DICHOTOMIES FOR LINEAR DIFFERENTIAL
EQUATIONS WITH DELAYS:

THE CARATHÉODORY CASE

by

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1. Introduction.

We consider on $[0,\infty[$ an equation of the form

$$\dot{u} + Mu = r \tag{1.1}$$

in a Banach space E, and the corresponding homogeneous equation

$$\dot{u} + Mu = 0; \tag{1.2}$$

here r is a locally integrable vector-valued function; the "solution" u is defined on $[-1,\infty[$, and M, the "memory" functional, takes a continuous function u into a locally integrable function Mu in such a way that the values of Mu on an interval $[a,b]$ depend on the values of u on $[a-1,b]$ only. The equations are to be satisfied "locally in $\underset{\sim}{L}^1$ ".

The purpose of our investigation, which continues the work in [9] (and also in [2] and [3]) is to relate properties of (1.1) such as "admissibility" ("for every r in some given function space there is a solution in some given function space") and certain forms of conditional stability behaviour ("dichotomies") of the solutions of (1.2) and of its restrictions to intervals of the form $[m,\infty[$. The method consists, as in [9], in reducing this problem to a similar problem about a linear difference equation in a function space; this difference equation can then be studied by means of the theory developed in [1]. We refer to the introduction of [9] for further comments on method and significance, and to the work of Pecelli [6] for some related results obtained under more special assumptions and by a different method.

In [9] a special instance of the "continuous case" was considered: that is, r and Mu were assumed to be continuous, and the equations were to hold everywhere; and (Mu)(t) depended only on the values of u in [t-1,t]. In this paper we describe instances of the "Carathéodory case", in which continuity is replaced by local integrability. The reduction of the problem to one about difference equations is much simpler in the Carathéodory case (contrast Theorem 6.2 with [9; Theorem 6.2]). The more basic question of the existence, uniqueness and growth of solutions, which is almost trivial in the continuous case, becomes, on the other hand, a major issue if we wish to make our "Carathéodory assumptions" as natural as possible, and consequently quite a bit more general than those usually encountered as, e.g., in [6]. The relevant results were obtained in [4] in a form suitable for use here (see Section 5).

This paper is best read in conjunction with [9], although the formal dependence on that paper consists only in the use of some proofs. On the other hand, our present approach does depend, especially in Section 8, on material in [1] and [8].

2. Spaces.

Throughout this paper, E shall denote a real or complex Banach space. The norm in E, as in all normed spaces other than the scalar fields and the function and sequence spaces described below, is denoted by $\| \ \|$. If X and Y are Banach spaces, $[X \to Y]$ denotes the Banach space of operators (bounded linear mappings) from X to Y, and we set $\mathfrak{X} = [X \to X]$.

In this paper spaces of sequences occur together with spaces of functions on certain intervals of the real line. For the former, we adopt without elaboration the notation described in [1; Sections 2 and 3]. In particular: $\omega = \{0,1,\ldots\}$, and $\underset{\sim}{s}_{[m]}(X)$ denotes the Fréchet space of all functions on $\omega_{[m]} = \{m, m+1, \ldots\}$ with values in the Banach space X, where $m \in \omega$; and notations such as $\underset{\sim}{1}^{p}_{[m]}(X)$ are to be understood by the obvious analogy. If $f \in \underset{\sim}{s}_{[m]}(X)$ and $m' \geqq m$, then $f_{[m']} \in \underset{\sim}{s}_{[m']}(X)$ is the restriction of f to $\omega_{[m']}$.

The intervals that will occur as domains of measurable functions will be $[-1,0]$ and $[m,\infty[$ for real numbers m. We shall in general follow the notation and terminology of [5; Chapter 2] for spaces consisting of such functions, with some special simplifying conventions.

Spaces of functions on $[-1,0]$ will have no label indicating the domain. For instance, $\underset{\sim}{L}^1(E)$ is the Banach space of (equivalence classes modulo null sets of) Bochner integrable functions $f\colon [-1,0] \to E$, with the norm $\|f\|_1 = \int_{-1}^{0} \|f(t)\|dt$. The space $\underset{\sim}{C}(E)$ of continuous functions $f\colon [-1,0] \to E$ with the norm $\|f\| = \max\|f(t)\|$, which plays a central part in our work, is abbreviated to $\underset{\sim}{E}$, and its norm written without a subscript.

As indicated in these examples, thick hollow bars are used for the norms of function spaces with $[-1,0]$ as domain. This convention permits the following arrangement: suppose that, e.g., $g \in \underset{\sim}{1}^{q}_{[m]}(\underset{\sim}{L}^{p}(E))$, where $1 \leqq p,q \leqq \infty$ and $m \in \omega$; then $\|g\|$ is the element of $\underset{\sim}{1}^{q}_{[m]}(\underset{\sim}{L}^{p})$ - the argument R is omitted, as usual - given by $\|g\|(n) = \|g(n)\|$, $n \in \omega_{[m]}$ (where $\|g(n)\|(t) = \|(g(n))(t)\|$ for all $t \in [-1,0]$; the latter norm is the norm in E); $\|g\|_p$ is the element of $\underset{\sim}{1}^{q}_{[m]}$ given by $\|g\|_p(n) = \|g(n)\|_p = \|\,\|g(n)\|\,\|_p$, $n \in \omega_{[m]}$; thus $\|g\|_p = \|\,\|g\|\,\|_p$; and $\|g\|_q = \|\,\|g\|_p\|_q$ is the norm of g as an element of $\underset{\sim}{1}^{q}_{[m]}(\underset{\sim}{L}^{p}(E))$.

We recall from [5; Chapter 2] that $b\mathfrak{J}$ is the class of all

Banach spaces $\underset{\sim}{F}$ of (equivalence classes of) measurable functions $\varphi: [-1,0] \to \mathbb{R}$ such that

(N): $\underset{\sim}{F}$ is stronger than $\underset{\sim}{L}^1$, i.e., $\underset{\sim}{F}$ is algebraically contained in $\underset{\sim}{L}^1$ and there exists a number $\alpha_{\underset{\sim}{F}} > 0$ such that $\|\varphi\|_1 \leq \alpha_{\underset{\sim}{F}} \|\varphi\|_{\underset{\sim}{F}}$ for all $\varphi \in \underset{\sim}{F}$;

(F): if $\varphi \in \underset{\sim}{F}$ and $\psi: [-1,0] \to \mathbb{R}$ is measurable and $|\psi| \leq |\varphi|$, then $\psi \in \underset{\sim}{F}$ and $\|\psi\|_{\underset{\sim}{F}} \leq \|\varphi\|_{\underset{\sim}{F}}$.

If $\underset{\sim}{F} \in \mathfrak{B}$, then $\underset{\sim}{F}(E)$ denotes the Banach space of (equivalence classes of) measurable functions $f: [-1,0] \to E$ such that $\|f\| \in \underset{\sim}{F}$, with the norm $\|f\|_{\underset{\sim}{F}} = \|\,\|f\|\,\|_{\underset{\sim}{F}}$.

In considering spaces of functions defined on intervals of the type $[m,\infty[$, we shall use the following conventions. If $m \leq m'$ and f is some function defined on $[m,\infty[$, $f_{[m']}$ shall denote its restriction to $[m',\infty[$. The subscript $[m]$ is also used when the fact that $[m,\infty[$ is the domain has to be recorded (these usages are compatible). Thus $\underset{\sim}{L}_{[m]}(E)$ denotes the space of all (equivalence classes of) measurable functions $f: [m,\infty[\to E$ that are Bochner integrable on each compact interval; $\underset{\sim}{K}_{[m]}(E)$ denotes the space of <u>all</u> continuous functions $f: [m,\infty[\to E$ (cf. [9]); and similarly for the space $\underset{\sim}{M}_{[m]}(E)$ of all functions $f \in \underset{\sim}{L}_{[m]}(E)$ with $\|f\|_{\underset{\sim}{M}} = \sup_{t \geq m} \int_t^{t+1} \|f(s)\|\,ds < \infty$; for the spaces $\underset{\sim}{L}^p_{[m]}(E)$, $1 \leq p \leq \infty$; and for the space $\underset{\sim}{C}_{[m]}(E)$ of bounded continuous functions $f: [m,\infty[\to E$ with the supremum norm, and the subspace $\underset{\sim}{C}_{0[m]}(E)$ of those that tend to zero at infinity. The norms of all normed spaces of this kind will be indicated, as in [5], by thick bars with the appropriate subscript; the subscript is omitted for the supremum norm.

3. Slicing operations.

Let $m \geq 0$ be a given real number. For each $t \geq m$ we define the linear mapping $\Pi(t): L^1_{[m-1,\infty[}(E) \to L^1(E)$ by

(3.1) $(\Pi(t)f)(s) = f(t+s), \quad s \in [-1,0], \quad f \in L^1_{[m-1,\infty[}(E).$

Thus $\Pi(t)$ maps f into the "slice" of f between $t-1$ and t, transplanted to $[-1,0]$ for convenience. (Note that indication of m is omitted; this will not cause any confusion.)

When m is an integer, we define $\varpi f \in s_{[m]}(L^1(E))$ for each $f \in L^1_{[m-1,\infty[}(E)$ by

(3.2) $(\varpi f)(n) = \Pi(n)f, \quad n = m, m+1, \ldots .$

Thus $\varpi: L^1_{[m-1,\infty[} \to s_{[m]}(L^1(E))$ is a linear bijective mapping. This mapping has obvious restrictions to linear mappings of $K_{[m-1]}(E)$ into $s_{[m]}(E)$, of $C_{[m-1]}(E)$ into $l_{[m]}(E)$, and of $C^\infty_{[m-1]}(E)$ into $l^\infty_{[m]}(E)$.

The mapping ϖ has other restrictions that are "natural" isomorphisms between certain normed function spaces: e.g., $\varpi: L^p_{[m-1,\infty[}(E) \to l^p_{[m]}(L^p(E))$ is a congruence (linear isometry) for $1 \leq p \leq \infty$; $\varpi: M_{[m-1]}(E) \to l^\infty_{[m]}(L^1(E))$ is an isomorphism with norm 1, the norm of the inverse being 2; $\varpi: T_{[m-1]}(E) \to l^1_{[m]}(L^\infty(E))$ is another isomorphism, with norm 2, the norm of the inverse being 1 (for the space T see [5; pp. 61-62]). We might indeed define new normed spaces of functions on $[m,\infty[$ in this way, but we shall not do this here.

4. Memories.

In this section we shall make precise some of the assumptions on the "memory functional" M that appears in (1.1). We express the linearity of the functional and the fact that the scope of the memory extends at most one unit of time into the past by the following definition.

A memory is a linear mapping $M: \underset{\sim}{K}_{[-1]}(E) \to \underset{\sim}{L}_{[0]}(E)$ such that

(4.1) $\chi_{[a-1,b]}u = 0$ implies $\chi_{[a,b]}(Mu) = 0$

for all $u \in \underset{\sim}{K}_{[-1]}(E)$ and each interval $[a,b] \subset [0,\infty[$.

It is clear that a memory is uniquely determined by its restriction to $\underset{\sim}{C}_{[-1]}(E)$.

Condition (4.1) permits, for each $m \geqq 0$, the "cutting down" of M to a linear mapping $M_{[m]}: \underset{\sim}{K}_{[m-1]}(E) \to \underset{\sim}{L}_{[m]}(E)$: indeed, each $u \in \underset{\sim}{K}_{[m-1]}(E)$ can be written as $u = v_{[m-1]}$ for some $v \in \underset{\sim}{K}_{[-1]}(E)$, and we may set $M_{[m]}u = (Mv)_{[m]}$; since $v'_{[m-1]} = u = v_{[m-1]}$ implies $\chi_{[m-1,t]}(v'-v) = 0$ for each $t \geqq m$, (4.1) yields $(M(v'-v))_{[m]} = 0$; thus the definition does not depend on the choice of v. We have $M_{[0]} = M$; if $m' \geqq m \geqq 0$, these "cut-down" memories satisfy

(4.2) $$M_{[m']}u_{[m'-1]} = (M_{[m]}u)_{[m']} \qquad u \in \underset{\sim}{K}_{[m-1]}(E).$$

A memory is usually assumed to have some continuity or boundedness properties; it is typical to assume (or imply by the assumptions on M) that the restriction of M to $\underset{\sim}{C}_{[-1]}(E)$ is continuous (equivalently, closed) as a mapping from $\underset{\sim}{C}_{[-1]}(E)$ to the Fréchet space $\underset{\sim}{L}_{[0]}(E)$. For our purposes, we shall usually require a uniform condition of this type, namely:

(M): The restriction of the memory M to $\underset{\sim}{C}_{[-1]}(E)$ is a bounded linear mapping $M_{\underset{\sim}{C}}: \underset{\sim}{C}_{[-1]}(E) \to \underset{\sim}{M}_{[0]}(E)$.

Thus $M \mapsto \|M_{\underset{\sim}{C}}\|$ is a norm on the linear space of all memories satisfying (M).

Remark 1. Our definition of "memory" coincides with that of "short memory" in [4; Section 5], and our condition (M) is identical with condition (M') in that paper. The results in [4; Sections 5 and 6] are therefore applicable here (see Section 5).

A special kind of memory (one with no recall!) is described as follows: if $L \in \mathbf{M}_{[0]}(E)$, the mapping $M_L : \mathbf{K}_{[-1]}(E) \to \mathbf{L}_{[0]}(E)$ defined by

$$(4.3) \qquad (M_L u)(t) = L(t)u(t) \qquad t \in [0,\infty[, \quad u \in \mathbf{K}_{[-1]}(E)$$

is a memory satisfying (M), with $\|(M_L)_{\mathbf{C}}\| \leqq \|L\|_{\mathbf{M}}$.

We shall wish to investigate equation (1.1) by allowing r to range over a suitable function space. Our methods will be applicable if the behaviour of the memory M is adapted to the local properties of the functions of such a space.

For a memory M, Condition (M) may be rephrased as follows: *The restriction of the composite mapping* $\varpi M: \mathbf{K}_{[-1]}(E) \to \mathbf{s}_{[1]}(\mathbf{L}^1(E))$ *to* $\mathbf{C}_{[-1]}(E)$ *is a bounded linear mapping from* $\mathbf{C}_{[-1]}(E)$ *to* $\mathbf{l}^{\infty}_{[1]}(\mathbf{L}^1(E))$; the norm of this mapping, incidentally, lies between $\frac{1}{2}\|M_{\mathbf{C}}\|$ and $\|M_{\mathbf{C}}\|$. The condition we now envisage is a more restrictive assumption of the same type on the slices of Mu. For each given space $\mathbf{F} \in b\mathfrak{F}$ (see Section 2), we consider the following condition on a memory M:

$(M_{\mathbf{F}})$: *The restriction of* ϖM *to* $\mathbf{C}_{[-1]}(E)$ *is a bounded linear mapping from* $\mathbf{C}_{[-1]}(E)$ *to* $\mathbf{l}^{\infty}_{[1]}(\mathbf{F}(E))$. *The norm of this mapping shall be denoted by* $\|\varpi M\|_{\mathbf{F}}$.

We remark that, if M satisfies (M), the Closed Graph Theorem reduces the verification of $(M_{\mathbf{F}})$ to ascertaining that ϖM maps $\mathbf{C}_{[-1]}(E)$ into $\mathbf{l}^{\infty}_{[1]}(\mathbf{F}(E))$. Certain special cases of Condition $(M_{\mathbf{F}})$ are easier to state. We have already observed that $(M_{\mathbf{L}^1})$ is equivalent to (M); and since every space $\mathbf{F} \in b\mathfrak{F}$ is stronger than $\mathbf{L}^1$, each condition $(M_{\mathbf{F}})$ implies (M). In the same vein, $(M_{\mathbf{L}^\infty})$ may be rephrased as follows: *The restriction of* M *to* $\mathbf{C}_{[-1]}(E)$ *is a bounded linear mapping from* $\mathbf{C}_{[-1]}(E)$ *to* $\mathbf{L}^{\infty}_{[0]}(E)$. Similar rephrasings, involving other translation-invariant function spaces, are of interest for $\mathbf{F} = \mathbf{L}^p$, $1 < p < \infty$, among others, and may be supplied by the reader.

Remark 2. An important special kind of memory is, of course, the autonomous or time-independent memory; i.e., more precisely, a memory that commutes with left-translations. It will be shown elsewhere that if E is isomorphic to a Hilbert space (in particular finite-dimensional), an autonomous memory satisfies $(\underset{\sim}{M}_L 2)$.

We have spoken as if the memory functional M apearing in (1.1) were to be itself subjected to Condition $(\underset{\sim}{M}_F)$. In actual fact, however, it is typical of the problems we are dealing with that the condition need only be imposed on the dependence of Mu on the past of u, while its dependence on the current value of u is less restricted: for a given space $\underset{\sim}{F} \in b\mathfrak{F}$, we shall say that the memory M satisfies the standard assumptions with respect to $\underset{\sim}{F}$ if $M = M_L + M'$, where M_L is given by (4.3) for some $L \in \underset{\sim}{M}_{[0]}(\mathring{E})$, and M' is a memory satisfying $(\underset{\sim}{M}_F)$. Under these conditions, M obviously satisfies (M). We remark that these conditions include as a very special case those considered in [6]. The results in [6] can, as a consequence, be obtained by a specialzation of the methods and results of the present work.

5. Solutions.

We say that a function $f \in \underset{\sim}{K}_{[m]}(E)$ is a primitive (function) if there exists $g \in \underset{\sim}{L}_{[m]}(E)$ such that $f(t) - f(m) = \int_m^t g(s)ds$ for all $t \in [m,\infty[$; then g is unique, is denoted by $\mathring{f}$, and is called the derivative of f.

Assume that we are given a memory M and, in addition, a function $r \in \underset{\sim}{L}_{[0]}(E)$. A solution of the "differential equation with delay"

$$\mathring{u} + Mu = r \tag{5.1}$$

is a function $u \in \underset{\sim}{K}_{[-1]}(E)$ whose restriction $u_{[0]}$ to $[0,\infty[$ is a primitive whose derivative $\mathring{u}_{[0]}$ satisfies $\mathring{u}_{[0]} + Mu = r$ in

$L_{[0]}(E)$. More generally, for each $m \geq 0$, a solution of

$$(5.1)_{[m]} \qquad \dot{u}_{[m]} + M_{[m]}u = r_{[m]}$$

is a function $u \in K_{[m-1]}(E)$ whose restriction $u_{[m]}$ to $[m,\infty[$ is a primitive whose derivative $\dot{u}_{[m]}$ satisfies $(5.1)_{[m]}$ in $L_{[m]}(E)$.

These definitions of course also apply to the homogeneous equations

$$(5.2) \qquad \dot{u} + Mu = 0$$

$$(5.2)_{[m]} \qquad \dot{u}_{[m]} + M_{[m]}u = 0.$$

As usual, it is preferable to deal with integral equations equivalent to these differential equations.

5.1. Lemma. Let the memory M and $r \in L_{r}(E)$ ~~be given~~. A function $u \in K_{[m-1]}(E)$ is a solution of $(5.1)_{[m]}$ if and only if it satisfies

$$(5.3) \qquad u(t) - u(m) - \int_m^t ((M_{[m]}u)(s) - r(s))ds$$

for all $t \geq m$. If $m' \geq m \geq 0$ and u is a solution of $(5.1)_{[m]}$, then $u_{[m']}$ is a solution of $(5.1)_{[m']}$.

Proof. Definition of "solution" and (4.2).

We quote from [4] the results relative to the existence and uniqueness of solutions of $(5.1)_{[m]}$, estimates on their growth, and the compactness of certain "transition operators", that we shall require here. The applicability of these theorems to our present situation was pointed out in Remark 1 in Section 4.

5.2. Theorem. Let M be a memory satisfying (M). Then there exists a number $a > 0$ and, for each $m \geq 0$, there are linear mappings $P(m): E \to K_{[m]}(E)$ and $Q(m): L_{r}(E) \to K_{[m]}(E)$ such that for every $v \in E$ and $r \in L_{r}(E)$:

(1): $u = P(m)v + Q(m)r$ is the unique solution of $(5.1)_{[m]}$, with $H(m)u = v$;

(2): if $t \geq m$, then $u \in K_{[m]}(E)$ satisfies $H(m)u = v$

and (5.3) *for* $m \leqq t \leqq t_0$ *if and only if* u *and* $P(m)v + Q(m)r$ *agree on* $[m-1, t_0]$;

(3): *for all* $t \geqq m$,

$$\|(P(m)v)(t)\| \leqq e^{\sigma[t-m+1]}\|v\|$$

$$\|(Q(m)r)(t)\| \leqq \|M_C\|^{-1}(e^{\sigma}-1)\int_m^t e^{\sigma[t-s]}\|r(s)\|ds$$

(*if* $M = 0$, *read* 1 *for* $\|M_C\|^{-1}(e^{\sigma}-1)$); *here* $[\ ]$ *is the "greatest integer" function*;

(4): *if* E *is finite-dimensional*, $\Pi(m+1)P(m): E \to E$ *is compact*.

Proof. [4; Theorems 5.1, 5.3, 6.2].

5.3. *Corollary*. *Let* M *be a memory satisfying* (M). *If* u *is a solution of* $(5.2)_{[m]}$ *for some* $m \geqq 0$, *then*

$$\|\Pi(t)u\| \leqq e^{\sigma(t-t_0+1)}\|\Pi(t_0)u\| \quad \text{for all} \quad t \geqq t_0 \geqq m,$$

where σ *is as in Theorem* 5.2.

Proof. $u_{[t_0-1]}$ is a solution of $(5.2)_{[t_0]}$ (Lemma 5.1); the conclusion follows by applying Theorem 5.2,(1),(3) to this solution. (Cf. [4; Corollaries 5.2, 5.4].)

6. *The associated difference equation*.

Let us assume that the memory M satisfies (M). We construct a linear difference equation in E in such a way that the values of a solution of this equation are the slices of a solution of (5.1). For this purpose, we define the linear mappings

$$(6.1) \qquad \begin{aligned} A(n) &= -\Pi(n)P(n-1): E \to E \\ B(n) &= \Pi(n)Q(n-1): L_{[0]}(E) \to E \end{aligned} \qquad n = 1,2,\ldots$$

and observe that Theorem 5.2,(3) implies

$$(6.2) \qquad \begin{aligned} &A(n) \in \tilde{E}, \qquad \|A(n)\| \leqq e^{\sigma} \qquad n = 1,2,\ldots \\ &\|B(n)r\| \leqq \|M_C\|^{-1}(e^{\sigma}-1)\|(\varpi r)(n)\|_1, \quad n = 1,2,\ldots, \quad r \in L_{[0]}(E). \end{aligned}$$

We set $A = (A(n)) \in l^{\infty}_{[1]}(\tilde{E})$ and define the linear mapping $B: L_{[0]}(E) \to s_{[1]}(E)$ by $(Br)(n) = B(n)r$, $n = 1,2,\ldots$, $r \in L_{[0]}(E)$.

With A thus defined, we consider the following difference equations in $\underset{\sim}{E}$:

$$(6.3) \qquad x(n) + A(n)x(n-1) = f(n) \qquad n = 1,2,\ldots$$

$$(6.4) \qquad x(n) + A(n)x(n-1) = 0 \qquad n = 1,2,\ldots$$

and their restrictions $(6.3)_{[m]}$ and $(6.4)_{[m]}$ to $n = m+1, m+2,\ldots$ for each $m \in \omega$. Here $f \in \underset{\sim}{s}_{[1]}(\underset{\sim}{E})$.

The fact that (6.3) and (6.4) are, in some sense, reduced forms of (5.1) and (5.2) is expressed by the following proposition.

6.1. *Lemma*. *Let* $m \in \omega$ *and* $r \in \underset{\sim}{L}_{[0]}(E)$ *be given*. *A function* $x \in \underset{\sim}{s}_{[m]}(\underset{\sim}{E})$ *is a solution of* $(6.3)_{[m]}$ *with* $f = Br$ *if and only if* $x = \varpi u$ *for some solution* u *of* $(5.1)_{[m]}$. *In particular*, x *is a solution of* $(6.4)_{[m]}$ *if and only if* $x = \varpi u$ *for some solution* u *of* $(5.2)_{[m]}$.

Proof. This is a direct consequence of Theorem 5.2,(1) and (6.1), and a straightforward computation. The details may be found in the proof of [9; Lemma 6.1], which could be reproduced verbatim.

As usual, the main problem in applying difference-equation theory via Lemma 6.1 to our equations (5.1) and (5.2) is that not every $f \in \underset{\sim}{s}_{[1]}(\underset{\sim}{E})$ is of the form $f = Br$. Our fundamental result (Theorem 6.2) states that it is still possible, however, to relate equation (6.3) with arbitrary f to equation (5.1) with a suitable r.

In what follows, we assume that $M = M_L + M'$ satisfies the standard assumptions with respect to a given space $\underset{\sim}{F} \in b\mathfrak{F}$.

Let $V \in \underset{\sim}{K}_{[0]}(\tilde{E})$ be the unique solution of the operator differential equation $\dot{V} + LV = 0$ that satisfies $V(0) = I$ (I is the identity on E). We refer to [5; Section 31] for a detailed account of this operator-valued function. In particular, the values of V are invertible, and we write $V^{-1} \in \underset{\sim}{K}_{[0]}(\tilde{E})$ for the function defined by

$V^{-1}(t) = V(t)^{-1}$, $t \in [0,\infty[$. We also have

$$(6.5) \qquad \|V(t)V^{-1}(s)\| \leqq \exp\left|\int_s^t \|L(s')\|ds'\right|, \qquad s,t \geqq 0.$$

6.2. Theorem. *Assume that* $M = M_L + M'$ *satisfies the standard assumptions with respect to a given space* $\mathbf{F} \in b\mathfrak{F}$, $\mathbf{F} \neq \{0\}$. *For each* $f \in \mathbf{s}_{[1]}(\mathbf{E})$ *there exists* $r \in \mathbf{L}_{[0]}(E)$ *with* $\varpi r \in \mathbf{s}_{[1]}(\mathbf{F}(E))$ *and such that*

$$(6.6) \qquad \|(\varpi r)(n)\|_{\mathbf{F}} \leqq k(\|f(n-1)\| + \|f(n)\|), \qquad n = 1,2,\ldots,$$

and the solution w *of*

$$(6.7) \qquad w(n) + A(n)w(n-1) = f(n) - (Br)(n), \qquad n = 1,2,\ldots$$

with $w(0) = 0$ *satisfies*

$$(6.8) \qquad \|w(n)\| \leqq (1 + \exp\|L\|_{\mathbf{M}})\|f(n)\|, \qquad n = 0,1,\ldots,$$

where we set $f(0) = 0$, *and* $k > 0$ *depends only on* $\mathbf{F}$, $\|L\|_{\mathbf{M}}$, *and* $\|\varpi M'\|_{\mathbf{F}}$.

Proof. There exists $\varphi \in \mathbf{F}$ such that $\varphi \geqq 0$ and $\int_{-1}^0 \varphi(s)ds = 1$. We define $w \in \mathbf{s}_{[0]}(\mathbf{E})$ by $w(0) = 0$ and

$$(6.9) \qquad (w(n))(s) = (f(n))(s) - (\int_{-1}^s \varphi(s')ds')V(n+s)V^{-1}(n)(f(n))(0),$$
$$-1 \leqq s \leqq 0, \qquad n = 1,2,\ldots .$$

It is obvious that each $w(n)$ is continuous, hence in $\mathbf{E}$. Also,

$$(6.10) \qquad (w(n))(-1) = (f(n))(-1), \quad (w(n))(0) = 0, \qquad n = 0,1,\ldots;$$

and (6.9) and (6.5) yield

$$\|w(n)-f(n)\| \leqq \|f(n)\|\exp\|L\|_{\mathbf{M}}, \qquad n = 1,2,\ldots,$$

so that (6.8) holds (it is trivial for $n = 0$).

We now construct r. For this purpose we choose, for each $n \in \omega_{[1]}$, a function $z_n \in \mathbf{C}_{[n-2]}(E)$ such that

$$(6.11) \qquad \Pi(n-1)z_n = -\,w(n-1) \qquad \Pi(n)z_n = f(n) - w(n)$$

and such that z_n is constant on $[n,\infty[$; this is possible on account of (6.10). Then

$$(6.12) \qquad \|z_n\| = \max\{\|w(n-1)\|,\|f(n)-w(n)\|\} \leqq$$
$$\leqq \max\{\|f(n-1)\|(1+\exp\|L\|_{\mathbf{M}}),\ \|f(n)\|\exp\|L\|_{\mathbf{M}}\}.$$

We now define $r \in L_\infty(E)$ by

$$(6.13) \qquad r(t) = \varphi(t-n)V(t)V^{-1}(n)(f(n))(0) + (M'_{[n-1]}z_n)(t),$$
$$n-1 < t \leq n, \qquad n=1,2,\ldots .$$

From (6.5) and the fact that M' satisfies $(M_{\tilde{F}})$ it follows that $(\omega r)(n) \in \tilde{F}(E)$ and

$$\|(\omega r)(n)\|_{\tilde{F}} \leq \|\varphi\|_{\tilde{F}}\|f(n)\|\exp\|L\|_{\tilde{M}} + \|\omega M'\|_{\tilde{F}}\|z_n\|, \qquad n = 1,2,\ldots;$$

combining this with (6.12) we find (6.6) with

$$k = \|\omega M'\|_{\tilde{F}}\exp\|L\|_{\tilde{M}} + \max\{\|\omega M'\|_{\tilde{F}}, \|\varphi\|_{\tilde{F}}\exp\|L\|_{\tilde{M}}\}.$$

It remains for us to prove that w and r thus constructed satisfy (6.7). For this purpose, let $n \in \omega_{[-1]}$ and t, $n-1 < t \leq n$, be fixed for the time being. In the following computation, we use in succession: (6.11) and (6.10); (6.9); differentiation of products and the definition of V; (6.13) and (6.9); (6.11); the definition of M and (4.3).

$$z_n(t) - z_n(n-1) = (f(n)-w(n))(t-n) = (\int_{-1}^{t-n}\varphi(s')ds')V(t)V^{-1}(n)(f(n))(0) =$$
$$= \int_{n-1}^{t}(\varphi(s-n)V(s) - (\int_{-1}^{s-n}\varphi(s')ds')L(s)V(s))V^{-1}(n)(f(n))(0)ds =$$
$$= -\int_{n-1}^{t}((M'_{[n-1]}z_n)(s) - r(s) + L(s)(f(n)-w(n))(s-n))ds =$$
$$= -\int_{n-1}^{t}((M'_{[n-1]}z_n)(s) - r(s) + L(s)z_{n s})ds = -\int_{n-1}^{t}((M_{[n-1]}z_n)(s)-r(s))ds .$$

Since this equality holds for all $t \in]n-1,n]$, it follows from Theorem 5.2,(2) that z^n agrees on $[n-2,n]$ with $P(n-1)\Pi(\omega-1)z^n + Q(n-1)r = -P(n-1)w(n-1) + Q(n-1)r$. Combining this with (6.11) and (6.1) we find

$$f(n) - w(n) = \Pi(n)z^n = \Pi(n)(-P(n-1)w(n-1) + Q(n-1)r) =$$
$$= A(n)w(n-1) + B(n)r;$$

that is, $w(n-1)$ and $w(n)$ satisfy (6.7) for the given n. Since n was arbitrary, the proof is complete.

7. Admissibility.

The purpose of Theorem 6.2 was to allow us to replace the study of equations (5.1), (5.2) by that of the difference equations (6.3), (6.4); in this section and the next we propose to show how the method works. We shall assume that the memory M satisfies the standard assumptions with respect to a given space $\underset{\sim}{F} \in b\mathfrak{F}$, $\underset{\sim}{F} \neq \{0\}$ (this extra assumption excludes a trivial case in which the equations are ordinary differential equations). A and B are defined as in Section 6.

We suppose that the reader is acquainted with the concept of $\mathcal{t}$-pairs and $\mathcal{t}^{\rightarrow}$-pairs of sequence spaces, i.e., pairs $(\underset{\sim}{b},\underset{\sim}{d})$ of sequence spaces with $\underset{\sim}{b} \in b\mathcal{t}$ or $\underset{\sim}{b} \in b\mathcal{t}^{\rightarrow}$, respectively, and $\underset{\sim}{d} \in b\mathcal{t}$ in either case (the classes $b\mathcal{t}$ and $b\mathcal{t}^{\rightarrow}$ of translation-invariant Banach sequence spaces are discussed in [1; Section 3]). We recall that such a pair $(\underset{\sim}{b},\underset{\sim}{d})$ is admissible for A if (6.3) has a solution $x \in \underset{\sim}{d}_{[0]}(\underset{\sim}{E})$ for every $f \in \underset{\sim}{b}_{[1]}(\underset{\sim}{E})$. For details, see [1; Section 8].

7.1. Theorem. Assume that the memory $M = M_L + M'$ satisfies the standard assumptions with respect to a given space $\underset{\sim}{F} \in b\mathfrak{F}$, $\underset{\sim}{F} \neq \{0\}$. For each given $\mathcal{t}^{\rightarrow}$-pair (or, in particular, $\mathcal{t}$-pair) $(\underset{\sim}{b},\underset{\sim}{d})$ the following statements are equivalent:

(a): $\underset{\sim}{b}$ is stronger than $\underset{\sim}{d}$; and for every $r \in \underset{\sim}{L}_{[0]}(E)$ with $\varpi r \in \underset{\sim}{b}_{[1]}(\underset{\sim}{F}(E))$ equation (5.1) has a solution u with $\varpi u \in \underset{\sim}{d}_{[0]}(\underset{\sim}{E})$;

(b): $(\underset{\sim}{b},\underset{\sim}{d})$ is admissible for A.

Proof. (a) implies (b): Let $f \in \underset{\sim}{b}_{[1]}(\underset{\sim}{E})$ be given, and let r, w be as provided by Theorem 6.2. Since $\underset{\sim}{b} \in b\mathcal{t}^{\rightarrow}$, (6.6) implies $\varpi r \in \underset{\sim}{b}_{[1]}(\underset{\sim}{F}(E))$. Further, (6.8) implies $w \in \underset{\sim}{b}_{[0]}(\underset{\sim}{E})$, whence $w \in \underset{\sim}{d}_{[0]}(\underset{\sim}{E})$, since $\underset{\sim}{b}$ is stronger than $\underset{\sim}{d}$.

By the assumption, (5.1) with this r has a solution u such that $\varpi u \in \underset{\sim}{d}_{[0]}(\underset{\sim}{E})$. By Lemma 6.1 we have $(\varpi u)(n) + A(n)(\varpi u)(n-1) =$

$= (Br)(n)$, $n = 1,2,\ldots$; since w is a solution of (6.7), we conclude that $x = \varpi u + w \in \underset{\sim}{d}_{[0]}(\underset{\sim}{E})$ is a solution of (6.3). Thus $(\underset{\sim}{b},\underset{\sim}{d})$ is admissible for A.

(b) implies (a): By (6.2), $A \in \underset{\sim}{l}^{\infty}_{[1]}(\tilde{\underset{\sim}{E}})$; since $(\underset{\sim}{b},\underset{\sim}{d})$ is admissible for A, we conclude that $\underset{\sim}{b}$ is stronger than $\underset{\sim}{d}$ [7; Lemma 4.1]. Let now $r \in \underset{\sim}{L}_{[0]}(E)$ be given with $\varpi r \in \underset{\sim}{b}_{[1]}(\underset{\sim}{F}(E))$. Then (6.2) and the fact that $\underset{\sim}{F}$ satisfies Condition (N) (see Section 2) imply $\|Br\| \leq \|M_{\underset{\sim}{C}}\|^{-1}(e^{\sigma}-1)\alpha_{\underset{\sim}{F}}\|\varpi r\|_{\underset{\sim}{F}}$, so that $Br \in \underset{\sim}{b}_{[1]}(\underset{\sim}{E})$. Since $(\underset{\sim}{b},\underset{\sim}{d})$ is admissible for A, there exists a solution $x \in \underset{\sim}{d}_{[0]}(\underset{\sim}{E})$ of $x(n) + A(n)x(n-1) = (Br)(n)$, $n = 1,2,\ldots$, and by Lemma 6.1 there exists a solution u of (5.1) with $\varpi u = x \in \underset{\sim}{d}_{[0]}(\underset{\sim}{E})$, as asserted in (a).

If $\underset{\sim}{B}$ is a subset of $\underset{\sim}{L}_{[0]}(E)$ and $\underset{\sim}{D}$ is a subset of $\underset{\sim}{K}_{[-1]}(E)$, it is in keeping with earlier terminology to say that the pair $(\underset{\sim}{B},\underset{\sim}{D})$ is admissible for M - more loosely, for (5.1) - if for every $r \in \underset{\sim}{B}$ there exists a solution $x \in \underset{\sim}{D}$ of (5.1). Thus Statement (a) in Theorem 7.1 expresses the admissibility of a certain pair $(\underset{\sim}{B},\underset{\sim}{D})$ for M. To exemplify the uses of Theorem 7.1, we shall now specify $\underset{\sim}{B}$ to be one of the spaces $\underset{\sim}{L}^{p}_{[0]}(E)$, $1 \leqq p \leqq \infty$, or $\underset{\sim}{M}_{[0]}(E)$ or $\underset{\sim}{T}_{[0]}(E)$, and $\underset{\sim}{D}$ to be either $\underset{\sim}{C}_{[-1]}(E)$ or $\underset{\sim}{C}_{0[-1]}(E)$; but the choices may easily be extended in the spirit of [5; Chapter 2] and the remark at the end of Section 3. Following earlier practice, the name of a pair of such spaces is abbreviated, as, e.g., $(\underset{\sim}{L}^{p},\underset{\sim}{C}_0)$ for $(\underset{\sim}{L}^{p}_{[0]}(E),\underset{\sim}{C}_{0[-1]}(E))$, since there is no ambiguity.

We now record some special cases covered by Theorem 7.1.

7.2. Corollary. *Assume that the memory* $M = M_L + M'$ *satisfies the standard assumptions with respect to a given space* $\underset{\sim}{F}$. *With* $\underset{\sim}{F}$, $(\underset{\sim}{B},\underset{\sim}{D})$, $(\underset{\sim}{b},\underset{\sim}{d})$ *as specified in the following table*, $(\underset{\sim}{B},\underset{\sim}{D})$ *is admissible for* M *if and only if* $(\underset{\sim}{b},\underset{\sim}{d})$ *is admissible for* A.

F	(B,D)	(b,d)	
L^p	(L^p, C)	(l^p, r)	$1 \leqq p \leqq \infty$
L^p	(L^p, C_0)	$(l^p, c_0$	$1 \leqq p \leqq \infty$
L^1	(M, C)	(l^{∞}, l^{∞})	
$\underset{\sim}{L}^{\infty}$	$(\underset{\sim}{T}, \underset{\sim}{C})$	$(\underset{\sim}{l}^1, \underset{\sim}{l}^{\infty})$	
$\underset{\sim}{L}^{\infty}$	$(\underset{\sim}{T}, \underset{\sim}{C}_0)$	$(\underset{\sim}{l}^1, \underset{\sim}{l}^{\infty}_0)$	

Proof. Theorem 7.1 and the remarks on the slicing operator tsr in Section 3.

8. *Admissibility and the solutions of the homogeneous equation*.

The admissibility of certain pairs $(\underset{\sim}{b}, \underset{\sim}{d})$ of sequence spaces for A implies, under some additional assumptions, an (*ordinary*) *dichotomy* or an *exponential dichotomy* of the solutions of the homogeneous equations $(6.4)_{[m]}$ (see [1; Section 7]). An exponential dichotomy, for instance, may roughly be described thus: the bounded solutions tend uniformly exponentially to 0, there exists a "complementary" manifold of solutions of (6.4) tending uniformly exponentially to infinity, solutions of the two kinds remain uniformly apart, and together they span all solutions. Since Lemma 6.1 provides a bijective correspondence between solutions of $(5.2)_{[m]}$ and solutions of $(6.4)_{[m]}$, for integral m, Theorem 7.1 and Corollary 7.2 will alow us to translate that result into an analogous implication for differential equations with delays.

In order to avoid unenlightening complications, we restrict ourselves in this section to the case in which $\underset{\sim}{d}$ is specified to be $\underset{\sim}{l}^{\infty}$, i.e., in which *bounded* solutions of (5.1) and of (6.3) are sought. The case in which $\underset{\sim}{d}$ is $\underset{\sim}{l}^{\infty}_0$, so that attention is centred on solutions of (5.1) and of (6.3) that tend to 0, can easily be treated in a similar fashion; as can also cases with more general $\underset{\sim}{d}$ *e bt,* with appropriate use of [1].

We assume given a memory $M = M_L + M'$ that satisfies the standard assumptions with respect to a given space $\underset{\sim}{F} \in b\mathfrak{F}$, $\underset{\sim}{F} \neq \{0\}$. We denote by $\underset{\sim}{E}_0(0) \subset \underset{\sim}{E}$ the set of "initial slices" $\Pi(0)u$ of the bounded solutions u of (5.2); by Lemma 6.1, $\underset{\sim}{E}_0(0)$ is the set of values at $n = 0$ of the bounded solutions of (6.4).

We now state the main "direct" theorem, to the effect that the admissibility of certain pairs of function spaces for M implies a behaviour of the solutions of $(5.2)_{[m]}$ that may be described as an ordinary or an exponential dichotomy.

8.1. *Theorem. Assume that the memory* $M = M_L + M'$ *satisfies the standard assumptions with respect to a given space* $\underset{\sim}{F} \in b\mathfrak{F}$, $\underset{\sim}{F} \neq \{0\}$. *Assume that* $E_0(0)$ *is closed in* E. *Assume that* $\underset{\sim}{b} \in b\vec{\ell}$ *(in particular* $\underset{\sim}{b} \in b\ell$*) is [not stronger than* $\underset{\sim}{1}^1$ *and] such that for every* $r \in \underset{\sim}{L}_{[0]}(E)$ *with* $\varpi r \in \underset{\sim}{b}_{[1]}(\underset{\sim}{F}(E))$ *equation* (5.1) *has a bounded solution.*

Then there exists [a number $\nu > 0$ *and] a number* $N > 0$ *such that, for every real* $m \geqq 0$, *every bounded solution* v *of* $(5.2)_{[m]}$ *satisfies*

(i): $\|\Pi(t)v\| \leqq N\|\Pi(t_0)v\|$ $\quad$ [$\|\Pi(t)v\| \leqq Ne^{-\nu(t-t_0)}\|\Pi(t_0)v\|$]

for all $t \geqq t_0 \geqq m$;

There further exists a set $\underset{\sim}{W}$ *of solutions of* (5.2), *[a number* $\nu' > 0$ *] and numbers* $N' > 0$, $\lambda_0 > 1$ *such that, for every real* $m \geqq 0$, *every solution* u *of* $(5.2)_{[m]}$ *is of the form* $u = v + w_{[m-1]}$, *where* v *is a bounded solution and* $w \in \underset{\sim}{W}$, *and such that every solution* $w \in \underset{\sim}{W}$ *satisfies*

(ii): $\|\Pi(t)w\| \geqq N'^{-1}\|\Pi(t_0)w\|$ $\quad$ [$\|\Pi(t)w\| \geqq N'^{-1}e^{\nu'(t-t_0)}\|\Pi(t_0)w\|$]

for all $t \geqq t_0 \geqq 0$;

(iii): $\|\Pi(t)w\| \leqq \lambda_0\|\Pi(t)w - \Pi(t)v\|$ *for all* $t \geqq m \geqq 0$ *and all* bounded solutions v of $(5.2)_{[m]}$.

If E is finite-dimensional, then the assumption that $E_-(0)$ is closed is redundant, and W may be chosen to be a finite-dimensional linear manifold.

Proof. 1. By Theorem 7.1, (b, ℓ^{∞}) is admissible for A. We now refer to [1] and [8] in order to deal with equations (6.3), (6.4). Specifically, Condition (d) of [8; Lemma 4.2] is satisfied with $\underset{\sim}{d} = \underset{\sim}{\ell}^{\infty}$. We consider the covariant sequence E_- (whose general term is $E_-(n)$, the set of initial values of the bounded solutions of $(6.4)_{[n]}$). Since $E_-(0)$ is closed by assumption, [8; Theorem 4.3,(a)] shows that the covariant sequence E_- is closed and regular. We can therefore apply the fundamental "direct" results [1; Theorems 9.1 and 10.1] for difference equations, and find that this covariant sequence induces a dichotomy [an exponential dichotomy] for A.

2. To make this result manageable, we use the description of a dichotomy [an exponential dichotomy] given by [1; Theorem 7.1,(c)]. We observe that in the proof of that theorem we are free to choose the splitting q (a "non-linear projection" in E annihilating $E_-(0)$); this will be important in Part 3 of this proof. We choose q and denote its range by Z. Thus $E = E_-(0) + Z$. Now the covariant sequence E_- is regular; therefore we have, by [1; Lemma 5.2,(b) and (5.2)], for every integer $n \geq 0$,

$$E = E_-(n) + U(n,0)E = E_-(n) + U(n,0)E_-(0) + U(n,0)Z = E_-(n) + U(n,0)Z.$$

This means that if x is a given solution of $(6.4)_{[n]}$ there exists a solution z of (6.4) with $z(0) \in \tilde{Z}$ such that $y = x - z_{[n]}$ is a bounded solution of $(6.4)_{[n]}$.

We define $\tilde{W}$ to be the set of those solutions w of (5.2) that satisfy $\Pi(0)w \in \tilde{Z}$. The remainder of the proof of the main conclusion of the theorem is now identical to that of [9; Theorem 7.3] (from the last paragraph of Part 2) with the following changes: [1; Theorem 9.1] is used, and the exponential factors deleted, in the "ordinary dichotomy" case; and Corollary 5.3 and the factor e^{σ} are used instead of [9; Lemma 5.2] and the factor $\exp|M_C|$.

3. If E is finite-dimensional, then each $A(n)$ is compact, by (6.1) and Theorem 5.2,(4). Therefore [8; Theorem 4.3,(b)] is applicable and $E_0(0)$ is closed and has finite co-dimension in E. We may therefore choose the splitting q in the preceding proof to be a linear projection of E along $E_0(0)$ onto some finite-dimensional complementary subspace Z. Then W is a finite-dimensional linear manifold of solutions of (5.2).

8.2. Corollary. Assume that the memory $M = L + M'$ satisfies the standard assumptions with respect to a given space F. Assume that $E_0(0)$ is closed in E. Assume that (B,C) is admissible for M, where $F = L_p$ and $B = L_1$, or $F = L^\infty$ and $B = T$ [$F = L^p$ and $B = L$, $1 < p \leq \infty$, or $F = L$ and $B = M$]. Then the conclusions of Theorem 8.1 hold.

Proof. Use Corollary 7.2 instead of Theorem 7.1 to enter the proof of Theorem 8.1.

To conclude, we state a reasonably strong form of a "converse" theorem to Theorem 8.1, and sketch its proof.

8.3. Theorem. Assume that the memory M satisfies (M). If the main conclusion of Theorem 8.1 holds for the solutions of (5.2)[m], then the pairs (L^1,C) and (T,C) [the pairs (L^p,C), $1 \leq p \leq \infty$, and (M,C)] are admissible for M.

Proof. The assumption on M implies that M satisfies the standard assumptions with $L = 0$, $M^1 = M$, and $F = L^1$. The main conclusion of Theorem 8.1 implies, via Lemma 6.1 and a little computation, that E_0 is indeed a regular covariant sequence for A and induces a dichotomy [an exponential dichotomy] for A [1; Theorem 7.1]. From the "converse" theorems for difference equations [1; Theorems 9.2 and 10.3] it follows that the pair $(\underset{\sim}{l}^1, \underset{\sim}{l}^\infty)$ [the pair $(\underset{\sim}{l}^\infty, \underset{\sim}{l}^\infty)$] is admissible for A. From Corollary 7.2 we conclude that the pair $(\underset{\sim}{L}^1, \underset{\sim}{C})$ [the pair $(\underset{\sim}{M}, \underset{\sim}{C})$] is admissible for M. The other pairs of the statement are then obviously admissible, since $\underset{\sim}{T}_{[0]}(E)$ is stronger than $\underset{\sim}{L}^1_{[0]}(E)$ [since every $\underset{\sim}{L}^p_{[0]}(E)$ is stronger than $\underset{\sim}{M}_{[0]}(E)$].

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