FILTER SPACE MONADS, REGULARITY, COMPLETIONS

by

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1. Introduction

Many completions and compactifications in general topology follow a rigid pattern, known as the Wallman type. A space X is embedded into a space $T X$ of filters on X by mapping every point $x \in X$ into the point filter $\dot{x} \in T X$. The space $T X$ turns out to be complete or compact because every filter in $T (T X)$ converges, for the topology of $T X$, to its contraction in $T X$. The completion of a uniform space by Bourbaki [3 ; 1st ed.] and the Wallman compactification [28] are early, and typical, examples.

In recent years, the same filter constructions have occured in different settings. Point filters and contractions of filters of filters were used by Cook and Fischer [6], [4] and by Fleischer [10] to define and discuss regular convergence spaces, and by Sjöberg [27] to discuss regular uniform convergence spaces. Following this work, the author [32] showed that regularity can be interpreted as continuity of filter convergence. Manes [18], [19] used point filters and contractions to construct an ultrafilter monad on the category of sets, and he showed that algebras for this monad and their homomorphisms are compact Hausdorff spaces and their continuous maps.

In the present paper, we try to bring these trends together. We define filter space monads for categories of convergence spaces and categories of uniform convergence spaces, and we give numerous examples of such monads. We call a space X separated, with respect to a filter space monad (T, η, μ), if a filter in T X converges to at most one point in X, complete if every filter in T X converges, and regular if filter convergence is a continuous relation from T X to X. "Separated" clearly means T_2, and we show that "regular" means T_3 without T_0. If the points of T X are the ultrafilters on X, then "complete" means "compact". If (X, ξ) is an algebra for a filter monad, then every filter $\varphi \in T X$ converges to $\xi(\varphi)$ in X. Conversely, if X is separated, regular, complete, then X has a unique algebra structure for the monad, given by filter convergence $q : T X \longrightarrow X$. It follows that separated, regular, complete spaces are categorically very well behaved. We also discuss continuous extensions of functions, generalizing in particular results of Sjöberg [27].

The outline just given requires some supportive work. Thus we include two sections on filter algebra and one on continuous relations. We define categories of convergence spaces and categories of uniform convergence spaces, and we obtain functors connecting these categories. Topological spaces and uniform spaces are among the examples. The plural "categories" is motivated by the desire to include these examples, and by the fact that different contexts may require different axioms for convergence spaces and uniform convergence spaces. We shall use the language of top categories [30], [31] freely, but not essentially. The reader is referred to [17] for categorical terms.

Some questions connected with this work remain open. We have been unable to define filter space monads for categories of generalized proximity spaces or syntopogenous spaces [7]. Our work cannot be extended in its present form to categories of topological algebras. Operations can be lifted easily from points to subsets to filters, but most formal laws do not survive this process. Formal laws survive for operations on nets, but contraction for nets, i.e. the construction of diagonal nets for double nets, presents problems. Filters with special bases usually do not have nice functorial properties; thus our theory cannot be applied, in its present state, to compactifications of the Wallman type.

In order to keep the length of this paper in reasonable bounds, we suppress many proofs which we consider straightforward. Supplying these proofs will provide the reader with some healthy exercise in filter algebra. A final warning: in our effort to use coherent notations for filter algebra, we have discarded and sometimes reversed traditional notations which are incompatible with this effort.

2. Some filter algebra

2.1. We define a filter on a set S as a set F of subsets of S which satisfies the following two condition.

2.1.1. Every intersection of finitely many sets in F is in F.

2.1.2. If $A \subset B \subset S$ and $A \in F$, then $B \in F$.

It follows from 2.1.1 that $S \in F$.

If $A \subset S$, then the subsets of S containing A form a filter on S which we denote by $[A]$. More generally, we denote by $[G]$ the filter generated by a filter base G. We note that $F = [\emptyset] \Longleftrightarrow \emptyset \in F$ for a filter F on S. This filter is called the null filter on S; all other filters on S are called proper.

2#2. If + is a monotone binary operation, from subsets A of a set S^1 and subsets 6 of a set S^M to subsets A + B of a set S_9 and if F is a filter on S^1 and G one on S" , then the sets A + B with A£ F and B £ G form a filter base; we denote by P + C the filter on S with this base. This notation requires of course that the sets S^f , S^W , S are given by the context. We note that the sets A + B already form a filter base of P • G if we restrict A to some base of P and B to a base of G . It may happen that A + B can be empty for non-empty sets A and B ; in this case F + G may be the null filter for proper filters P and G .

One obvious law: [A] + [B] « [A • B] .

We use the convention introduced above for operations of any finite arity. Formal laws for operations on subsets can then be extended easily to the corresponding operations for filters*

Example: for filters F and G on a set S , the filters *F %jQ* and P n O on S are generated by, in fact eonsist of, all sets A u B and *An* B respectively with A £ F and B £ G . We note that F u G is the intersection of the sets F and G of subsets of S ; this coincidence should not stand in the way of consistent notation,

2.5. If U , V are comparable structures on a set S_9 then we write U ^ V if U is the finer structure than V ; this ensures that monotone mappings from structures of one kind to structures of another kind usually preserve order. In particular, we write F^ G for filters on a set S if F is the finer filter, i.e. if the set F contains the set G . Thus [A] ^ [fl] , for subsets A and B of S , if and only if A *c* B_f and F ^ $[A]_f$ for a filter F and a subset A , if and only if A £ P .

Filters on a set S form a complete lattice, with the null filter as finest and the filter $[S]$ as coarsest element. The mapping $A \mapsto [A]$ preserves all suprema, and finite infima. The complete lattice of filters on S is atomic, its atoms are called ultrafilters on S. General inequalities for finitary monotone operations on subsets imply the corresponding inequalities for the corresponding operations on filters.

2.4. If $f : S \longrightarrow S'$ is a mapping and F a filter on S, then the sets $f(A)$ with $A \in F$ form a base of the filter $f(F)$ on S', by 2.2. $f^{-1}(G)$ is defined similarly for a filter G on S'; we note that $f(F) \leqslant G \iff F \leqslant f^{-1}(G)$. It follows that f preserves all suprema, and f^{-1} all infima, of families of filters. We note that

$$B \in f(F) \iff f^{-1}(B) \in F \quad , \tag{2.4.1}$$

for $B \subset S'$ and a filter F on S. This is a very useful law.

2.5. Let now S^* be a set of filters on a set S. For $x \in S$, we put $\dot{x} = [\{x\}]$; this filter consists of all $A \subset S$ such that $x \in A$. We assume that $\dot{x} \in S^*$ for every $x \in S$.

If $A \subset S$, then we denote by A^* the set of all $\varphi \in S^*$ with $A \in \varphi$. In particular, $x \in A \iff \dot{x} \in A^*$ for $x \in S$. We note that always

$$(A \cap B)^* = A^* \cap B^* \quad ,$$

and that $\emptyset^* = \emptyset$ if S^* consists of proper filters on S. We put

$$\Phi_* = \{A \subset S : A^* \in \Phi\} \; ,$$

and call Φ_* the contraction of Φ, for a filter Φ on S^*. One sees easily that this is a filter on S, and that Φ_* is proper if Φ is proper and S^* consists of proper filters on S . We note that

(2.5.1) $$\Phi_* = \inf_{P \in \Phi} \sup_{\varphi \in P} \varphi \; ;$$

see [25]. Thus Φ_* is essentially the contraction defined by Kowalsky [15].

2.6. Let $j : S \longrightarrow S^*$ be defined by $j(x) = \dot{x}$. Let S^{**} be a set of filters on S^* such that $\Phi_* \in S^*$ for every $\Phi \in S^{**}$; we denote by $k : S^{**} \longrightarrow S^*$ the resulting contraction mapping. We note the following formal laws, omitting the straightforward proofs.

2.6.1. $j^{-1}(A^*) = A$, for $A \subset S$.

2.6.2. $(\dot{\varphi})_* = \varphi$ for $\varphi \in S^*$, and $(j(F))_* = F = (F^*)_*$, for a filter F on S .

2.6.3. $\Phi_* \leq F \iff \Phi \leq F^*$, for filters F on S and Φ on S^* . It follows that $F \mapsto F^*$ preserves all infima, and $\Phi \mapsto \Phi_*$ all suprema, of families of filters.

2.6.4. $k^{-1}(A^{**}) = (A^*)^*$, for $A \subset S$.

2.6.5. $(k(\mathcal{F}))_* = (\mathcal{F}_*)_*$, for a filter $\mathcal{F}$ on S^{**} .

2.7. Consider now a mapping $f : R \longrightarrow S$, and sets R^* of filters on R and S^* of filters on S such that f maps R^* into S^* . We denote by $f^* : R^* \longrightarrow S^*$ the resulting filter mapping, and we note the following formal laws.

2.7.1. $f(x) = y \iff f^*(\dot{x}) = \dot{y}$, for $x \in R$ and $y \in S$.

2.7.2. $(f^{-1}(B))^* = (f^*)^{-1}(B^*)$, for $B \subset S$.

2.7.3. $f(\Phi_*) = (f^*(\Phi))_*$, for a filter Φ on R^* .

3. Categories of convergence spaces

3.1. We define a <u>convergence structure</u> on a set S as a relation q from proper filters on S to S, subject to the two Fréchet axioms:

L 1. If $x \in S$, then $\dot{x}\, q\, x$;

L 2. If $\mathcal{F}\, q\, x$ and $\mathcal{F}' \supset \mathcal{F}$, then $\mathcal{F}'\, q\, x$.

A <u>convergence space</u> (S, q) consists of a set S and a convergence structure q on S ; we may put $q = q_X$ and $S = |X|$ if $X = (S, q)$.

We call q' <u>finer</u> than q, and put $q' \leq q$, for convergence structures q and q' on the same set, if $\mathcal{F}\, q'\, x$ always implies $\mathcal{F}\, q\, x$. With this notation, convergence structures on S form a complete lattice, with $\mathcal{F}\, (\inf q_i)\, x$, for a family $(q_i)_{i \in I}$ of convergence structures on S, if and only if $\mathcal{F}\, q_i\, x$ for every $i \in I$.

If $f : S \to S'$ is a mapping and q' a convergence structure on S', then $\mathcal{F}\, (f^* q')\, x \iff f(\mathcal{F})\, q'\, f(x)$, for $x \in S$ and a proper filter $\mathcal{F}$ on S, defines a convergence structure $f^* q'$ on S . This mapping f^* preserves infima, and thus

$$(3.1.1) \qquad q \leq f^* q' \iff f_* q \leq q' ,$$

for a mapping f_* from convergence structures on S to convergence structures on S' . We say that $f : (S, q) \to (S', q')$ is <u>continuous</u> if these inequalities are satisfied.

This defines a category CONV of convergence spaces and continuous functions; the word <u>map</u> will always refer to a continuous function.

3.2. The category CONV is too large for many purposes; many authors have considered additional axioms. We list some axioms which have been proposed,

J>»2_.1. A convergence space (S,q) is called a limit space if ? q x and G q x always juaply (P *KJ* G) q x •

J5.2.2. A convergence structure q on a set S is called a pseudotopology* and the space (S^q) a Choquat space, if P q x whenever eveiy ultrafilter finer than P converges to x •

3»2«3» A convergence space (S_tq) is called a neighborhood space or a closure space if every x € S has a neighborhood filter N_x such that P q x <p=^ P <£ H_x t for *every* x € S and every proper filter P on S „

*3*2Am k* convergence structure q on S is called topological if q is a neighborhood structure, and every neighborhood filter N_x has a base of open sets. Hei^ U *a* 3 is called open for q if P q x and x € U always imply U £ P • One sees easily that q is topological if and only if q is filter convenience for a topology on S .

The following two axioms are of different nature.

^•2#5. A convergence space (S,q) is called uniform!zable if the relations P q x_t G q x_f G q y always imply P q y ,

3*2.6. A convergence space (S,q) is called Quasi-uniformizable if the relations P q x and i q y always imply F ^ y .

^«3« We *do* not want to specify a particular system of axioms for convergence spaces, and thus we proceed as follows. We specify for every set S a Bet Q S of convergence structures of S_t subject to the following two conditions.

5»'5ol« If U j) ^ ! *s a family of structures in Q S_f then inf $q._1$ is a structure in Q S #

3.3.2. If f i S ^ S^1 is a mapping and q^f € Q S^f $_f$ then f^fq ' G Q S .

We denote by ENS^Q the category of all convergence spaces (S,q) with $q \in Q\,S$ and their continuous functions, and we call such a category ENS^Q a <u>category of convergence spaces</u>.

<u>3.4</u>. If ENS^Q is a category of convergence spaces and S a set, then $Q\,S$ is a complete lattice, with the indiscrete convergence structure of S as its coarsest element. If $f : S \longrightarrow S'$ is a mapping, then we denote by $Q\,f : Q\,S' \longrightarrow Q\,S$ the mapping obtained by restriction of $f^{\leftarrow}$. The mappings $Q\,f$ preserve infima and define a contravariant functor.

In the language of [30] and [31], every category ENS^Q of convergence spaces is a top category, and $ENS^{Q'}$ is a top subcategory of ENS^Q if $Q'\,S \subset Q\,S$ for every set S. If this is the case, then $ENS^{Q'}$ is a reflective subcategory of ENS^Q, with reflections $id\,S : (S,q) \longrightarrow (S,\rho q)$, for $q \in Q\,S$ and ρq the finest structure in $Q'\,S$ which is coarser than q.

<u>3.5</u>. Let ENS^Q be a category of convergence spaces. If r is a relation, from proper filters on a set S to S, then there is a finest structure q in $Q\,S$ such that $F\,r\,x$ always implies $F\,q\,x$. We say that this structure q <u>generated</u> by r, or by the convergences $F\,r\,x$.

<u>3.5.1. Proposition</u>. <u>Let</u> q <u>in</u> $Q\,S$ <u>be generated by a relation</u> r. <u>If</u> $f : S \longrightarrow S'$ <u>is a mapping and</u> q' <u>in</u> $Q\,S$, <u>then</u> $f : (S,q) \longrightarrow (S',q')$ <u>is continuous if and only if</u> $F\,r\,x$ <u>always implies</u> $f(F)\,q'\,f(x)$.

We omit the simple proof of this useful result.

<u>3.6</u>. The logical connections between the axioms of 3.2 are mostly obvious; we note only that every topological convergence structure is quasi-uniformizable.

Every combination of axioms in 3.2 leads to sets $Q\,S$ of convergence structures which satisfy 3.3.1 and 3.3.2, and hence to a category ENS^Q of convergence spaces. In particular, we shall regard the category TOP of topological spaces as a category of convergence spaces.

Many possible axioms for convergence spaces do not lead to a top category ENS^Q of convergence spaces. We list only two important examples.

T_1 . If $\dot{x}\; q\; x$, then $x = y$.

T_2 . If $F\; q\; x$ and $F\; q\; y$ for some filter F , then $x = y$.

In both cases, 3.3.2 is not valid, and 3.3.1 fails for empty families.

4. Convergence spaces of filters

4.1. We work in a category ENS^Q of convergence spaces. If S^* is a set of proper filters on a set S , with $\dot{x} \in S^*$ for every $x \in S$, then a convergence structure q^* in $Q\,S^*$ will be called compatible with a structure q in $Q\,S$ if q^* satisfies the following three conditions.

4.1.1. If $F\; q\; x$, then $F^*\; q^*\; \dot{x}$.

4.1.2. If $\Phi\; q^*\, \varphi$ and $\Phi_* = \Psi_*$, then $\Psi\; q^*\, \varphi$.

4.1.3. If $\Phi\, q^*\, \varphi$ and $\varphi\; q\; x$, then $\Phi_*\; q\; x$.

We note that properties 4.1.1 and 4.1.2 are preserved by infima in $Q\,S^*$, for a given $q \in Q\,S$. If one structure q^* in $Q\,S^*$ satisfies 4.1.3, then every finer structure in $Q\,S^*$ satisfies 4.1.3. Thus if $Q\,S^*$ admits a structure compatible with q in $Q\,S$, then $Q\,S^*$ admits a finest structure which is compatible with q .

4.2. Proposition. *If a structure* q^* *in* $Q\,S^*$ *is compatible with a*

<u>structure</u> q <u>in</u> Q S , <u>then</u> q <u>is quasi-uniformizable</u> (3.2.6), <u>and</u>

(4.2.1) $$\Phi\, q^* \,\dot{x} \iff \Phi_*\, q\, x \,,$$

<u>for</u> $x \in S$ <u>and a filter</u> Φ <u>on</u> S^* . <u>If</u> q^* <u>in</u> $Q\,S^*$ <u>is quasi-uniformizable</u> <u>and satisfies</u> (4.2.1) <u>and</u> 4.1.2, <u>then</u> q^* <u>is compatible with</u> q <u>in</u> Q S .

<u>Proof</u>. If F q x and $\dot{x}$ q y , then $F^*\, q^*\, \dot{x}$ by 4.1.1, and F q y follows from 4.1.3 and 2.6.2. If $\Phi\, q^*\, \dot{x}$, then $\Phi_*\, q\, x$ by 4.1.3 and L 1. Conversely, if $\Phi_*\, q\, x$, then $\Phi\, q^*\, \dot{x}$ by 4.1.1, 4.1.2 and 2.6.2.

4.1.1 for q^* follows from (4.2.1) by 2.6.2. If $\Phi\, q^* \varphi$ and $\varphi\, q\, x$, then $\dot{\varphi}\, q^*\, \dot{x}$ by (4.2.1) and 2.6.2, and $\Phi\, q^*\, \dot{x}$ and $\Phi_*\, q\, x$ follow if q^* is quasi-uniformizable.

<u>4.3</u>. Because of 4.2, we look for examples only if ENS^Q consists of quasi-uniformizable convergence spaces.

If ENS^Q is the category of all quasi-uniformizable convergence spaces, then we put $\Phi\, q^* \varphi$, for φ in S^* and a proper filter Φ on S^* , if and only if either $\Phi_* \leq \varphi$ or $\Phi_*\, q\, x$ for some $x \in S$ such that $\dot{x} \leq \varphi$.

This clearly defines a convergence structure q^* which satisfies (4.2.1) and 4.1.2. If $\Phi\, q^* \varphi$ and $\dot{\varphi}\, q^* \psi$, then we must prove $\Phi\, q^* \psi$. The only nontrivial case is $\Phi_*\, q\, x$, $\varphi\, q\, y$, with $\dot{x} \leq \varphi$ and $\dot{y} \leq \psi$. But then also $\dot{x}$ q y , and $\Phi_*\, q\, y$ since q is quasi-uniformizable.

If q^* in $Q\,S^*$ is compatible with q in Q S , then clearly $\Phi\, q^* \varphi$ if $\Phi_* \leq \varphi$. If $\dot{x} \leq \varphi$ and $\Phi_*\, q\, x$, then $\Phi\, q^* \psi$ and $\dot{\psi}\, q^* \varphi$ for $\psi = \dot{x}$, and $\Phi\, q^* \varphi$ follows if q^* is quasi-uniformizable. Thus we have obtained the finest structure q^* in $Q\,S^*$ which is compatible with q .

4.4. If ENS^Q is the category of quasi-uniformizable limit spaces, then we put $\Phi q^* \varphi$, for $\varphi \in S^*$ and a proper filter Φ on S^*, if $\Phi \leq (\bigcup F_i)^*$ for a finite family of filters F_i on S such that $(F_i)^* q^* \varphi$ by 4.3. This defines the finest structure q^* in $Q S^*$ compatible with q in $Q S$. The proof proceeds as in 4.3; we omit it.

4.5. If ENS^Q is the category of quasi-uniformizable neighborhood spaces, and if N_φ is the neighborhood filter of $\varphi \in S^*$ for q^*, then $N_\varphi = (F_\varphi)^*$ for $F_\varphi = (N_\varphi)_*$, by 4.1.2 and 2.6.2, with $F_{\dot{x}} = N_x$ for $x \in S$ by (4.2.1). Conversely, putting $N_\varphi = (F_\varphi)^*$ for $\varphi \in S^*$, with $\varphi \leq F_\varphi$ and $F_{\dot{x}} = N_x$ for $x \in S$, defines a neighborhood structure q^* on S^* which satisfies 4.1.2 and (4.2.1).

If we say that a filter F on S is q-saturated if $\dot{x} \leq F$ always implies $N_x \leq F$, then the infimum of a family of q-saturated filters is q-saturated. Thus there is a finest q-saturated filter F on S which is coarser than φ; we choose this filter as F_φ for $\varphi \in S^*$.

The neighborhood filter N_x of $x \in S$ is q-saturated if q is quasi-uniformizable, and $\dot{x} \leq N_x$. Thus $F_{\dot{x}} = N_x$. If $\dot{\varphi} q^* \psi$, then $\varphi \leq F_\psi$, and $F_\varphi \leq F_\psi$ follows by our construction. But then $\Phi q^* \varphi$ implies $\Phi q^* \psi$. Thus q^* is quasi-uniformizable, and compatible with q by 4.2.

On the other hand, if q^* is compatible with q and $\dot{x} \leq F_\varphi$, for $x \in S$ and $\varphi \in S^*$, then $(N_x)^* q^* \psi$ and $\dot{\psi} q^* \varphi$ for $\psi = \dot{x}$, and $(N_x)^* q^* \varphi$ and $N_x \leq F_\varphi$ follow. Thus F_φ must be q-saturated, and the structure q^* constructed above is the finest structure in $Q S^*$ which is compatible with q.

4.6. Let now ENS^Q be the category of topological spaces. Neighborhood filters for q^* must again be of the form $N_\varphi = (F_\varphi)^*$, with $\varphi \leq F_\varphi$, and with $F_{\dot{x}} = N_x$ for $x \in S$. These conditions are satisfied for the topology of S^* constructed in [32], with the sets U^* for q-open sets U as a basis of open sets. For this topology, F_φ is generated by the q-open sets in φ. We claim that this topology of S^* is the finest structure q^* in $Q\,S^*$ which is compatible with q.

If U is q-open, then U^* is q^*-open since q^* is finer than the topology of S^* described above. Thus $U \in F_\varphi$ if U in φ is q-open. On the other hand, if $V \in F_\varphi$, then there are sets $P \subset S^*$ and $W \in F_\varphi$ such that P is q^*-open and $W^* \subset P \subset V^*$. If $x \in W$, then $\dot{x} \in W^*$, and P is in $N_{\dot{x}} = (N_x)^*$. Thus $(U_x)^* \subset P$ for some open q-neighborhood U_x of x, and $U_x \subset V$ follows. If U is the set union of these open sets U_x, then $W \subset U \subset V$, and $U \in F_\varphi$ follows. Thus F_φ is generated by the q-open sets in F_φ, and our claim is verified.

5. Filter space monads for convergence spaces

5.1. Let ENS^Q be a category of convergence spaces, and let $T : ENS^Q \longrightarrow ENS^Q$ be a functor. We put $T\,X = (X^*, q_{TX})$ for an object X of ENS^Q, and we say that T is a filter space functor on ENS^Q if the following three conditions are satisfied.

5.1.1. For every object X of ENS^Q, the set X^* is a set of proper filters on $|X|$, with $\dot{x} \in X^*$ for every $x \in |X|$.

5.1.2. For every object X of ENS^Q, the structure q_{TX} of $T\,X$ is compatible (4.1) with the structure q_X of X.

5.1.3* If f : X —> Y in BKS^Q f then T f maps every filter $\varphi \in X^*$ into the filter f(<f?) on |Yl •

Ve say that a filter space functor T on ENS is fine if q^ is the finest structure in Q $X^\#$ which is compatible with q 9 for every object X of ENS^Q .

5.2. Theorem, Let ENS^Q be a category of convergence spaces» and assume that a set X* of proper filters on IXl is assigned to every object X of ENS^Q # This assignment deter^i"^s a fine filter space functor T jjn EHB if and only if the following three conditions are satisfied.

5.2.1. I£ x € |X| f then always * € X* .

5.2#2# For every object X jg£ $E1B^H$ f there is in Q $X^\#$ a structure q* which is compatible with q_x *

5.2.3. II f : X — ? Y ia $B»S^Q$ t then the filter fty fla ^| is in Y* for every filter <p € X* .

Proof. The conditions obviously are necessary. Conversely, they determine a fine filter space functor T on ENS^ uniquely, provided only that the induced filter mapping T f : (X*?<w) —^ (^ttlmy) is continuous for evexy map f : X —} Y in ENS^Q .

We note that q^ is the finest structure q* in Q X* which satisfies

$$\Phi_* q_X x \Longrightarrow \Phi q^* \dot{x} ,$$

for x € |X] and a filter φ on X* f and 4.1.2# Thus all we have to do is to show that (if) q^j satisfies these conditions• This follows immediately from 2.7.3 and the definitions; we omit the details.

5.3. Let T be a filter space functor on a category ENS^Q of convergence spaces. For a space X in ENEH, we define $\eta_X : X \to TX$ by putting $\eta_X(x) = \dot{x}$ for every $x \in |X|$.

Proposition. $\eta_X : X \to TX$ is an embedding, and natural in X.

Proof. η_X is injective and natural in X by 2.1.1. It follows from 2.6.2 and (4.2.1) that always ?qxX <***$> yx(P) q^ x, for x £ |X| and a filter F on Ul. Thus ^ is an embedding.

5.4. We say that a filter space functor T on a category ENS^ of convergence spaces defines a filter space monad if T satisfies the condition:

5.4.1. If Cj) & (T X)» f then always C|>$_{\#}$ ^ X* .

for every object X of ENS^Q. If this is the case, then we denote by μ_X : T T X → T X the contraction map given by $\mu_X(\phi) = \phi_{\#\#}$ for $\phi \in |TTX|$.

5.5. Theorem. If a filter space functor T on a category EKS^n of convergence spaces defines a filter space monad, then (T, η, μ) is a monad an ENS^Q.

We call a monad (T, η, μ) obtained in this way a ~~filter space monad~~.

Proof. ^ is natural in X by 2.7.3. If 4>£(T X)# and <p • 0# t then (p qjjj cp ty 2.6.2, 4.1.2 and L 1. Thus &qy^ty implies F* q^j <p, by 4.1.3. Since (^(^))# « (^#)# iqr 2.6.5, yux(J=~) q^ y follows ly 4.1.2. Thus μ_X is continuous. The foxnal laws for a monads

$$\mu_X (T\eta_X) = id_{TX} = \mu_X \eta_{TX}, \quad \mu_X (T\mu_X) = \mu_X \mu_{TX}.$$

follow immediately from 2.6.2 and 2.6.5.

5.6. By 4.2, a category ENS^Q of convergence spaces can have a filter space functor only if every space X in ENS^Q is quasi-uniformizable. On the other hand, ENS^Q always admits a trivial filter space monad if this condition is satisfied: let X^* be the set of all filters $\dot{x}$ for $x \in |X|$. For the resulting filter space monad, η and μ are natural equivalences.

For the categories ENS^Q discussed in 4.3 - 4.6, filter space functors and monads are easily obtained. 5.2.2 is automatically satisfied, and thus only 5.2.1, 5.2.3 and 5.4.1 have to be verified. We list some obvious examples.

5.6.1. Let X^* be the set of all proper filters on $|X|$.

5.6.2. Let X^* be the set of all filters on $|X|$ which converge for q_X.

5.6.3. Let X^* be the set of all ultrafilters on $|X|$.

5.6.4. Let X^* be the set of all ultrafilters on $|X|$ which converge for q_X.

5.6.5. Let X^* be the set of all proper filters on $|X|$ with the countable intersection property.

6. Categories of uniform convergence spaces

6.1. We define a pre-uniform convergence structure on a set S as a set $\mathcal{U}$ of filters on $S \times S$ which satisfies the following two axioms.

6.1.1. If $x \in S$, then $\dot{x} \times \dot{x} \in \mathcal{U}$.

6.1.2. If $\Phi \in \mathcal{U}$ and $\Psi \leq \Phi$, then $\Psi \in \mathcal{U}$.

We include the null filter on $S \times S$ in $\mathcal{U}$. A pre-uniform convergence space $(S, \mathcal{U})$ consists of a set S and a pre-uniform convergence structure $\mathcal{U}$ on S; we may put $\mathcal{U} = \mathcal{U}_X$ and $S = |X|$ if $X = (S, \mathcal{U})$.

We order pre-uniform convergence structures on S by calling $\mathcal{U}'$ <u>finer</u> than $\mathcal{U}$, and putting $\mathcal{U}' \leq \mathcal{U}$, if $\mathcal{U}' \subset \mathcal{U}$. This defines a complete lattice of pre-uniform convergence structures on S, with set intersections as infima.

If $f : S \longrightarrow S'$ is a mapping and $\mathcal{U}'$ a pre-uniform convergence structure on S', then we denote by $f^{\leftarrow}\mathcal{U}'$ the pre-uniform convergence structure on S consisting of all filters Φ on $S \times S$ such that $(f \times f)(\Phi)$ is in $\mathcal{U}'$. The mapping $f^{\leftarrow}$ thus defined preserves infima.

These data define a category of pre-uniform convergence spaces; a map $f : (S,\mathcal{U}) \longrightarrow (S',\mathcal{U}')$ is a mapping $f : S \longrightarrow S'$ such that $\mathcal{U} \leq f^{\leftarrow}\mathcal{U}'$. A map of this category is also called a <u>uniformly continuous</u> function.

<u>6.2</u>. If U and V are subsets of $S \times S$, then we put

$$U^{-1} = \{(x,y) : (y,x) \in U\}$$

and
$$V \circ U = \{(x,y) : (\exists z)((x,z) \in U \text{ and } (z,y) \in V)\} ;$$

the corresponding operations for filters on $S \times S$ are then defined by 2.2. See [5] for some laws satisfied by these filter operations.

<u>6.3</u>. We list some additional axioms for pre-uniform convergence spaces. $(S,\mathcal{U})$ will be a pre-uniform convergence space.

<u>6.3.1</u>. We call $(S,\mathcal{U})$ a <u>quasi-uniform convergence space</u> if $\Psi \circ \Phi$ is in $\mathcal{U}$ for every pair of filters Φ and Ψ in $\mathcal{U}$.

<u>6.3.2</u>. We call $(S,\mathcal{U})$ a <u>semi-uniform convergence space</u> if Φ^{-1} is in $\mathcal{U}$ for every filter Φ in $\mathcal{U}$.

<u>6.3.3</u>. We call $(S,\mathcal{U})$ a <u>demi-uniform convergence space</u> if $\Phi \circ \Phi^{-1} \circ \Phi$ is in $\mathcal{U}$ for every filter Φ in $\mathcal{U}$.

6.3.4. We call $(S,\mathcal{U})$ a uniform convergence space if $(S,\mathcal{U})$ is both semi-uniform and quasi-uniform.

The following two axioms belong to another group of axioms.

6.3.5. We call $(S,\mathcal{U})$ a pre-uniform limit space if $\Phi \cup \Psi$ is in $\mathcal{U}$ for every pair of filters Φ and Ψ in $\mathcal{U}$.

6.3.6. We call $(S,\mathcal{U})$ a pre-uniform space if there is a filter Φ_o in $\mathcal{U}$ such that $\Phi \in \mathcal{U} \iff \Phi \leq \Phi_o$, for a filter Φ on $S \times S$.

We shall combine the two groups of names freely; thus $(S,\mathcal{U})$ will be called a quasi-uniform limit space if 6.3.1 and 6.3.5 are satisfied. We shall write $\mathcal{U} = [\Phi]$ if $\mathcal{U}$ is a pre-uniform structure with coarsest filter Φ .

6.4. As in 3.3, we avoid choosing a particular system of axioms for uniform convergence spaces as follows. We assign to every set S a set $U\,S$ of pre-uniform convergence structures on S , subject to the following two conditions.

6.4.1. If $(\mathcal{U}_i)_{i \in I}$ is a family of structures in $U\,S$, then $\inf \mathcal{U}_i$ is a structure in $U\,S$.

6.4.2. If $f : S \to S'$ is a mapping and $\mathcal{U}' \in U\,S'$, then $f^{\leftarrow} \mathcal{U}' \in U\,S$. We denote by ENS^U the category of all pre-uniform convergence spaces $(S,\mathcal{U})$ with $\mathcal{U} \in U\,S$ and their uniformly continuous functions, and we call such a category ENS^U a category of uniform convergence spaces.

We note that the indiscrete uniform structure of S , consisting of all filters on $S \times S$, is in $U\,S$ for every set S , by 6.4.1. The considerations of 3.4 and 3.5 can be taken over almost verbatim; we consider this done.

6.5. If S is a set and Φ_o a filter on $S \times S$, then the set $[\Phi_o]$ of all filters $\Phi \leq \Phi_o$ on $S \times S$ is a uniform structure on S in our sense if and only if Φ_0 is a uniform structure of S in the Bourbaki sense. Thus uni* form spaces define a category of unifoxm convergence spaces* The same remark applies to quasi-uniform and semi-uniform spaces*

We have tried to adopt a standardized and consistent taxonomy for convergence 8paces and uniform convergence spaces. 3*5 and 6.4 enable readers who so desire to substitute their terminology for ours. A uniform convergence space in the sense of Cook and Fischer [5] is a uniform limit space in our sense, with 6.1,1 replaced by the stronger axiom $[\Delta] \in \mathcal{U}$, where A is the diagonal of S X S. These spaces define a category of uniform convergence spaces in our sense. The main effect of $[\Delta] \in \mathcal{U}$ seems to be that the null filter on S X S can be avoided in computations. On the other hand, examples become harder to construct, and our theory of spaces of Cauchy filters has to be modified, if this axiom is adopted.

Demi-uniformity (6."5.3) seems to be the appropriate axiom for generalised epsilonties. We note that a demi-unifonn limit space (S,t# with $[\Delta] \in \mathcal{U}$ is already a unifona limit space, and that always $\Phi \subset \Phi^{-1} \circ \Phi$.

7. Induced and fine structure functors

7.1» If $(S,\mathcal{U})$ is a pre-uniform convergence space, then

$$\mathcal{F}\, q_{\mathcal{U}}\, x \iff \mathcal{F} \times \dot{x} \in \mathcal{U} ,$$

for a proper filter F on S and $x \in S$, defines a convergence structure $q_{\mathcal{U}}$ on S. We say that $q_{\mathcal{U}}$ is induced by $\mathcal{U}$, and we write $q(\mathcal{U})$ for $q_{\mathcal{U}}$ if this notation is more convenient*

If $f : (S,\mathcal{U}) \to (S',\mathcal{U}')$ is uniformly continuous, then $f : (S,q_{\mathcal{U}}) \to$

$(S', q_{\mathcal{U}'})$ is continuous. This follows immediately from

$$(f \times f)(F \times \dot{x}) = f(F) \times f(\dot{x}) \quad .$$

This formula is also used in the proof of our next result; we omit this proof.

7.2. Proposition. The mapping $\mathcal{U} \mapsto q_{\mathcal{U}}$ from pre-uniform convergence structures to convergence structures preserves infima, and it satisfies

$$q(f^{\leftarrow}\mathcal{U}') = f^{\leftarrow}(q_{\mathcal{U}'}) \quad ,$$

for a mapping $f : S \longrightarrow S'$ and a structure $\mathcal{U}'$ on S' .

7.3. Putting $P(S,\mathcal{U}) = (S, q_{\mathcal{U}})$ defines a functor P which preserves not only underlying sets and mappings, but also infima of structures and inverse image structures. We call this functor P an induced structure functor. In the terminology of [31], P is a top functor.

We need in fact not one but many induced structure functors. If ENS^Q is a category of convergence spaces and ENS^U a category of uniform convergence spaces, then we may denote by $U'S$, for a set S , the set of all structures $\mathcal{U}$ in US such that $q_{\mathcal{U}} \in QS$. It follows from 7.2 that the sets $U'S$ satisfy 6.4.1 and 6.4.2. Thus a category $ENS^{U'}$ of uniform convergence spaces and an induced structure functor $P : ENS^{U'} \longrightarrow ENS^Q$ are defined.

7.4. Every induced structure functor $P : ENS^{U'} \longrightarrow ENS^Q$ has a left adjoint $F : ENS^Q \longrightarrow ENS^{U'}$ which also preserves underlying sets and mappings. We call such a left adjoint F a fine structure functor. In the terminology of [31], F is a cotop functor.

If $P : ENS^{U'} \longrightarrow ENS^Q$ is given, then an object (S,q) of ENS^Q will be called uniformizable, with appropriate prefixes or constraints to indicate P or ENS^U , if $q = q_{\mathcal{U}}$ for some $\mathcal{U} \in U' S$. One sees easily that (S,q) is of this form if and only if $(S,q) = P F (S,q)$.

7.5. If (S,q) is a convergence space, then the filters $F \times \dot{x}$ on $S \times S$ for which $F q x$ generate a demi-uniform convergence structure on S which induces q . Thus every convergence space is demi-uniformizable.

A pre-uniform limit structure $\mathcal{U}$ induces a limit structure $q_{\mathcal{U}}$. Conversely, if $P : ENS^U \longrightarrow ENS^Q$ goes from demi-uniform limit spaces to limit spaces, then every limit space is (demi-)uniformizable for P .

If $P : ENS^U \longrightarrow ENS^Q$ goes from uniform convergence spaces to convergence spaces, or from uniform limit spaces to limit spaces, then an object (S,q) of ENS^Q is uniformizable for P if and only if q satisfies the uniformizability condition of 3.2.5, by results of Ramaley [23], [24] and Keller [13].

7.6. It is well known that every topology is induced by a quasi-uniform structure; see [22] or [21] . The following result seems to be new.

Proposition. A convergence structure or limit structure q on a set S is induced by a quasi-uniform convergence structure or a quasi-uniform limit structure on S if and only if q satisfies the condition of 3.2.6.

Proof. A composition $(G \times \dot{y}) \circ (F \times \dot{x})$ is null if $G \cap \dot{x}$ is the null filter, and $F \times \dot{y}$ if $\dot{x} \leqslant G$. Thus $F q_{\mathcal{U}} x$ and $\dot{x} q_{\mathcal{U}} y$ imply $F q_{\mathcal{U}} \dot{y}$ if $\mathcal{U}$ is quasi-uniform. On the other hand, one sees easily that the fine pre-uniform convergence structure generated by the filters $F \times \dot{x}$ such that $F q x$, and the fine pre-uniform limit structure generated by finite joins of such filters if q is a limit structure, are quasi-uniform if q satisfies 3.2.6.

8. More filter algebra

8.1. Let again S^* be a set of proper filters on a set S, with $\dot{x} \in S^*$ for every $x \in S$. We use the notations of 2.5, and the following notations.

If $U \subset S \times S$, then we denote by U^* the set of all pairs (φ,ψ) in $S^* \times S^*$ such that $U \in \varphi \times \psi$. We note that

$$(U \cap V)^* = U^* \cap V^* \quad \text{and} \quad (\dot{x},\dot{y}) \in U^* \iff (x,y) \in U ,$$

for subsets U and V of $S \times S$ and $(x,y) \in S \times S$, and that $\emptyset^* = \emptyset$.

We define the compression $\mathcal{F}_*$ of a filter $\mathcal{F}$ on $S^* \times S^*$ by putting

$$\mathcal{F}_* = \{V \subset S \times S : V^* \in \mathcal{F}\} .$$

One verifies easily that $\mathcal{F}_*$ is a filter on $S \times S$, and that $\mathcal{F}_*$ is proper if $\mathcal{F}$ is proper.

We plead now guilty to using the same notations simultaneously for different concepts, but we contend that this should not cause any confusion.

8.2. We define $j : S \longrightarrow S^*$ as in 2.6, and we note the following formal laws.

8.2.1. $(A \times B)^* = A^* \times B^*$ and $(F \times G)^* = F^* \times G^*$, for subsets A, B and filters F, G on S.

8.2.2. $\mathcal{F}_* \leq \Phi \iff \mathcal{F} \leq \Phi^*$, and $(\Phi^*)_* = \Phi$, for filters Φ on $S \times S$ and $\mathcal{F}$ on $S^* \times S^*$.

8.2.3. $(\dot{\varphi} \times \dot{\psi})_* = \varphi \times \psi$, for φ and ψ in S^*.

8.2.4. $(j \times j)^{-1}(U^*) = U$ for $U \subset S \times S$.

8.2.5. $((j \times j)(\Phi))_* = \Phi$ for a filter Φ on $S \times S$.

We omit the straightforward proofs of these statements.

8.3. The following formal laws involve the operations defined in 6.2.

8.5.1. (IT^1)* « (U*)"^{1} for U d S X S .

8.3.2. (J " ^{1})$_\#$ = (F*)"^{1} for a filter F **on** S* x S* .

8.5.3. V* o u* **c (V c** u)* for subsets U and V of S x S ,

8.5.4. (^ - J=)$_\#$ ^Q^oT^ for filters f and ^ on **S » x S ».**

8.3.5. U* <£ <£ X 4-> =£ U o u*^{1} o U <£(p* x %, for O C S X S and filters <£> and ^ on S» .

8.3.6. (<t>*V)$_\#$ $\leq$ 0>*xq^ $(\Phi\times\Psi)_* \circ (\Psi\times\Phi)_* \circ (\Phi\times\Psi)_*$, for filters <p and ^ on S* .

The proofs of the first four laws and of the first half of 8.3.6 **are easy.** The second half of 8.3.6 follows directly from 8.3.5 which we **now prove.**

8.3.5 is trivial if 4> or 4^ is the null filter. Otherwise, choose $P \in \Phi$ and Q&V so that P x Q C H* , fix ^ 6 P and ^G Q , and choose X_1 t VT$_1$ and Y_1 6 4-$_1$, so that L_1 x Y_1 C U . For every <^>£P there is X,p ccp and Y_φ ^y$_1$ so that X^ X Y«$_\varphi$C U , and for every y c£Q **there** is X$_\psi$pfc*^ and Y$_f$^y so that X^xY^tT U . If X - U**$_\varphi$ and Y = $\bigcup Y_f$, for all $\varphi \in$ P and all y 6 Q , then P d X* and QC Y* . Thus Xe<£»$_\#$ and Y<S^$_\#$. If xtX^ and yC-Iy, then U,y') , (x',y') and (x^1,y) are in U for x' e X'$_\psi$ n ^ — which is in Cf>^ — and y^1 <r Y^nY . Thus (x,y) € 0 <a U"^{1} o U , and X X Y C. U o U^{-1}o U which proves 8.3.5.

8.4. Assume now that f : R —*S induces f* : R*—* S* as in 2.7. We note the following formal laws.

£.4*1.. ((f^frtv))* « (fXf*)'V) for V C S*S .

8.4.2. ((f*K f*)(J))$_\#$ - (fX f)(f») for a filter F on RXR.

The proofs are straightforward.

8.5. Let nan S** be a set of proper filters on S* f with <peS** for eveiy <p £ S* and <E># € S* for eveiy 0 € S** f and let k : S** —> S* be the resulting contraction mapping. We note the following formal laws*

8.5.1. (kXk)""V) C (U*)* C $(kxkr^1((U o u^o u)«)_t$ for U a SX s #

8.5.2. (jt«)» < $((kXk)(^))_t$ ^ $(<K_\#)_\#$ o ((jfJJ^o $(tf_\#)_{\# f}$ fora filter <# on S** X S** .

The first part of 8.5.1 follows from the first part of $8.3«6_f$ the second part from 8.3.5, and 8.5.2 follows from 8.5.1 and the definitions.

8.6» Let now (IL be a pre-uniform convergence structure of S f and denote by U* the set of all filters J=* on S*)(S* such that J=; eiU* By $8.2.3_f$ 2/* is a pre-unlform convergence structure of S* if and only if QPX<p GL XL for every filter <p in S* . A filter <p on S with this property is called a Cauchy filter of the space (S_fU) #

If x £ S f then i is a Cauchy filter of (S_fO) . If tL is a uniform convergence structure, then every filter P on S which converges for q^ is a Cauchy filter of (S_fli) #

If IL* is a pre-uniform structure, then j : (S_tO) —> (S%UF) is an embedding, by 8.2.5, and it follows easily from 8.2 and 8*3 that every property listed in 6.3 which U has is inherited by U* .

9, Filter spacejnonadsjE;orjxniform convergence spaces

9.1. We assume in this section that a set X* of proper filters on IXI is assigned to every object X of a category ENS^U of uniform convergence spaces, with the following three properties.

9.1.1. If x 6 |Xl , then always i £ X* .

9.1.2. If f : X —* T in BNS^U , then the filter f(cp) on | YI always is in Y* for a6 X* .

9.1.5. If X* consists of Cauchy filters of X , and if T X « (X*,U*) for the structure U* of X* defined by *J^GU** *<?=*>* *JF^e.11^* (see 8.6), then T X is an object of ENS^U, and (T X) * consists of Cauchy filters of T X .

9.2. We say that an object X of EMS^U is precomplete. for the given assignment X I—? X* f if X* consists of Cauchy filters of X • By 9«1.3* a precomplete object TX = (X*f £X*) of ENS^U is defined for every precomplete object X of ENS^U . We define *y^* : X —> T X \y putting y_x(x) « x for x£ |Xl , if X is precomplete. If f : X — > Y is a map between precomplete objects of EHS^U f then we define T f : T X —> T Y by putting (T f)(O>) * f(<p) for every filter a> in X* .

9.3. Theoreine For the data of 9«1» precomplete ob.iects of ENS^U define a category J^5 of uniform convergence spaces, a top subcategorv of ENS^U # The data of 9.2 define a functor T : *T —r'p* t and a natural embedding η_X : X — ^ T X for every ob.iect X *g£ P .*

Proof. Consider a family of objects $X_\pm$ *m* $(S_f U_\pm)$ of ENS^U f and put X « (S_f inf $U_\pm$) • If f 6 X * , then cp£ ($X_\pm$)* for every $H_\pm$ by 9^1*2f since id S : X — > X_i in ENS^U . If every X_i is precomplete, then *<PXQ>* is in every II_i and thus in inf It_i , and X is precomplete. A similar argument proves 6*4,2 for precomplete objects of ENS^U .

By 8.4.2, $T f : T X \longrightarrow T Y$ is uniformly continuous for $f : X \longrightarrow Y$ in $\mathcal{P}$, and thus the data of 9.2 define a functor T as claimed. η_X is an embedding by 8.6, and natural in X by 2.7.1.

9.4. We say that the data of 9.1 define a filter space monad in ENS^U if every precomplete object X of ENS^U is a demi-uniform convergence space (6.3.3) and satisfies the following condition.

9.4.1. If $\Phi \in (T X)^*$, then always $\Phi_* \in X^*$.

If this is the case, then we denote by $\mu_X : T T X \longrightarrow T X$ the resulting contraction map, given by $\mu_X(\Phi) = \Phi_*$, for $\Phi \in (T X)^*$.

9.5. Theorem. If every precomplete object of ENS^U is a demi-uniform convergence space and satisfies 9.4.1, for the data of 9.1, then (T, η, μ) is a monad on the category $\mathcal{P}$ of precomplete objects of ENS^U.

We call this monad (T, η, μ) a filter space monad in ENS^U.

Proof. μ_X is uniformly continuous by 8.5.2 and the definitions, and natural by 2.7.3. The monadic laws (see 5.5) follow from 2.6.2 and 2.6.5.

9.6. If an induced structure functor $P : ENS^U \longrightarrow ENS^Q$ (see 7.3) is defined for a category ENS^Q of convergence spaces, then the following result relates Cauchy filter spaces to the filter spaces of section 4.

Proposition. If X is a precomplete object of ENS^U and a uniform convergence space, then the structure $q_{PTX} = q(\mathcal{U}^*)$ of $P T X$ is compatible with the structure $q_{PX} = q(\mathcal{U}_X)$ of $P X$.

Proof. By 8.3.6 and the definitions, $\Phi \, q_{PTX} \, \varphi \iff \Phi_* \times \varphi \in \mathcal{U}_X$, for

$\varphi \in X^*$ and a proper filter Φ on X^*. This clearly satisfies 4.1.1 and 4.1.2, and 4.1.3 follows from

$$(\varphi \times \dot{x}) \circ (\Phi_* \times \varphi) = \Phi_* \times \dot{x}$$

and the definitions.

We do not know whether q_{PTX} is the finest structure in $Q\, X^*$ which is compatible with q_{PX} in $Q\,|X|$.

9.7. Examples are easily obtained. By 8.6, the topological part of 9.1.3 presents no problems for the axioms listed in 6.3. Since 9.1.1 and 9.1.2 are 5.1.1 and 5.1.3, and 9.4.1 is 5.4.1, the assignments of 5.6 work. We note that T X is automatically precomplete, by the first part of 8.3.6, if X is precomplete and satisfies 9.4.1. For the example 5.6.3 of all ultrafilters on $|X|$, a precomplete space is called precompact or totally bounded. In addition to the examples of 5.6, we list two examples for which every space is precomplete.

9.7.1. X^* is the set of all Cauchy filters of X .

9.7.2. X^* is the set of all Cauchy ultrafilters of X .

10. Continuous relations

10.1. We define and discuss in this section continuous relations in a top category ENS^t over sets, using for ENS^t the notations introduced in 3.1 for convergence spaces, and in 6.1 for uniform convergence spaces.

Continuous relations in this sense were introduced in [32] for topological spaces; see [32] for a comparison with continuous relations as defined e.g. in [20] or in [1]. Recently, Grimeisen [11] introduced a different continuity concept for relations between topological spaces. A continuous relation in our sense is continuous in his sense, but not conversely. Klein [14] discussed relations for a large class of categories. 10.5 provides a connection between his concept and ours; the two concepts may have a common generalization.

10.2. Sets and relations form a category REL ; the composition $g f$ of relations $f : S \longrightarrow S'$ and $g : S' \longrightarrow S''$ is defined by putting $x (g f) z$, for $x \in S$ and $z \in S''$, if and only if $x f y$ and $y g z$ for some $y \in S'$. If (S,u) and (S',u') are objects of the top category ENS^t, then we say that a relation $f : S \longrightarrow S'$ is <u>continuous</u>, from (S,u) to (S',u'), if the following condition is satisfied.

10.2.1. If $g : R \longrightarrow S$ and $g' : R \longrightarrow S'$ are mappings such that always $g(x) f g'(x)$ for $x \in R$, and if $g : (R,v) \longrightarrow (S,u)$ in ENS^t, for a structure $v \in t R$, then $g' : (R,v) \longrightarrow (S',u')$ in ENS^t.

For given g and g', it is sufficient to test 10.2.1 for the coarsest structure $v \in t R$ for which g is continuous. Thus 10.2.1 is equivalent to the following condition.

10.2.2. If $g : R \longrightarrow S$ and $g' : R \longrightarrow S'$ are mappings such that always $g(x) f g'(x)$ for $x \in R$, then $g^{\leftarrow} u \leq (g')^{\leftarrow} u'$.

10.3. Proposition. (i) *If* $f : (S,u) \longrightarrow (S',u')$ *and* $g : (S',u') \longrightarrow (S'',u'')$ *are continuous relations, then* $g f : (S,u) \longrightarrow (S'',u'')$ *is continuous.* (ii) *A mapping* $f : S \longrightarrow S'$ *defines a continuous relation* $f : (S,u) \longrightarrow (S',u')$ *if and only if* $f : (S,u) \longrightarrow (S',u')$ *in* ENS^t.

Proof. If $h : R \longrightarrow S$ and $h'' : R \longrightarrow S''$ are such that $h(x) (g f) h''(x)$ for every $x \in R$, choose $h'(x)$ so that $h(x) f h'(x)$ and $h'(x) g h''(x)$, for every $x \in R$. Then $h^{\leftarrow} u \leq (h')^{\leftarrow} u' \leq (h'')^{\leftarrow} u''$, and $g\,f$ is continuous.

For (ii), we note that $g' = f\,g$ in 10.2.1. Thus $g' \in \mathrm{ENS}^t$ if f and g are in ENS^t , and $f : (S,u) \longrightarrow (S',u')$ in ENS^t is continuous as a relation. For the converse, use $g = \mathrm{id}\,(S,u)$ in 10.2.1, with $g' = f$.

10.4. We need the following definition. A map $f : (S,u) \longrightarrow (S',u')$ in ENS^t is called <u>coarse</u> if $u = f^{\leftarrow} u'$. We note the following properties of coarse maps, omitting the straightforward proofs.

10.4.1. <u>If</u> $f : S \longrightarrow S'$ <u>is a mapping and</u> $u \in t\,S$, <u>and if</u> $g : (S',u') \longrightarrow (S'',u'')$ <u>is coarse in</u> ENS^t , <u>then</u> $f : (S,u) \longrightarrow (S',u')$ <u>in</u> ENS^t <u>if and only if</u> $g\,f : (S,u) \longrightarrow (S'',u'')$ <u>in</u> ENS^t .

10.4.2. <u>If</u> $f : X \longrightarrow X'$ <u>in</u> ENS^t , <u>and if</u> $g : X' \longrightarrow X''$ <u>is coarse in</u> ENS^t , <u>then</u> $g\,f$ <u>is coarse in</u> ENS^t <u>if and only if</u> f <u>is coarse</u>.

10.4.3. <u>Every subspace inclusion in</u> ENS^t <u>is coarse</u>.

10.5. The graph of a relation $f : S \longrightarrow S'$ is a subset of $S \times S'$. If we replace S and S' by (S,u) and (S',u') , then this subset defines a subspace Γ_f of the product space $(S,u) \times (S',u')$ in ENS^t ; we regard Γ_f as the graph of $f : (S,u) \longrightarrow (S',u')$. The two projections $p : \Gamma_f \longrightarrow (S,u)$ and $p' : \Gamma_f \longrightarrow (S',u')$ are then maps in ENS^t .

<u>Proposition</u>. <u>A relation</u> $f : S \longrightarrow S'$ <u>is continuous from</u> (S,u) <u>to</u> (S',u') <u>if and only if the projection</u> $p : \Gamma_f \longrightarrow (S,u)$ <u>is coarse</u>.

<u>Proof</u>. The subspace structure of Γ_f is $p^{\leftarrow} u \cap (p')^{\leftarrow} u'$. If f is continuous, then this is $p^{\leftarrow} u$ by 10.2.2, and p is coarse. Conversely, if $g : R \longrightarrow S$ and $g' : R \longrightarrow S'$ are such that $g(x)\,f\,g'(x)$ for every $x \in R$,

then $g = p\,h$ and $g' = p'\,h$ for a unique mapping $h : R \longrightarrow \Gamma_f$. If $g : (R,v) \longrightarrow (S,u)$ in ENS^t and p is coarse, then $h : (R,v) \longrightarrow \Gamma_f$ by 10.4.1, and $g' : (R,v) \longrightarrow (S',u')$ follows. Thus f is continuous.

10.6. Proposition. A top functor $P : ENS^t \longrightarrow ENS^{t'}$ preserves continuous relations.

Proof. This follows immediately from 10.5; a top functor preserves products, subspaces and coarse maps.

10.7. We list without proof some useful properties of continuous relations which we shall not need in this paper. We note for 10.7.1 that relations $f : S \longrightarrow S'$ form a complete lattice, with $f' \leqslant f$ if the graph of f' is contained in the graph of f.

10.7.1. If $f : (S,u) \longrightarrow (S',u')$ is continuous and $f' : S \longrightarrow S'$ is finer than f, then $f' : (S,u) \longrightarrow (S',u')$ is continuous.

10.7.2. If $f : (S,u) \longrightarrow (S',u')$ is a map in ENS^t, then the inverse relation $f^{-1} : (S',u') \longrightarrow (S,u)$ is continuous if and only if f is coarse.

10.7.3. Every continuous relation is of the form $g\,f^{-1}$ for a coarse map f and a map g in ENS^t.

11. Separated, regular and complete spaces

11.1. A definition of a filter space functor $T : ENS^t \longrightarrow ENS^t$ on a top category ENS^t can easily be abstracted from sections 4 and 9. We assume in this section that such a functor, and a top functor $P : ENS^t \longrightarrow ENS^Q$ from ENS^t to a category of convergence spaces, are given. The objects of ENS^t will be called spaces. For a space X_t we denote by $q_X : X^* \to X$ the relation obtained by restricting the structure q_{PX} of the induced convergence space PX to the underlying set X^* of IX ,

A space X will be called separated if q_X is functional, i.e. if a filter in X^* converges to at most one point of X_t complete if every $\Phi \in X^*$ converges to at least one point of X_t and regular if $q_X : TX \to X$ is continuous* These properties are defined relative to a given filter space functor T_9 but different filter space functors may produce the same separated, regular or complete spaces*

If X is a space and $A \subset |X|$, then $4_X U^*)$ will be called the closure of A in X_f relative to I, and A will be called closed if $q_X(A^*) = A$. Closure is monotone, and $q_X(0) = 0$ and $A \subset q_X U)$ • The other two Kuratowski laws are not necessarily satisfied. The intersection of closed sets is closed* A will be called dense in X if |X| is the only closed set containing A •

11»2« Proposition* A product space of separated spaces is separated* If $f : X \to Y$ in EMS* with f infective and Y separated* then X jy£ separated* Thus separated spaces define an epireflective subcategory of ENS^t ; all reflections for this subcate^ory are quotient maps in ENS^t *

Proof* If $\varphi \in X^*$ for a product space X and $\varphi\, q_X\, x$ • then the projections of φ converge to the projections of x • If X is the product of separated 8paces, this determines x uniquely; thus X is separated* The second statement is proved similarly* Now separated spaces form an epireflective subcategory of ENS^t by [iZl 10*2*l]$_9$ and the reflections are quotient maps by [33i 5-5] •

!!•?• Proposition. Regular spaces define a top subcategory of ESS^{t} #

Proof. Let $X_{\pm}$ = (S,u$_{\pm}$) for i € I , and let X « (S$_f$ inf u$_{\pm}$) • We must show that X is regular if all *7L$_{\pm}$ are* regular. Thus let g : (R,v) --^T X in ENS* , and let g^f 5 R —> S be a mapping such that g(x) $$_x$ g^f(x) for every i £ R . We note that id S : X *–f X$_{\%}$.* If *g$_{\pm}$* « (T id S) g : (R$_f$v) —^ T X $_{i}$, then g.$_{i}$(x) q$_x$ g'$_{i}$(x) follows for every x £R • But then g^f t (R$_f$v)—>X$_{i}$ since X$_{i}$ is regular, and g^f : (R,v) — * X follows* Thus X is regular*

If f : S —> S^f is a mapping and (S^f$_f$u^f) a regular space$_9$ then we must show that the space (S, f$^{\leftarrow}$u) is regular. The method of the preceding paragraph can be used for this; we omit the details.

^•4. Proposition, The product of complete spaces is complete, and every closed subspace of a complete space is complete.

Proof. If *cpCz* X* for a product space X $_f$ and if every projection of O> converges, then *CP* converges. This proves the first part.

Let now j : A —^S be an inclusion and X » (S$_f$u) a space, and let Y * (A$_f$ j*~u) be the resulting subspace. If <p 6 Y* , then *i((b)e* X* and A *G i(<j>)* . Thus if ofo>) 4$_X$ x and A is closed, then x £A , and *<p q^ %* follows. This proves the second part.

11J). Lemma. _I £ f : X — ^ Y and g : X —> Y in ENS* and Y is separated, then the set of all x £jXJ such that f(x) » g(x) is closed in X .

Proof.- Let A be this set. If 0>eA$^{\#}$ and <p q$_x$ x , then f(<p) « g(p) , and this filter converges to f(x) and to g(x) by q$_Y$ • Thus x £ A #

11.6. By 11.5, every map $f : X \longrightarrow Y$ of separated spaces with dense range is epimorphic in the category of separated spaces. However, we cannot use [12; 10.2.1] to conclude from this and 11.4 that separated complete spaces or separated regular complete spaces define an epireflective subcategory of separated spaces. The reason for this is that the category of separated spaces in ENS^t may fail to be co-well-powered with respect to maps with dense range. See [34] for examples.

Epireflectors may still exist. For completeness in the usual sense that X^* is the set of all Cauchy filters of X, the epireflector from separated uniform limit spaces to complete separated uniform limit spaces has been constructed in [29], but we do not know whether separated regular complete spaces form an epireflective subcategory of separated spaces for this example.

Every space is complete if X^* always consists of filters on $|X|$ which converge for q_{PX}, but 11.2 and 11.3 are still useful in this situation. Even the trivial case that X^* consists of all filters $\dot{x}$ for $x \in |X|$, and $\eta_X : X \longrightarrow T\,X$ is an isomorphism of ENS^t for every space X, has some interest. Separated spaces are T_1 spaces in this situation.

11.7. We consider now the situation that T indices a filter space monad (T, η, μ) on ENS^t, i.e. 5.4.1 is satisfied, and $\mu_X : T\,T\,X \longrightarrow T\,X$ in ENS^t for the resulting contraction mapping $\mu_X : (T\,X)^* \longrightarrow X^*$, for every space X. We assume, moreover, that the convergence structure q_{PTX} in $Q\,X^*$ is compatible (4.1) with q_{PX} in $Q\,|X|$, for every space X.

Proposition. If (X, ξ) is an algebra for (T, η, μ), then $\varphi\ \hat{q}_X\ \xi(\varphi)$ for every filter $\varphi \in X^*$.

Proof. We have ^x(<p) qipx <p for d>6X* , by 4.1.2 and 2.6.2. Since ξ is continuous and ξ «x « id X t <p qx |(<p) follows.

This result has an important converse.

11.8. Theorem, if Y is a separated regular complete space, under the assumptions of 11.1ftAfl11.7, then (Y, q̂Y) is an algebra for (T, ^, j, t) • Moreover. XL (*tξ) is an algebra for this monad and f : X — > Y i& ENS, then f induces a homomorphism f : (X, ξ) —^ (Y, q̂Y) of algebras for (T, η, μ) •

Proof. q̂Y is a map from T Y to Y by the definitions, and q̂Y ηY - id Y since f q̂y y for ye Ul • If (p 6 (T X)» , then <I> q̂^ (f for cp ** (fj+ ** fa(<p) f by 4.1.2 and 2«6.2# But then q̂Y((|)) converges to q̂^(φ) since q̂X is continuous. This means that q̂X (T q̂X) = q̂X μX • ^hus (Y, q̂X) is an algebra.

If f : X —^ Y and (X, ξ) is an algebra, then w q« ^(cp) for cp ∈ X* by 11.7, and f(cp) qY f(ξ(<?>)) results since f is continuous. But this says that q̂Y (T f) = f ξ , and f is a hoaomorphism of algebras as claimed.

12. Stone-Čech compactifications

12.1. We consider in this section the important case that X* is the set of all ultrafilters on |X| , for every space X . It is well known that every Mapping f : S — > S' maps ultrafilters on S into ultrafilters on S' . Thus 5.1.1 and 5.1.3 are satisfied in this situation.

If X* consists of ultrafilters on |X| , then (A u B)# « A* ^ B* for any subsets A and B of |X| , and it follows that (j)# is an ultrafilter on

$|X|$ for every ultrafilter Φ on X^*. Thus 5.4.1 is satisfied in our present situation.

We assume now that the assignment $X \longmapsto$ ultrafilters on $|X|$ can be lifted to a filter functor $T : ENS^t \longrightarrow ENS^t$ which satisfies the assumptions of 11.7. A complete space will be called *compact* for this example. Separated and regular spaces relative to T are separated and regular spaces in the usual sense, and closure relative to T is closure in the usual sense.

12.2. Theorem. *Under the assumptions of 12.1, separated regular compact spaces define an epireflective subcategory of the category of separated spaces.*

Proof. Let X be a space; X need not be separated. If $f : X \longrightarrow Y$ is a map from X to a separated regular compact space Y, then $f = g\,\eta_X$ for a unique homomorphism $g : (T\,X, \mu_X) \longrightarrow (Y, \hat{q}_Y)$ of monadic algebras, by the general theory of monads. By the second part of 11.8, g is also the unique map in ENS^t for which $f = g\,\eta_X$.

By 11.2 and 11.3, separated regular spaces define an epireflective subcategory of ENS^t; let $r_{TX} : T\,X \longrightarrow R\,T\,X$ be the reflection for this situation. It follows that $g = h\,r_{TX}$ for a unique map $h : R\,T\,X \longrightarrow Y$. Since r_{TX} is epimorphic in ENS^t and thus surjective, every ultrafilter on $|R\,T\,X|$ is the image by r_{TX} of an ultrafilter Φ in $(T\,X)^*$. Now $\Phi\ \hat{q}_{TX}\ \mu_X(\Phi)$ by 11.7, and thus $r_{TX}(\Phi)$ converges for $R\,T\,X$. This shows that $R\,T\,X$ is compact. Now $h : R\,T\,X \longrightarrow Y$ is uniquely determined by $f = h\,r_{TX}\,\eta_X$, and thus $r_{TX}\,\eta_X : X \longrightarrow R\,T\,X$ is a reflection for regular separated compact spaces.

The range of the reflection $r_{TX}\,\eta_X$ is dense in $R\,T\,X$ since every closed subspace of a compact space is compact, by 11.4. If X is separated, it follows that $r_{TX}\eta_X$ is an epimorphism in the category of separated spaces, by 11.5.

12.3. Theorem 12.2 is known for topological spaces, precompact uniform spaces, and limit spaces [25]. Precompact uniform convergence or limit spaces provide an application of 12.2 which is new. We do not know whether the Smirnov compactification of a proximity space results from an ultrafilter space monad on the category of proximity spaces.

The epireflection constructed in the proof of 12.2 is usually not a compactification in the usual sense, i.e. a dense embedding into a compact space. For convergence spaces and limit spaces, Kent and Richardson [26] have answered fully the author's question: when is the Stone-Čech compactification of Theorem 12.2 a dense embedding? Their conditions are necessary for any example, but we do not know whether more restrictive conditions are needed for uniform convergence spaces.

13. Regular convergence spaces

13.1. The condition by which Cook and Fischer [6] and Fleischer [10] defined regularity for convergence spaces is clearly a special continuity condition for filter convergence. Biesterfeldt [2] pointed out that it is equivalent to the topological axiom T_3 adapted to convergence spaces. However, his proof is valid only for separated spaces. We shall close this gap.

We work with a category ENS^Q of convergence spaces in which a filter space functor T, in the sense of 5.1, is given. For a space X in ENS^Q, we denote by $\hat{q}_X : X^* \longrightarrow |X|$ the restriction of the structure q_X to X^*. We recall that $\hat{q}_X(A^*)$ is, by definition, the closure of $A \subset |X|$, and we define the closure $\hat{q}_X(F^*)$ of a filter F on $|X|$ by 2.2: $\hat{q}_X(F^*)$ is generated by the sets $\hat{q}_X(A^*)$ with A in some filter base of F. Note that (2.4.1) cannot be used here since $\hat{q}_X$ is in general not a mapping.

<u>13.2. Theorem</u>. <u>Under the assumptions of</u> 13.1, <u>the following two statements are logically equivalent for a space</u> X <u>in</u> ENS^Q.

(i) X <u>is regular, i.e.</u> $\hat{q}_X : T\,X \longrightarrow X$ <u>is continuous</u>.

(ii) X <u>satisfies</u> T_3, <u>i.e. if</u> $F\, q_X\, x$, <u>then always</u> $\hat{q}_X(F^*)\, q_X\, x$.

<u>Proof</u>. Let Γ_q be the graph of $\hat{q}_X$, with projections $p : \Gamma_q \longrightarrow T\,X$ and $p' : \Gamma_q \longrightarrow X$. For $A \subset |X|$, let $S_A = p^{-1}(A^*)$, and define S_F accordingly for a filter F on X. Then $p(S_A) \subset A^*$ and $p'(S_A) = \hat{q}_X(A^*)$, with corresponding results for filters. If $F\, q_X\, x$, then $F^*\, q_X\, \dot{x}$, and it follows that S_F converges to $(\dot{x},x)$ in Γ_q if $\hat{q}_X$ is continuous, i.e. p coarse (10.5). But then $\hat{q}_X(F^*)\, q_X\, x$ by continuity of p', and (i) $\Longrightarrow$ (ii).

Conversely, consider $g : (R,q) \longrightarrow T\,X$ and $g' : R \longrightarrow |X|$ such that $g(x)\, \hat{q}_X\, g'(x)$ for every $x \in R$. If $F\, q\, x$ for (R,q), then $g(F)\, q_{TX}\, g(x)$ and $g(x)\, q_X\, g'(x)$. Thus $G\, q_X\, g'(F)$ for $G = (g(F))_*$, by 4.1.3. Now if $B \in G$, then $g(A) \subset B^*$ for some $A \in F$ by the definition of G, and it follows that $g'(A) \subset \hat{q}_X(B^*)$. Thus $g'(F) \leq \hat{q}_X(G^*)$, and $g'(F)\, q_X\, x$ follows if X satisfies T_3. Thus (ii) $\Longrightarrow$ (i).

<u>13.3</u>. For the first four examples in 5.6, $\hat{q}_X(A^*)$ is the usual closure of A in X. Thus the four corresponding filter functors define the same regularity for spaces in ENS^Q. They also provide the same separated spaces, but complete spaces are not the same. Regularity for 5.6.5 seems to be different from regularity for the other examples in 5.6.

Continuity of $\hat{q}_X$ depends on the existence of $T X$, and $T X$ can exist only if X is quasi-uniformizable, by 4.2. On the other hand, T_3 and separated and complete spaces depend only on the assignment $X \longmapsto X^*$ which must of course satisfy 5.1.1 and 5.1.3. This has the added advantage that spaces in ENS^Q need not be quasi-uniformizable. The proof of 13.2 can be remodeled easily into a proof that T_3 is equivalent to the regularity condition of Cook and Fischer [6] and Fleischer [10] . The first part of this proof closes the gap in [2] mentioned above.

13.4. One equivalence in [32; Thm. 3] is a special case of 13.2. The whole theorem can easily be adapted to the more general case considered here, with T_3 formulated as in 13.2. We digress from the main theme of this paper by adding the following equivalence to [32; Thm. 3], using our present notation.

Proposition. A topological space X is regular if and only if X satisfies R_1 and $\hat{q}_X : T X \longrightarrow X$ is lower semi-continuous on its domain.

Proof. R_1 is one of the axioms of Davis [8] and can be stated as follows. If $F\, q_X\, x$ and $F\, q_X\, y$ for some filter F , then always $\dot{x}\, q_X\, y$. This follows from T_3 ; see [8] . Continuity of $\hat{q}_X$ implies lower semi-continuity by [32; Thm. 2]. This proves the Proposition in one direction.

On the other hand, let $x \in U$ with U open in X . $(\hat{q}_X)^{-1}(U)$ is relatively open and thus contains all convergent filters in some neighborhood V^* of $\dot{x}$ in $T X$, with V open in X . If $\varphi \in V^*$ and $\varphi\, \hat{q}_X\, x$, then also $\varphi\, \hat{q}_X\, y$ for some $y \in U$. But then $\dot{x}\, q_X\, y$ by R_1 , and $x \in U$ follows. Thus $\bar{V} \subset U$, and X satisfies T_3 in its usual form.

13.5. We give two examples. For the first example, let S be infinite, with two points x, y singled out. We define a neighborhood structure of S as follows. N_x consists of all sets $A \subset S$ with x and y in A and with $S \setminus A$ finite. $N_y = \dot{x} \cup \dot{y}$, and $N_z = \dot{z}$ for all other points. One sees easily that these filters are closed; thus the given neighborhood space (S,q) is regular. On the other hand, the space is not quasi-uniformizable, as $\dot{x}\, q\, y$, but N_x does not converge to y.

The second example was suggested by J. J. Schäffer. Let $X = (S,q)$ be an infinite set S with the coarse T_1 topology, and let X^* consist of all convergent filters on S. Then $q^{-1}(U) = U^*$ for every open set U, and thus q is lower semi-continuous. This example shows that 13.4 is "best possible".

14. Regular uniform convergence spaces

14.1. We assume in this section that ENS^U is a category of uniform convergence spaces, or more exactly of pre-uniform convergence spaces, with a filter space functor $T : ENS^U \longrightarrow ENS^U$. The objects of ENS^U will be called spaces. Thus we assume the conditions of 9.1 for ENS^U, with the added condition that X^* consists of Cauchy filters of X, for every space X.

For a space $X = (S,\mathcal{U})$, we denote by $\hat{q}_X : X^* \longrightarrow S$ the restriction of the induced convergence structure $q_{\mathcal{U}}$ (7.1). For $U \subset S \times S$, we call the set $(\hat{q}_X \times \hat{q}_X)(U^*)$ the closure of U in X, relative to T. We shorten this to $\hat{q}_X(U^*)$; this abus de langage will not lead to confusion. We carry this notation over to filters on $S \times S$ by the standard procedure. Note again that (2.4.1) is not applicable because $\hat{q}_X$ is in general a relation.

14f2# Theorem* Undey the assumptions of 13•!$ the following two ptatementa are logically equivalent for a space X in ENS^U .

(i) X is regular. i.e. λ : TX — > X is continuous.

(ii) X satisfies T_3 f i.e. if 0 ^ ^ , then always $\hat{q}_x$(cf>*)^U_x *

Proof. Let P_q be the graph of $\hat{q}_{Y_A}$, regarded as subspace of the product space T X X X f with projections p : p - > T X and p^f : T —> X . For U CZ |Xlx (Xj , let S_u = (p x p)$^{-1}$(U^f and define S^ accordingly for a filter <p . Then (p x:p)(S_u) C U* and (p^f K p^t)(S_u) - $\hat{q}_x$(U*) f with correspooding results for filters• If <ty € U^ 9 it follows that S^ is in U(r_q) if q_λ is continuous, i.e. p coarse (lO«5). But then ^qJ^tp*) £lλ by continuity of p^f $ and (i) « ^ (ii).

Conversely, consider g : (RftO —^ T X and g^f : R — > | X) such that g(x) $\hat{q}_x$ g^f(x) for every x eR . If 4 > 6 ^ and l^» ((gx g)(<$>))* , then M^d7$X_x$. If V £ ip, then (g x g)(u) cZ V* for some U 6 <p 9 and (g^fX g^f)(u) C $\hat{q}_x$(V*) follows. Thus (g^f X g')((!>) ^ q_x(^*) • » X satisfies T_5 , it follows that g^1 : (R&) —^ X . Thus (ii) ==$> (i) o

14.3# „Proposition. Every uniform space in ESS^U is regular.

Proof. For an entourage U of the uniform space X , choose a symmetric entourage V such that V o V o V C T U . If (xfy)<£ 4_X($^{V\#}$) t then there are filters cp and ψ> in X* such that

$$V \in \dot{x} \times \varphi , \quad V \in \varphi \times \psi , \quad V \in \psi \times \dot{y} .$$

It follows that U g i x $\dot{y}$ f i.e. (xfy) 6 U . Thus q_x(V*) CT U f and X satisfies T_3 •

14.4. We assume for the following result that an induced structure functor $P : ENS^U \to ENS^Q$ from ENS^U to a category ENS^Q of convergence spaces, and a filter space functor T' on ENS^Q are given, $\hat{q}$ then refers to T'.

Proposition. If a space X in ENS^U is regular and if $\varphi \hat{q}_{PX} x$ always implies $\varphi \hat{q}_X x$, then the induced space $PX \in ENS^Q$ is regular.

Proof. By the second hypothesis,

$$\hat{q}_{PX}((F \times F)_*) \Rightarrow \hat{q}_X((F \times F)_*) ,$$

for a filter F on $|X|$ and $x \in X$. Thus T_3 for X implies T_3 for PX.

Using 10.6 in this situation would require a stronger hypothesis.

14.5. For the first four examples in 5.6, and for the additional examples of 9.7, closure as defined in 14.1 is closure in the usual sense. Thus these examples produce the same regularity for pre-uniform convergence spaces.

Closure and T_3 depend only on the assignment $X \mapsto X^{\#}$, and not on the filter space functor T. If this functor is not available, e.g. if one uses the axioms of Cook and Fischer [5] (see 6.5), then one can reformulate continuity of $\hat{q}_X$ as follows.

14.5.1. If $g : R \to X$ and $g' : R \to |X|$ are mappings such that $g(x) \hat{q}_X g'(x)$ for every $x \in R$, then

$$((g \times g)(\Phi))_* \in \mathcal{U}_X \implies (g' \times g')(\Phi) \in \mathcal{U}_X \ ,$$

for every filter Φ on $R \times R$.

The proof of 14.2 can be transformed easily into a proof that 14.5.1 and T_3 are logically equivalent.

15* Extensions of uniformly continuous functions

15JL. We use again the assumptions and notations of 14«1* We call a space X a (S_fL0 in ENE^U diagonal, relative to the filter space functor T_f if for every mapping u : S –> X* such that u(x) q_{λ} x for all x € S * and for every filter <$£ U.. the filter $((uXu)(Cp))_f$ is again in *XL.*

Our first result connects diagonal uniform convergence spaces with the diagonal limit spaces defined by Kowalsky [16].

15*2* Proposition* Let X » (S.ti.) be a uniform convergence space in ENS^U • If, X is diagonal and u : S –> X * is a mapping such that u(x) q^ x for all x £ S_f then P q^x always implies $(u(F))_\#$ q $_{\mathcal{U}}$x . It follows that a subset A of S is dense in X if and only if q_x(A*) « S •

Proof. If P *c^x* and *(p=* u(x) , then $(u(P) \times cp)_\#$ and *4f>xX* are in *Vi.* It follows from this with 8.3.6 and 2.6*2 that

$$(\varphi \times \dot{x}) \circ ((u(F))_* \times \varphi) \; - \; (u(P))_\# X^*$$

is in 2Z. . This proves the first part* Now the closure operator A H-> q^(A*) in X is idempotent, by *[>b* \ Satz 8]. and the second part follows*

13*3* Let j : X –*j>Y* be a map in EMS^U such that q_Y((T j)(X*)) - $JY|_f$ and let Z be a space* We say that a mapping $\bar{f}$: |Y| –> |z| is a weak extent sion by continuity of a map f : X –> Z if always

(1) jdp) q_Y y = » f(cp) q_Z $\bar{f}$(y) ,

for cp£ X* and y € |Yl • If in addition $\bar{f}$ j = f , then we call $\bar{f}$ an extension of f by continuity*

If Z is a T_1 space in this situation, then every weak extension by continuity is in fact an extension; try $\varphi = \dot{x}$ in (1) for $y = f(x)$. In any case, (1) is satisfied for $y = j(x)$ if j is coarse and $\bar{f}(y) = f(x)$. Every map $g : Y \longrightarrow Z$ is an extension by continuity of the map $g\,j : X \longrightarrow Z$. If Z is separated, then a map $f : X \longrightarrow Z$ has at most one extension by continuity.

15.4. Theorem. If j is coarse, Y a diagonal space and Z a regular space, in the situation of 15.3, then every weak extension by continuity of a map $f : X \longrightarrow Z$ is a uniformly continuous map $\bar{f} : Y \longrightarrow Z$.

Proof. For each $y \in |Y|$, choose $u(y) \in X^*$ such that $j(u(y))\ \hat{q}_Y\ y$. Then $f(u(y))\ \hat{q}_Z\ \bar{f}(y)$ for a weak extension $\bar{f}$ of a map $f : X \longrightarrow Z$. Now let Φ be in $\mathcal{U}_Y$, and put

$$\Psi = (f \times f)(((u \times u)(\Phi))_*) = ((T\,f \times T\,f)(u \times u)(\Phi))_* .$$

As Y is diagonal, the filter

$$(j \times j)(((u \times u)(\Phi))_*) = ((T\,j \times T\,j)(u \times u)(\Phi))_*$$

is in $\mathcal{U}_Y$. Thus $((u \times u)(\Phi))_*$ is in $\mathcal{U}_X$, and hence Ψ in $\mathcal{U}_Z$, if j is coarse.

If $W \in \Psi$, then $(T\,f \times T\,f)(u \times u)(V) \subset W^*$ for some $V \in \Phi$. It follows that

$$(\bar{f} \times \bar{f})(V) \subset \hat{q}_Z(W^*) .$$

Thus $(\bar{f} \times \bar{f})(\Phi) \leqslant \hat{q}_Z(\Psi^*)$, and $\bar{f}$ is uniformly continuous if Z is regular.

<u>15.5. Theorem</u>. <u>If</u> j <u>is coarse</u>, Y <u>a diagonal space</u>, <u>and</u> Z <u>a separated regular complete space</u>, <u>in the situation of</u> 15.3, <u>then every map</u> $f : X \longrightarrow Z$ <u>has a unique extension to a map</u> $\bar{f} : Y \longrightarrow Z$ <u>such that</u> $f = \bar{f}\, j$.

<u>Proof</u>. Construct $u : |Y| \longrightarrow X^*$ as in the proof of 15.4. If $y \in |Y|$, then $f(u(y))\ \hat{q}_Z\ z$ for a unique $z \in |Z|$ in the present situation, with $z = f(x)$ for $y = j(x)$. We must put $\bar{f}(y) = z$. Now the proof of 15.4 can be carried through for this mapping $\bar{f}$, and thus $\bar{f}$ is a map.

<u>15.6</u>. Weak extensions by continuity can be defined in the general situation of 11.1, and the remarks in 15.3 remain valid in this situation. Extensions by continuity have usually been considered only if $j : X \longrightarrow Y$ is a dense embedding. 15.4 is well known for topological spaces, and 15.4 and 15.5 are well known for uniform spaces. In these two cases, every space Y is diagonal. Cook [4] proved 15.4 for convergence spaces.

Sjöberg [27] proved 15.4 and 15.5 for uniform convergence spaces, with the following condition for Y .

(A). For every filter Φ in $\mathcal{U}_Y$, there is an open filter Ψ in $\mathcal{U}_Y$ such that $\Phi \leqslant \Psi$.

Here $U \subset |Y| \times |Y|$ is open if the complement of U is closed, in the sense of 14.1, and a filter Ψ is open if Ψ has a base of open sets. Every uniform space satisfies condition (A). If a uniform convergence space Y satisfies condition (A), then Y is diagonal, and the induced convergence structure $q(\mathcal{U}_Y)$ is a topology.

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