uniform sparseness of S has not been used.

3. Let $u \in K_{r,[m-1]}(E)$ satisfy $\Pi(m)u = v$ and (5.3) for $m \le t \le t_0$ and set $\alpha_n = \max\{\|u(t)\|: m-1 \le t \le b_n\}$, $n = 0,\ldots,h$, so that $\alpha_0 = |v|$. Using (4.2) we may now apply Lemma 5.2 to b_{n-1}, b_n, and $u|_{[b_{n-1}-1]}$ instead of a, b, and u, and find, with K as in (5.9),

$$\|u(t)\| = \|u_{[b_{n-1}-1]}(t)\| \le K\|\Pi(b_{n-1})u_{[b_{n-1}-1]}\| + (1-k_0)^{-1}\int_{b_{n-1}}^{b_n}\|r(s)\|ds =$$

$$= K\|\Pi(b_{n-1})u\| + (1-k_0)^{-1}\int_{b_{n-1}}^{b_n}\|r(s)\|ds, \quad b_{n-1} \le t \le b_n, \quad n = 1,\ldots,h.$$

This implies $\alpha_n \le K\alpha_{n-1} + (1-k_0)^{-1}\int_{b_{n-1}}^{b_n}\|r(s)\|ds$, $n = 1,\ldots,h$, and hence

(5.15) $$\|u(t_0)\| \le \alpha_h \le K^h|v| + (1-k_0)^{-1}\sum_{n=1}^{h} K^{h-n}\int_{b_{n-1}}^{b_n}\|r(s)\|ds.$$

From the definition of $b_0,\ldots,b_h$, and from Lemma 4.2 we have

$$h-n = (h-1) - n + 1 \le N(a_{n+1} - a_n + 2) + 1 \le N(t_0-b_n+2) + 1,$$

$$n = 1,\ldots,h,$$

$$h = (h-1) + 1 \le N(t_0-b_1+2) + 2 \le N(t_0-m+2) + 2.$$

Therefore (5.15) implies

$$\|u(t_0)\| \le K^{N(t_0-m+2)+2}|v| + (1-k_0)^{-1}\sum_{n=1}^{h}\int_{b_{n-1}}^{b_n} K^{N(t_0-s+2)+1}\|r(s)\|ds =$$

$$= Ce^{\sigma(t_0-m)}|v| + C'\int_m^{t_0} e^{\sigma(t_0-s)}\|r(s)\|ds,$$

where σ, C, C' are as given in (5.9).

5.4. Theorem. *Let the memory* M *satisfy* (M) *and* $\rho_0(M) < 1$. *Then there exist positive numbers* σ, C, C', *and, for each real* $m \geq 0$, *there exist linear mappings* $P(m): \underset{\sim}{E} \to \underset{\sim}{K}_{[m-1]}(E)$ *and* $Q(m): \underset{\sim}{L}_{[0]}(E) \to \underset{\sim}{K}_{[m-1]}(E)$ *such that for every* $v \in \underset{\sim}{E}$ *and* $r \in \underset{\sim}{L}_{[0]}(E)$:

(1): $u = P(m)v + Q(m)r$ *is the unique solution of* $(5.1)_{[m]}$ *with* $\Pi(m)u = v$;

(2): if $t_0 \geq m$, *then* $u \in \underset{\sim}{K}_{[m-1]}(E)$ *satisfies* $\Pi(m)u = v$ *and* (5.3) *for* $m \leq t \leq t_0$, *if and only if* u *and* $P(m)v + Q(m)r$ *coincide on* $[m-1, t_0]$;

(3): *for all* $t \geq m$,

$$\|(P(m)v)(t)\| \leq Ce^{\sigma(t-m)}|v|, \qquad \|(Q(m)r)(t)\| \leq C'\int_m^t e^{\sigma(t-s)}\|r(s)\|ds.$$

Proof. We choose σ, C, C' as in Lemma 5.3. Let $m \geq 0$, $v \in \underset{\sim}{E}$, $r \in \underset{\sim}{L}_{[0]}(E)$ be given. For each $t_0 \geq m$ there is, by Lemma 5.3, a function $u_{t_0} \in \underset{\sim}{K}_{[m-1]}(E)$ satisfying $\Pi(m)u_{t_0} = v$ and (5.3) for $m \leq t \leq t_0$; and the restriction of u_{t_0} to $[m-1, t_0]$ is uniquely determined by these conditions. It follows that if $t_1 \geq t_0 \geq m$, then u_{t_1} and u_{t_0} coincide on $[m-1, t_0]$. There exists, therefore, a function $u \in \underset{\sim}{K}_{[m-1]}(E)$ that coincides with u_{t_0} on $[m-1, t_0]$ *for each* $t_0 \geq m$; it follows that $\Pi(m)u = v$ and u satisfies (5.3) for all $t \geq m$; by Lemma 5.1, u is a solution of $(5.1)_{[m]}$.

Conversely, if u is a solution of $(5.1)_{[m]}$ with $\Pi(m)u = v$, it satisfies (5.3) for all $t \geq m$ (Lemma 5.1); by Lemma 5.3, its restriction to $[m-1, t_0]$ is uniquely determined by these assumptions for every $t_0 \geq m$; it is therefore itself unique.

This unique solution u depends linearly on v and r; the linear mappings $P(m)$ and $Q(m)$ such that $u = P(m)v + Q(m)r$ is this solution are therefore well defined. Part (1) of the conclusion has thus been proved, and Part (2) follows from Lemma 5.3.

From Part (2) and Lemma 5.3 we also have

$$\|(P(m)v+Q(m)r)(t)\| \leqq Ce^{\sigma(t-m)}|v| + C'\int_m^t e^{\sigma(t-s)}\|r(s)\|ds$$

for all $t \geqq m$. Since this holds for every v and every r, Part (3) of the conclusion follows.

5.5. Corollary. *Let the memory M satisfy (M) and $\rho_0(M) < 1$. If u is a solution of $(5.2)_{[m]}$ for some $m \geqq 0$, then*

$$|\Pi(t)u| \leqq Ce^{\sigma(t-t_0)}|\Pi(t_0)u| \quad \text{for all } t \geqq t_0 \geqq 0,$$

where σ, C are as in Theorem 5.4.

Proof. $u_{[t_0-1]}$ is a solution of $(5.2)_{[t_0]}$ (Lemma 5.1); the conclusion follows by applying Theorem 5.4, Parts (1) and (3), to this solution, and observing that $C \geqq 1$ by (5.9).

6. The associated difference equation.

Let us assume that the memory M satisfies (M) and $\rho_0(M) < 1$. We construct a linear difference equation in $\underset{\sim}{E}$ in such a way that the values of a solution of this equation are the slices of a solution of (5.1). For this purpose, we define the linear mappings

$$(6.1) \qquad \begin{aligned} A(n) &= -\Pi(n)P(n-1): \underset{\sim}{E} \to \underset{\sim}{E} \\ B(n) &= \Pi(n)Q(n-1): \underset{\sim}{L}_{[0]}(E) \to \underset{\sim}{E} \end{aligned} \qquad n = 1,2,\ldots$$

and observe that Theorem 5.4,(3) implies

$$(6.2) \qquad \begin{aligned} &A(n) \in \underset{\sim}{\tilde{E}}, \qquad \|A(n)\| \leqq Ce^{\sigma}, \qquad n = 1,2,\ldots \\ &|B(n)r| \leqq C'e^{\sigma}|(\varpi r)(n)|_1, \quad n = 1,2,\ldots, \quad r \in \underset{\sim}{L}_{[0]}(E). \end{aligned}$$

We set $A = (A(n)) \in \underset{\sim}{l}^{\infty}_{[1]}(\underset{\sim}{\tilde{E}})$ and define the linear mapping $B: \underset{\sim}{L}_{[0]}(E) \to \underset{\sim}{s}_{[1]}(\underset{\sim}{E})$ by $(Br)(n) = B(n)r$, $n = 1,2,\ldots$, $r \in \underset{\sim}{L}_{[0]}(E)$.

With A thus defined, we consider the following difference equations in $\underset{\sim}{E}$:

$$(6.3) \qquad x(n) + A(n)x(n-1) = f(n) \qquad n = 1,2,\ldots$$

$$(6.4) \qquad x(n) + A(n)x(n-1) = 0 \qquad n = 1,2,\ldots$$

and their restrictions $(6.3)_{[m]}$ and $(6.4)_{[m]}$ to $n = m+1, m+2, \ldots$ for each $m \in \omega$. Here $f \in \underset{\sim}{s}_{[1]}(\underset{\sim}{E})$.

The fact that (6.3) and (6.4) are, in some sense, reduced forms of (5.1) and (5.2) is expressed by the following proposition.

6.1. Lemma. *Let* $m \in \omega$ *and* $r \in \underset{\sim}{L}_{[0]}(E)$ *be given. A function* $x \in \underset{\sim}{s}_{[m]}(\underset{\sim}{E})$ *is a solution of* $(6.3)_{[m]}$ *with* $f = Br$ *if and only if* $x = \varpi u$ *for some solution* u *of* $(5.1)_{[m]}$. *In particular,* x *is a solution of* $(6.4)_{[m]}$ *if and only if* $x = \varpi u$ *for some solution* u *of* $(5.2)_{[m]}$.

Proof. This is a direct consequence of Theorem 5.4,(1) and (6.1), via a straightforward computation. The details can be found in the proof of [10; Lemma 6.1], which could be reproduced here verbatim.

As usual, the main problem in applying difference-equation theory via Lemma 6.1 to our equations (5.1), (5.2) is that not every $f \in \underset{\sim}{s}_{[1]}(\underset{\sim}{E})$ is of the form $f = Br$. Our fundamental theorem (Theorem 8.1) states that it is still possible, however, to relate equation (6.3) with arbitrary f to equation (5.1) with a suitable r.

The amount of information obtainable from the use of equations (6.3) and (6.4) is considerably greater when the operators $A(n)$ are known to be compact. It is easy to see that this can happen only when E is finite-dimensional; for this case we now provide a simple compactness criterion.

6.2. Lemma. *If* E *is finite-dimensional and* $\rho(M) = 0$, *then each* $A(n)$ *is a compact operator*.

Proof. Let $n \in \omega_{[1]}$ be given. Since $\rho(M) = 0$, there exists, for given $\varepsilon > 0$, a set $S \in \mathcal{S}$ such that $k(M;S) \leqq \frac{1}{2}\varepsilon$. Let $\delta > 0$ be the least distance between distinct points of the finite set $[n-1,n] \cap S$; then clearly $k(M;a,b) \leqq \varepsilon$ for $[a,b] \subset [n-1,n]$, $b-a \leqq \delta$.

For given $v \in E$ let $u \in C_{r,n}(E)$ coincide with $P(n-1)v$ on $[n-2,n]$ and be constant on either side of this interval. By Theorem 5.4,(3) and (5.9), we have $|u| \leq Ce^{Q}|v|$. By Theorem 5.4,(2) and (4.2) we have

$$u(t) = u(n-1) - \int_{n-1}^{t} (M_{[n-1]} u_{[n-2]})(s)ds = u(n-1) - \int_{n-1}^{t} (Mu)(s)ds,$$
$$n-1 \leq t \leq n.$$

Therefore (4.3), (6.1) and the preceding argument show that

$$\|(A(n)v)(s') - (A(n)v)(s)\| = \|(P(n-1)v)(n+s') - (P(n-1)v)(n+s)\| =$$
$$= \|u(n+s') - u(n+s)\| = \|\int_{n+s}^{n+s'} (Mu)(s)ds\| \leq |u|k(M;n+s,n+s') \leq Ce^{Q}|v|\varepsilon$$

for all $s,s' \in [-1,0]$, $0 < s'-s \leq \delta$. Since ε was arbitrarily small, we conclude from this and (6.2) that the image under $A(n)$ of the unit ball of $\tilde{E}$ is a bounded equicontinuous set of continuous functions $[-1,0] \to E$. When E is finite-dimensional, it follows from the Arzela-Ascoli Theorem that the closure in $\tilde{E}$ of that image is compact; hence $A(n)$ is a compact operator.

7. More conditions on the memory.

We shall wish to investigate equation (5.1) by allowing r to range over a suitable function space. Our methods will be applicable if the behaviour of the memory M is adapted to the local properties of the functions in such a space.

For a memory M, Condition (M) may be rephrased as follows: The restriction of the composite mapping $\mathcal{P}M$: $K_{r,n-1}(E) \to S_{r,n-1}(L^1(E))$ to $C_{r,n-1}(E)$ is a bounded linear mapping from $C_{r,n-1}(E)$ to $S_{r,n-1}(L^1(E))$; the norm of this mapping, incidentally, lies between $\frac{1}{2}\|M\|$ and $\|M\|$. The condition we now envisage is a more restrictive assumption of the same type on the slices of Mu. For each given space $F \in \mathfrak{F}$ (see Section 2), we consider the following condition on a memory M:

(M_F): The restriction of $\mathcal{P}M$ to $C_{r,1}(E)$ is a bounded linear mapping from $C_{r\eta}(E)$ to $L^{\infty}_{\eta}(F(E))$. The norm of this mapping shall be denoted by $\|M\|_{F}$.

Certain special cases of this condition are easier to state. We have already noted that (M_{L^1}) is equivalent to (M); and since every space $F \in \mathcal{B}$ is stronger than L^1, each condition (M_F) implies (M). In the same vein, (M_{L^∞}) may be rephrased as: The restriction of M to $C_{r\eta}(E)$ is a bounded linear mapping from $C_{r\eta}(E)$ to $L^{\infty}_{r\eta}(E)$. Similar rephrasings, involving other translation-invariant function spaces, are of interest for $F = L^p$, $1 < p < \infty$, among others, and may be supplied by the reader.

In addition to the part Condition (M_F) will play in making the memory amenable to our methods, this condition is also sometimes sufficient to ensure that $p_\eta(M) = p(M) = 0$, as we now show.

We shall say that a space $F \in \mathcal{B}$ is tame if for every $\varepsilon > 0$ there is a positive integer h such that

$$(7.1) \quad \int_{-i/h}^{-(i-1)/h} |\varphi(t)|dt \leq \varepsilon|\varphi|_F, \text{ for all } i = 1,\dots,h \text{ and all } \varphi \in F.$$

Since $[-1,0]$ is compact, this is equivalent to assuming that for each $t \in [-1,0]$ and $\varepsilon > 0$ there exists an interval $[a,b]$ such that $t \in [a,b] \subset [-1,0]$ and $\int_a^b |\varphi(s)|ds \leq \varepsilon|\varphi|_F$ for all $\varphi \in F$. We note, in particular, that L^p is tame for $1 < p \leq \infty$: indeed, it is sufficient to choose $h \geq \varepsilon^{-p/(p-1)}$.

We shall say that a memory M is tame if it satisfies (M_F) for some tame $F \in \mathcal{B}$. As observed above, a tame memory satisfies (M).

7.1. Lemma. If M is a tame memory, then $p_\eta(M) = p(M) = 0$.

Proof. Let $F \in \mathcal{B}$ be the tame space such that M satisfies (M_F). Let $\varepsilon > 0$ be given, and choose the positive integer h so as to satisfy (7.1). The set $S_h = \{n/h: n \in \omega\}$ is uniformly sparse and contains the positive integers. For given $n \in \omega$, choose $j \in \omega\setminus\{0\}$ so that $j-1 \leq n/h < (n+1)/h \leq j$; then (M_F) implies, for each $u \in C_{r\eta}(E)$,

$$\int_{n/h}^{(n+1)/h} \|(Mu)(t)\| dt = \int_{-(jh-n)/h}^{-(jh-n-1)/h} \|((\varpi Mu)(j))(s)\| ds \leqq$$

$$\leqq \varepsilon \|(\varpi Mu)(j)\|_{\underset{\sim}{F}} \leqq \varepsilon \| \|\varpi Mu\|_{\underset{\sim}{F}} \| \leqq \varepsilon \|\varpi M\|_{\underset{\sim}{F}} \|u\| ;$$

then $\rho(M) \leqq k(M;S_h) \leqq \varepsilon \|\varpi M\|_{\underset{\sim}{F}}$; but $\varepsilon > 0$ was arbitrary.

Remark 1. An important special kind of memory is, of course, the autonomous or time-independent memory; i.e., more precisely, a memory that commutes with left-translations. It will be shown in a future paper that if E is isomorphic to a Hilbert space (in particular finite-dimensional), an autonomous memory satisfies $(M_{\underset{\sim}{L}}2)$, so that such a memory is always tame.

In this section we have spoken as if the memory functional M appearing in (5.1) were to be itself subjected to Condition $(M_{\underset{\sim}{F}})$. In actual fact, however, it is typical of the problems we are dealing with that the condition need only be imposed on the dependence of Mu on the past of u, while its dependence on the current value of u is less restricted. The standard assumptions we shall make are stated in the following lemma.

7.2. Lemma. *Let $L \in \underset{\sim}{M}_{[0]}(\tilde{E})$ be given and define M_L by (4.7). Let $\underset{\sim}{F} \in b\mathfrak{F}$ be given, and assume that the memory M' satisfies $(M_{\underset{\sim}{F}})$ and $\rho_0(M') < 1$, Then the memory $M = M_L + M'$ satisfies (M) and $\rho_0(M) < 1$, so that the conclusions of Theorem 5.4, Corollary 5.5, and Lemma 6.1 hold. If, in addition, $\rho(M') = 0$ and E is finite-dimensional, the conclusion of Lemma 6.2 also holds.*

Proof. Lemmas 4.6 and 4.3.

7.3. Scholium. The condition $\rho_0(M') < 1$ in Lemma 7.2 appears to be the one most difficult to verify. However, we know that indeed $\rho_0(M') = 0$ if M' is a sum of memories each one of which is uniformly delayed, uniformly narrow, or tame (Lemmas 4.3, 4.4, 4.5, 7.1), hence in particular if the space $\underset{\sim}{F}$ is tame (since then M' is itself tame). If M' is a sum of uniformly narrow and tame memories, or in particular if $\underset{\sim}{F}$ is tame, then we also have $\rho(M') = 0$, so that the conclusion of Lemma 6.2 holds.

Remark 2. The assumptions of Lemma 7.2 with M' uniformly delayed are precisely those considered in [2] for the Carathéodory case (up to an obvious change in time-scale); the results pertaining to this case in [2] are thus subsumed in the present paper. We note, however, that [2; Lemma 8.1], asserting that the transition operators are compact for finite-dimensional E, is invalid on account of an error in the proof.

Remark 3. The assumptions of Lemma 7.2 with $L = 0$ and $M = M'$ uniformly narrow include those considered by Pecelli [7]. Most of the results in [7] can, as a consequence, be obtained by a specialization of the methods and results of the present paper.

8. The fundamental theorem.

We now return to the basic problem of using the difference equations (6.3), (6.4) to obtain information, via Lemma 6.1, about equations (5.1), (5.2). As in earlier work, the core of our method is a proposition that permits us to infer properties of (6.3) with arbitrary f - and not just those of the form $f = Br$ - from information on (5.1).

We assume throughout this section that $M = M_L + M'$ satisfies the assumptions of Lemma 7.2 with respect to some given space $\underset{\sim}{F} \in b\mathfrak{F}$, so that Theorem 5.4 and Lemma 6.1 are applicable. We assume that A, B are as defined in Section 6.

Let $V \in \underset{\sim}{K}_{[0]}(\widetilde{E})$ be the unique solution of the operator differential equation $\dot{V} + LV = 0$ that satisfies $V(0) = I$ (I is the identity on E). We refer to [6; Section 31] for a detailed account of this operator-valued function. In particular, V is invertible-valued, and as usual we write $V^{-1} \in K_{[0]}(\tilde{E})$ for the function defined by $V^{-1}(t) = (V(t))^{-1}$, $t \geqq 0$. We also have

$$(8.1) \qquad \|V(t)V^{-1}(s)\| \leqq \exp\left|\int_s^t \|L(\sigma)\| d\sigma\right|, \qquad s,t \geqq 0.$$

8.1. Theorem. _Assume that_ $M = M_L + M'$ _satisfies the assumptions of Lemma 7.2 with respect to a given space_ $F \in b\mathfrak{F}$. _For each_ $f \in s_{[1]}(E)$ _there exists_ $r \in L_{[0]}(E)$ _with_ $\varpi r \in s_{[1]}(F(E))$ _such that_

$$(8.2) \qquad |(\varpi r)(n)|_F \leqq c_0(|f(n-1)| + |f(n)|), \qquad n = 1,2,\ldots,$$

and such that the solution w _of_

$$(8.3) \qquad w(n) + A(n)w(n-1) = f(n) - (Br)(n), \qquad n = 1,2,\ldots$$

with $w(0) = 0$ _satisfies_

$$(8.4) \qquad |w(n)| \leqq (1 + \exp|L|_M)|f(n)|, \qquad n = 0,1,\ldots,$$

where we set $f(0) = 0$, _and_ $c_0 > 0$ _depends only on_ F, $|L|_M$, _and_ $\|\varpi M'\|_F$.

Proof. There exists $\varphi \in F$ such that $\varphi \geqq 0$ and $\int_{-1}^{0}\varphi(s)ds = 1$. We define $w \in s_{[0]}(E)$ by

$$(8.5) \quad (w(n))(s) = (f(n))(s) - \left(\int_{-1}^{s}\varphi(\sigma)d\sigma\right)V(n+s)V^{-1}(n)(f(n))(0),$$
$$-1 \leqq s \leqq 0, \quad n = 0,1,\ldots .$$

It is obvious that each $w(n)$ is continuous, hence in E, and that $w(0) = 0$, as required. Also,

$$(8.6) \quad (w(n))(-1) = (f(n))(-1), \quad (w(n))(0) = 0, \qquad n = 0,1,\ldots;$$

and (8.5) and (8.1) yield

$$|w(n)-f(n)| \leqq |f(n)|\exp|L|_M ,$$

so that (8.4) holds.

We now construct r. For this purpose we choose, for each $n \in \omega_{[1]}$, a function $z_n \in C_{[n-2]}(E)$ such that

(8.7) $\quad \Pi(n-1)z_n = -\, w(n-1) \qquad \Pi(n)z_n = f(n) - w(n)$

and such that z_n is constant on $[n,\infty)$; this is possible on account of (8.6). Then

$$(8.8)\quad \|z_n\| = \max\{\|w(n-1)\|,\|f(n)-w(n)\|\} \leqq \max\{\|f(n-1)\|(1+\exp\|L\|_{\underset{\sim}{M}}), \|f(n)\|\exp\|L\|_{\underset{\sim}{M}}\}.$$

We now define $r \in \underset{\sim}{L}_{[0]}(E)$ by

$$(8.9)\quad r(t) = \varphi(t-n)V(t)V^{-1}(n)(f(n))(0) + (M'_{[n-1]}z_n)(t), \qquad n-1 < t \leqq n, \quad n = 1,2,\ldots .$$

From (8.1) and the fact that M' satisfies $(M_{\underset{\sim}{F}})$ it follows that $(\varpi r)(n) \in \underset{\sim}{F}(E)$ and

$$\|(\varpi r)(n)\|_{\underset{\sim}{F}} \leqq \|\varphi\|_{\underset{\sim}{F}}\|f(n)\|\exp\|L\|_{\underset{\sim}{M}} + \|\varpi M'\|_{\underset{\sim}{F}}\|z_n\| \qquad n = 1,2,\ldots;$$

combining this with (8.8) we find (8.2) with

$$c_0 = \|\varpi M'\|_{\underset{\sim}{F}}\exp\|L\|_{\underset{\sim}{M}} + \max\{\|\varpi M'\|_{\underset{\sim}{F}}\ ,\ \|\varphi\|_{\underset{\sim}{F}}\exp\|L\|_{\underset{\sim}{M}}\}.$$

It remains for us to prove that w and r thus constructed satisfy (8.3). For this purpose, let $n \in \omega_{[1]}$ and t, $n-1 < t \leqq n$, be fixed for the time being. In the following computation we use in succession: (8.7) and (8.6); (8.5); differentiation of products and the definition of V; (8.9) and (8.5); (8.7); the definition of M and (4.7).

$$\begin{aligned} z_n(t) - z_n(n-1) &= (f(n) - w(n))(t-n) = \left(\int_{-1}^{t-n}\varphi(\sigma)d\sigma\right)V(t)V^{-1}(n)(f(n))(0) = \\ &= \int_{n-1}^{t}\left(\varphi(s-n)V(s) - \left(\int_{-1}^{s-n}\varphi(\sigma)d\sigma\right)L(s)V(s)\right) V^{-1}(n)(f(n))(0)ds = \\ &= -\int_{n-1}^{t}((M'_{[n-1]}z_n)(s) - r(s) + L(s)(f(n)-w(n))(s-n))ds = \\ &= -\int_{n-1}^{t}((M'_{[n-1]}z_n)(s) - r(s) + L(s)z_n(s))ds = -\int_{n-1}^{t}((M_{[n-1]}z_n)(s)-r(s))ds. \end{aligned}$$

Since this equality holds for all t, $n-1 < t \leqq n$, it follows from Theorem 5.4,(2) that z_n coincides on $[n-2,n]$ with $P(n-1)\Pi(n-1)z_n + Q(n-1)r = -\,P(n-1)w(n-1) + Q(n-1)r$. Combining this with (8.7) and (6.1) we find

$$f(n) - w(n) = \Pi(n)z_n = \Pi(n)(-P(n-1)w(n-1) + Q(n-1)r) =$$
$$= A(n)w(n-1) + B(n)r;$$

that is, $w(n-1)$ and $w(n)$ satisfy (8.3) for the given n. Since n was arbitrary, the proof is complete.

9. *Admissibility*.

The purpose of Theorem 8.1 was to allow us to replace the study of equations (5.1), (5.2) by that of the difference equations (6.3), (6.4); in this section and the next we propose to show how the method works. We shall assume that the memory M satisfies the assumptions of Lemma 7.2 and that A and B are as defined in Section 6.

We suppose that the reader is acquainted with the concept of $\mathfrak{t}$-*pairs* and $\mathfrak{t}^{\rightarrow}$-*pairs* of sequence spaces, i.e., pairs $(\underset{\sim}{b},\underset{\sim}{d})$ of sequence spaces with $\underset{\sim}{b} \in b\mathfrak{t}$ or $\underset{\sim}{b} \in b\mathfrak{t}^{\rightarrow}$, respectively, and $\underset{\sim}{d} \in b\mathfrak{t}$ in both cases (the classes $b\mathfrak{t}$ and $b\mathfrak{t}^{\rightarrow}$ of translation-invariant sequence spaces are discussed in [1; Section 3]. We recall that such a pair $(\underset{\sim}{b},\underset{\sim}{d})$ is *admissible for* A (or for (6.3)) if (6.3) has a solution $x \in \underset{\sim}{d}_{[0]}(E)$ for every $f \in \underset{\sim}{b}_{[1]}(E)$. For details see [1; Section 8].

9.1. *Theorem. Assume that the memory* $M = M_L + M'$ *satisfies the assumptions of Lemma 7.2 with respect to a given space* $\underset{\sim}{F} \in b\mathfrak{F}$. *For each given* $\mathfrak{t}^{\rightarrow}$*-pair (or, in particular,* $\mathfrak{t}$*-pair)* $(\underset{\sim}{b},\underset{\sim}{d})$ *the following statements are equivalent*:

(a): $\underset{\sim}{b}$ *is stronger than* $\underset{\sim}{d}$; *and for every* $r \in \underset{\sim}{L}_{[0]}(E)$ *with* $\varpi r \in \underset{\sim}{b}_{[1]}(\underset{\sim}{F}(E))$ *equation* (5.1) *has a solution* u *with* $\varpi u \in \underset{\sim}{d}_{[0]}(\underset{\sim}{E})$;

(b): $(\underset{\sim}{b},\underset{\sim}{d})$ *is admissible for* A.

Proof. (a) *implies* (b): Let $f \in \underset{\sim}{b}_{[1]}(E)$ be given, and let r, w be as provided by Theorem 8.1. Since $\underset{\sim}{b} \in b\mathfrak{t}^{\rightarrow}$, (8.2) implies

$\varpi r \in b_{[1]}(F(E))$. Further, (8.4) implies $w \in b_{[0]}(E)$, whence $w \in d_{[0]}(E)$ since b is stronger than d.

By the assumption, (5.1) with this r has a solution u such that $\varphi u \in d_{[0]}(E)$. By Lemma 6.1 we have $(\Phi u)(n) + A(n)(\Phi u)(n-1) = (Br)(n)$, $n = 1,2,\ldots$; since w is a solution of (8.3), we conclude that $x = \Phi u + w \in d_{[0]}(E)$ is a solution of (6.3). Thus (b,d) is admissible for A.

(b) implies (a): Since $A \in l^{\infty}_{[1]}(E)$ (by (6.2)) and (b,d) is admissible for A, we conclude that b is stronger than d [8; Lemma 4.1]. Let now $r \in L_{[0]}(E)$ be given with $\varpi r \in b_{[1]}(F(E))$. Then (6.2) and the fact that F satisfies Condition (N) (Section 2) imply $|Br| \le C e^{\theta} \|\varpi r\|_F$, so that $Br \in b_{[1]}(E)$. Since (b,d) is admissible for A, there exists a solution $x \in d_{[0]}(E)$ of $x(n) + A(n)x(n-1) = (Br)(n)$, $n = 1,2,\ldots$, and by Lemma 6.1 there exists a solution u of (5.1) with $\varphi u = x \in d_{[0]}(E)$, as asserted in (a).

If B is a subset of $L_{[0]}(E)$ and D is a subset of $K_{[0]}(E)$, it is in keeping with earlier terminology to say that the pair (B,D) <u>is admissible for</u> M - more loosely, <u>for</u> (5.1) - if for every $r \in B$ there exists a solution $x \in D$ of (5.1). Thus, statement (a) in Theorem 9.1 expresses the admissibility of a certain pair (B,D) for M. To exemplify the uses of Theorem 9.1, we shall here specify B to be one of the spaces $L^p_{[0]}(E)$, $1 \le p < \infty$, or $M_{[0]}(E)$ or $T_{[0]}(E)$, and D to be either $C_{[-1]}(E)$ or $C_{0[-1]}(E)$; but the choices may easily be extended in the spirit of [6; Chapter 2] and the remark at the end of Section 3. Following earlier practice, the name of a pair of such spaces is abbreviated, as, e.g., (T, C_0) for $(T_{[0]}(E), C_{0[-1]}(E))$, since there is no ambiguity.

We now record some of the special cases covered by Theorem 9.1.

9.2. *Corollary*. *Assume that the memory* $M = I_L + M^T$ *satisfies the assumptions of Lemma* [illegible] *with respect to a given space* F. *With* F, (B,D), (b,d) as specified in the following table, (B,D) *is admissible for* M *if and only if* (b,d) *is admissible for* A.

F	(B,D)	(b,d)	
L^p	(L^p, C)	(l^p, l^∞)	$1 \leqq p \leqq \infty$
L^p	(L^p, C_0)	(l^p, l^∞_0)	$1 \leqq p < \infty$
L^1	(M, C)	(l^1, l^∞)	
L^∞	(T, C)	(l^1, l^∞)	
L^∞	(T, C_0)	(l^1, l^∞_0)	

Proof. Theorem 9.1, and the remarks on the slicing operator in Section 3.

10. Admissibility and the solutions of the homogeneous equation.

The admissibility of certain pairs (b,d) of sequence spaces for A implies, under some additional assumptions, an (*ordinary*) *dichotomy* or an *exponential dichotomy* of the solutions of the homogeneous equations $(6.4)_{[m]}$, (see [1; Section 7]). An exponential dichotomy, for instance, may roughly be described thus: the bounded solutions tend uniformly exponentially to 0, there exists a "complementary" manifold of solutions of (6.4) tending uniformly exponentially to infinity, solutions of the two kinds remain uniformly apart, and together they span all solutions. Since Lemma 6.1 provides a bijective correspondence between solutions of $(5.2)_{[m]}$, and solutions of $(6.4)_{[m]}$, for integral m, Theorem 9.1 and Corollary 9.2 will allow us to translate that result into an analogous implication for differential equations with delays.

In order to avoid unenlightening complications, we restrict ourselves in this section to the case in which F is specified to be L^{∞}, i.e., in which bounded solutions of (5.1) and of (6.3) are sought. The case in which $\underset{\sim}{d}$ is $\underset{\sim}{l}_0^\infty$, so that attention is centred on solutions of (5.1) and of (6.3) that tend to 0, can easily be treated in a similar fashion; as can also cases with more general $\underset{\sim}{d} \in b\mathfrak{t}$, with appropriate use of [1].

We assume given a memory $M = M_L + M'$ that satisfies the assumptions of Lemma 7.2. We denote by $\underset{\sim}{E}_0(0) \subset \underset{\sim}{E}$ the set of "initial slices" $\Pi(0)u$ of the bounded solutions u of (5.2); by Lemma 6.1, $\underset{\sim}{E}_0(0)$ is the set of values at $n = 0$ of the bounded solutions of (6.4).

We now state the main "direct" theorem, to the effect that the admissibility of certain pairs of function spaces for M implies a behaviour of the solutions of $(5.2)_{[m]}$ that may be termed an ordinary or an exponential dichotomy.

10.1. Theorem. Assume that the memory $M = M_L + M'$ satisfies the assumptions of Lemma 7.2 with respect to a given space $\underset{\sim}{F} \in b\mathfrak{F}$. Assume that $\underset{\sim}{E}_0(0)$ is closed in $\underset{\sim}{E}$. Assume that $\underset{\sim}{b} \in b\vec{\mathfrak{t}}$ (in particular, $\underset{\sim}{b} \in b\mathfrak{t}$) is [not stronger than $\underset{\sim}{l}^1$ and] such that for every $r \in \underset{\sim}{L}_{[0]}(E)$ with $\varpi r \in \underset{\sim}{b}_{[1]}(\underset{\sim}{F}(E))$ equation (5.1) has a bounded solution.

Then there exists [a number $\nu > 0$ and] a number $N > 0$ such that, for every real $m \geqq 0$, every bounded solution v of $(5.2)_{[m]}$ satisfies

(i): $\|\Pi(t)v\| \leqq N\|\Pi(t_0)v\|$ $\quad [\ \|\Pi(t)v\| \leqq Ne^{-\nu(t-t_0)}\|\Pi(t_0)v\|\ ]$ for all $t \geqq t_0 \geqq m$;

there further exist a set $\underset{\sim}{W}$ of solutions of (5.2), [a number $\nu' > 0$] and numbers $N' > 0$, $\lambda_0 > 1$ such that, for every real $m \geqq 0$, every solution u of $(5.2)_{[m]}$ is of the form $u = v + w_{[m-1]}$, where v is a bounded solution and $w \in \underset{\sim}{W}$, and such that every solution $w \in \underset{\sim}{W}$

satisfies

(ii): $\|\Pi(t)w\| \le N^{-1}\|\Pi(t_0)w\|$ [$\|\Pi(t)w\| \le N^{-1}e^{-\nu(t-t_0)}\|\Pi(t_0)w\|$]

for all $t \ge t_0 \ge 0$;

(iii): $\|\Pi(t)w\| \le X_0\|\Pi(t)w - \Pi(t)v\|$ for all $t \ge m \ge 0$ and all bounded solutions $v \in (5.2)_{[m]}$.

If E is finite-dimensional and $p(M') = 0$, then the assumption that $E_0(0)$ is closed is redundant, and W may be chosen to be a finite-dimensional linear manifold.

Proof. 1. By Theorem 9.1, (b, l^∞) is admissible for A. We now refer to [1] and [9] in order to deal with equations (6.3), (6.4). Specifically, Condition (d) of [9; Lemma 4.2] is satisfied with $d = 1^\infty$. We consider the covariant sequence $E_\sim$ (whose general term is $E_\sim(n)$, the set of initial values of the bounded solutions of $(6.4)_{[n]}$). Since $E_0(0)$ is closed by assumption, [9; Theorem 4.3,(a)] shows that the covariant sequence E_0 is (closed and) regular. We can therefore apply the fundamental "direct" results [1; Theorems 9.1 and 10.1] for difference equations, and find that this covariant sequence induces a dichotomy [an exponential dichotomy] for A.

2. To make this result manageable, we use the description of a dichotomy [an exponential dichotomy] given by [1; Theorem 7.1,(c)]. We observe in the proof of that theorem that we are free to choose the splitting q (a "non-linear projection" in E annihilating $E_0(0)$); this will be important in Part 3 of this proof. We choose q and denote its range by Z. Thus $E = E_0(0) + Z$. Now the covariant sequence E_0 is regular; therefore we have, for every integer $n \ge 0$, by [1; Lemma 5.2,(b) and (5.2)],

$$E = E_0(n) + U(n,0)E = E_0(n) + U(n,0)E_0(0) + U(n,0)Z = E_0(n) + U(n,0)Z.$$

This means that if x is a given solution of $(6.4)_{[n]}$ there exists a solution z of (6.4) with $z(0) \in \underset{\sim}{Z}$ such that $y = x - z_{[n]}$ is a bounded solution of $(6.4)_{[n]}$.

We define $\underset{\sim}{W}$ to be the set of those solutions w of (5.2) that satisfy $\Pi(0)w \in \underset{\sim}{Z}$. The remainder of the proof of the main conclusion of the theorem is now identical to that of [10; Theorem 7.3] (from the last paragraph of Part 2) with the following changes: [1; Theorem 9.1] is used, and the exponential factors deleted, in the "ordinary dichotomy" case; and Corollary 5.5 and the factor Ce^{σ} are used instead of [10; Lemma 5.2] and the factor $\exp\|\underset{\sim}{M}_C\|$.

3. If E is finite-dimensional and $\rho(M') = 0$, then each $A(n)$ is compact (Lemmas 7.2 and 6.2). Therefore [9; Theorem 4.3,(b)] is applicable and $\underset{\sim}{E}_0(0)$ is closed and has finite co-dimension in $\underset{\sim}{E}$. We may therefore choose the splitting q in the preceding proof to be a linear projection of $\underset{\sim}{E}$ along $\underset{\sim}{E}_0(0)$ onto some finite-dimensional complementary subspace $\underset{\sim}{Z}$. Then $\underset{\sim}{W}$ is a finite-dimensional linear manifold of solutions of (5.2).

10.2. Corollary. *Assume that the memory $M = M_L + M'$ satisfies the assumptions of Lemma 7.2 with respect to a given space $\underset{\sim}{F} \in b\mathfrak{F}$. Assume that $\underset{\sim}{E}_0(0)$ is closed in $\underset{\sim}{E}$. Assume that $(\underset{\sim}{B},\underset{\sim}{C})$ is admissible for M, where $\underset{\sim}{F} = \underset{\sim}{L}^1$ and $\underset{\sim}{B} = \underset{\sim}{L}^1$, or $\underset{\sim}{F} = \underset{\sim}{L}^\infty$ and $\underset{\sim}{B} = \underset{\sim}{T}$ [$\underset{\sim}{F} = \underset{\sim}{L}^p$ and $\underset{\sim}{B} = \underset{\sim}{L}^p$, $1 < p \leqq \infty$, or $\underset{\sim}{F} = \underset{\sim}{L}^1$ and $\underset{\sim}{B} = \underset{\sim}{M}$]. Then the conclusions of Theorem 10.1 hold.*

Proof. Use Corollary 9.2 instead of Theorem 9.1 to enter the proof of Theorem 10.1.

10.3 *Scholium*. Since $\underset{\sim}{L}^p$ is tame when $1 < p \leqq \infty$, Scholium 7.3 implies that the conditions $\rho_0(M') = 0$, $\rho(M') = 0$ are automatically verified when $\underset{\sim}{F} = \underset{\sim}{L}^p$ for such a p.

To conclude, we state a reasonably strong form of a "converse" theorem to Theorem 10.1, and sketch its proof.

10.4. _Theorem_. _Assume that the memory_ M _satisfies_ (M) _and_ $\rho_0(M) < 1$. _If the main conclusion of Theorem 10.1 holds for the solutions of_ $(5.2)_{[m]}$, _then the pairs_ $(\underset{\sim}{L}^1, \underset{\sim}{C})$ _and_ $(\underset{\sim}{T}, \underset{\sim}{C})$ [_the pairs_ $(\underset{\sim}{L}^p, \underset{\sim}{C})$, $1 \leqq p \leqq \infty$, _and_ $(\underset{\sim}{M}, \underset{\sim}{C})$] _are admissible for_ M.

Proof. The assumption on M implies that it satisfies the assumptions of Lemma 7.2 with $L = 0$, $M' = M$, and $\underset{\sim}{F} = \underset{\sim}{L}^1$. The main conclusion of Theorem 10.1 implies, via Lemma 6.1 and a little computation, that $\underset{\sim}{E}_0(0)$ is indeed a regular covariant sequence for A and induces a dichotomy [an exponential dichotomy] for A [1; Theorem 7.1]. From the "converse" theorems for difference equations [1; Theorems 9.2 and 10.3] it follows that the pair $(\underset{\sim}{l}^1, \underset{\sim}{l}^\infty)$ [the pair $(\underset{\sim}{l}^\infty, \underset{\sim}{l}^\infty)$] is admissible for A. From Corollary 9.2 we conclude that the pair $(\underset{\sim}{L}^1, \underset{\sim}{C})$ [the pair $(\underset{\sim}{M}, \underset{\sim}{C})$] is admissible for M. The other pairs in the statement are then obviously admissible, since $\underset{\sim}{T}_{[0]}(E)$ is stronger than $\underset{\sim}{L}^1_{[0]}(E)$ [since every $\underset{\sim}{L}^p_{[0]}(E)$ is stronger than $\underset{\sim}{M}_{[0]}(E)$].

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