CLASSIFYING CONVEX EXTREMUM PROBLEMS OVER LINEAR TOPOLOGIES HAVING SEPARATION PROPERTIES

by

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Abstract

It is shown that any convex or concave extremum problem possesses a subsidiary extremum problem which has certain homogeneous properties. Analogous to the given problem, the "homogenized" extremum problem seeks the minimum of a convex function or the maximum of a concave function over a convex domain. By using homogenized extremum problems new relationships are developed between any given convex extremum problem (P) and a concave extremum problem (P*) (also having a convex domain), called the "dual" problem of (P). This is achieved by combining all possibilities in tabular form of (1), the values of the extremum functions and (2), the nature of the convex domains including perturbations of all problems (P), (P*), and each of their respective homogenized extremum problems.

This detailed and refined classification is contrasted to the relationships obtainable by combining only the possible values of the extremum functions of the problems (P) and (P*)

and the possible limiting values of these functions stemming from perturbations of the convex constraint domains of (P) and (P*) respectively.

The extremum problems in this paper and classification results are set forth in real topologically paired vector spaces having the Hahn-Banach separation property.

1. Introduction

This paper develops new relationships between a given convex extremum problem and another extremum problem, called its "dual" problem. A convex extremum problem (P) seeks to minimize a convex function over a convex constraint domain, while the dual problem (P*) seeks to maximize a concave function over another convex constraint domain. We study the structure between these two problems by classifying conceivable and permissible events on:

(α) the minimizing value of the convex function of (P) and the maximizing value of the concave function of (P*) and

(β) the nature of the convex constraint domains of (P) and (P*).

Some of these events are easily illustrated in the well-known finite elementary linear programming classification table. Here problem (P) and its dual (P*) are elementary finite linear programs.

(P*) \ (P)		CONS		INC
		BD	UBD	
CONS	BD	1	0	0
	UBD	0	0	2
	INC	0	3	4

Classification Table 0

The abbreviation " CONS" or " INC" denotes whether the convex constraint domain is non-empty or empty, respectively, called "consistent" or "inconsistent" in the linear programming literature. When the constraint domain is non-empty, then the linear functional value is either bounded "BD" or unbounded "UBD". Any given linear problem (P) and its dual (P*) may occur only in one of the joint-events 1, 2, 3, or 4, and all other joint-events are forbidden and thereby denoted by "0". Elementary examples show that the four events are realizable.

Returning to the general case, classification table 0 is not valid because of examples where problem (P) is consistent and bounded but (P*) is inconsistent, see [2]. When the convex constraint domain is not a finite dimensional polyhedral set or when the extremum problem is in an infinite dimensional space, various convergence complexities arise. Therefore various subclasses of events are required. Consequently more rows and columns are needed in classification table 0.

For example, the minimizing value of the functional of a convex extremum problem (P) may be finite, -oo, or +oo . Its constraint domain may be empty, or not. However, by slightly relaxing the constraint condition by a perturbation, the relaxed domain is another convex domain which may not be empty. In this case when the perturbations approach 0, one obtains special solutions termed "asymptotic solutions". By computing functional values of asymptotic solutions, further conceivable events for (P) arise. For the given problem (P) only certain combinations of conceivable events are permissible, and their total number is 7, Events analogously described arise for the dual problem (P*), and their total permissible number is also 7. However, when combining permissible events for both problems (P) and (P*) together, many of the conceivable joint-events are forbidden. In fact, out of a total of 49 conceivable joint-events only 11 are permissible. Some of these and related classifications are given in references [2], [3], [6], [11], [14], [15], and [23].

In this paper we carry the classification process much further by introducing a new construction which yields even more detailed and refined information about the original convex extremum problems. This is in the form of a pair of subsidiary problems associated with the given convex extremum problem and its dual problem, respectively. These new problems are also convex and concave respectively, but satisfy certain homogeneous properties. Because of this they are called "homogenized extremum problems".

In linear programming the existence of a homogeneous problem is rather transparent. In this case this construction has already been done by Duffin [6] in 1956. We illustrate this case.

Example 1. Let A be a continuous linear operator on a locally convex space E to a locally convex space F and let E^* be a space paired topologically with E. Let $c \in E^*$, $b \in F$, and let C be a closed convex cone in E.

Compute $\min(c,x)$ *for all* $x \in E$ *subject to the constraints* $Ax = b$ *and* $x \in C$.

This is an infinite linear program and its infinite linear homogenized program is:

Compute min(c,x) for all x e E subject to the constraints Ax = 0 and x € C«

Duffin [6] used the linear homogenized program to give information about the original linear program. For example, he used a linear homogenized program to characterize the existence of asymptotically consistent solutions as well as the existence of consistent solutions of the given linear program,,

in linear programming the existence of homogenized subsidiary problems permits classification refinements. These are achieved by first collecting information on the extremum values of the linear functions of the homogenized problems of (P) and (P*) and the nature of their convex domains including perturbations. This information is then joined with permissible events on (a) and (p) above, but for the given linear problem and its dual problem. This process increases the number of permissible events associated with a linear problem from 7 to 11. However, when considering joint-events of both a linear extremum problem and its dual, many conceivable joint-events are forbidden. In fact out of 121 conceivable joint-events only 11 are permissible. Classifications of this type are given in Kalina-Williams [14] and reference [15], for linear extremum problems.

In convex programming, however, the question arises as to whether there is a related homogenized problem associated with any convex extremum problem. For example, what, if any, is the convex homogenized problem of the following problem.

<u>Example 2</u>. Let $x \in R^1$ and $u \in R^1$ and define

$$g(u,x) = \begin{cases} -\log(x+u) & \text{if } x > -u \\ +\infty & \text{if } x \leq -u. \end{cases}$$

<u>Compute the minimum of</u> $g(0,x)$ <u>for all</u> $x \in R^1$.

This is a convex extremum problem with perturbations of the constraint condition by the variable u, which relaxes the convex constraint domain.

The answer to the question is that any convex extremum problem (P) has a convex homogenized subsidiary problem (H), and its dual problem (P*) has a concave homogenized subsidiary problem (H*). Problems (H) and (H*) are constructed from linearizations of problems (P) and (P*) respectively. Two theorems establish elementary equivalences between the dual of a convex extremum problem and the dual of its linearization. This aids in determining conceivable and permissible events on (a) and (p) but with respect to all the problems (P), (P*), (H), and (H*). The desired classification refinements are then obtained in this paper by applying four theorems in order to determine forbidden joint-events. The first theorem extends the 49 conceivable joint-event classification achieved for linear extremum problems to an arbitrary convex problem (P) and its dual (P*). The second theorem characterizes consistency of (P) [(P*)] by the existence of a specific permissible event of (H*)[(H)]. The third theorem characterizes asymptotic consistency of (P) [(P*)] by the existence of another specific permissible event of (H*) [(H)]. The fourth theorem ties certain permissible events of (P) [(P*)] to certain permissible events of (H) [(H*)].

The end result is a classification similar to the linear extremum problem case. It states that out of 121 conceivable joint-events between any convex extremum problem (P) and its dual (P*), only 11 are permissible.

In an analogous manner another pair of convex extremum subsidiary problems having homogeneous properties is introduced. When permissible events of these particular subsidiary problems are combined with those of the original (P) and (P*), then 400 joint-events are conceivable but only 93 are permissible.

Finally these classifications are contrasted to those obtainable from combining only the permissible values and limiting values of the extremum functions of a convex extremum problem (P) and its dual (P*). By using classification methods a new characterization is obtained for when 0 is in the closure of the domain of an arbitrary closed convex function.

To begin the process, we make a choice of a particular form of the dual problems to study. We present the dual convex problems in the underlying framework of Gale's work [10], but as generalized in the convex "bifunction" terminology of Rockafellar, [24], sections 29 and 30, which are all related to the conjugate function approach of Fenchel [9]; see also Stoer-Witzgall [28].

We turn now to the development of classification schemes for convex extremum problems over real topological vector spaces having the Hahn-Banach extension property. The first task is to introduce definitions which adequately encompass and discriminate the phenomena of "events" that occur on (α) and (β). This is done by defining duality states for any given convex extremum problem (P) and its dual (P*).

2. Duality States for Closed Convex Bifunction Dual Families.

Let E, F be real topological vector spaces which have the Hahn-Banach extension property (HBEP), see [12] and let E^*, F^* be topologically paired with E, F respectively, see [4]. Thus $F \times E$ is topologically paired with $F^* \times E^*$ in the product topology.

Let G be a bifunction from F to E, that is, to each $u \in F$ is associated an extended real valued function on E, $Gu : E \to [-\infty, \infty]$. The value of Gu at a point $x \in E$ is denoted by $(Gu)(x)$, see Rockafellar [24], sections 29 and 30. We assume throughout that G is a proper closed convex bifunction. This means that the graph function g is proper, closed, and convex on $F \times E$ where by definition:

$$g(u,x) = (Gu)(x). \tag{1}$$

Recall that proper means that $g(u,x)$ is not identically $+\infty$ and $g(u,x)$ is not allowed to take on the value $-\infty$.

The adjoint of G is defined as the bifunction, see Rockafellar [24], from E^* to F^* given by:

$$(G^*x^*)(u^*) = \inf_{\substack{u \in F \\ x \in E}} \{ (Gu)(x) - (x,x^*) + (u,u^*) \} \tag{2}$$

G^* is also proper closed and convex since G is. Then it follows that

$$(G^*x^*)(u^*) = -g^*(-u^*,x^*) \tag{3}$$

where g^* is the conjugate transform of g, valid in this infinite dimensional setting (Asplund [1], Brondsted [5] and Moreau [20], [21], see also Rockafellar [24], p.309).

<u>Remark</u>. Following the conventions of [24], the closure of a convex function is defined to be the lower semi-continuous hull of f if f nowhere has the value $-\infty$. If f assumes the value $-\infty$ somewhere, then its closure, $c\ell\ f$, is defined to be the constant function $-\infty$. These distinctions are given in [24], p.52-54, in particular the comment after Corollary 7.2.2.

Throughout this paper we use the definition of closure as given in Rockafellar [24], and thus for example, Corollary 30.2.2 is applicable to our analysis.

We consider the following pair of dual convex programs

<u>Program</u> (P) <u>Seek</u> $\inf (GO)x$ <u>subject</u> <u>to</u> $x \in E$ (4)

and

<u>Program</u> (P*) <u>Seek</u> $\sup (G^*O)(u^*)$ <u>subject</u> <u>to</u> $u^* \in F$. (5)

These dual programs are related to the dual family,

$$[P:P^*] = [(Pu) : (P^*x^*)]_{(u,x^*) \in F\times E^*} \tag{6}$$

which usually arises from perturbations of a given convex program.

Following [24] let

$$\mathrm{dom}(Gu) = \{x \in E \mid (Gu)(x) < \infty\} \tag{7}$$

and

$$\mathrm{dom}\, G = \{u \in F \mid Gu \text{ is proper on } E\}. \tag{8}$$

We say that Program (P) [(P*)] is

CONS (consistent)	if	$\mathrm{dom}\, G0 \neq \emptyset [\mathrm{dom}\, G^*0 \neq \emptyset]$	(9)
INC (inconsistent)	if	$\mathrm{dom}\, G0 = \emptyset [\mathrm{dom}\, G^*0 \neq \emptyset]$	(10)
AC (asymptotically consistent)	if	$0 \in c\ell(\mathrm{dom}\, G)\ [0 \in c\ell(\mathrm{dom}\, G^*)]$	(11)
SINC (strongly inconsistent)	if	$0 \notin c\ell(\mathrm{dom}\, G)\ [0 \notin c\ell(\mathrm{dom}\, G^*)]$	(12)

The <u>value</u> of Program (P) [(P*)] is

$$(\inf G)(0) \quad [(\sup G^*)(0)] \tag{13}$$

while the <u>subvalue</u> of Program (P) [(P*)] is

$$(c\ell(\inf G))(0) \quad [(c\ell(\sup G^*))(0)]. \tag{14}$$

Any of these may be finite or infinite. See Rockafellar [24] for definitions of these terms in R^n-space, which extend to the infinite dimensional setting here.

Using definitions (9) through (14) the following <u>states</u> are introduced for Program (P) [(P*)].

(i) Let Program (P) [(P*)] be CONS. Then it is BD (bounded if $(\inf G)(0) > -\infty$ [$(\sup G^*)(0) < \infty$]. Otherwise it is UBD (unbounded).

(ii) Let Program (P) [(P*)] be AC. Then it is PAC (properly AC) if $(c\ell(\inf G))(0) < \infty$ [$(c\ell(\sup G^*))(0) > -\infty$]. Otherwise it is IAC (improperly AC).

(iii) Let Program (P) [(P*)] be PAC. Then it is ABD (asymptotically BD) if $(c\ell(\inf G))(0) > -\infty$ [$(c\ell(\sup G^*))(0) < \infty$]. Otherwise it is AUBD.

<u>A duality state of the pair of programs</u> (P) <u>and</u> (P*) <u>is a pair of states, one of Program</u> (P) <u>and one of Program</u> (P*).

Duality states of the above type were developed by Duffin [6] and Ben-Israel-Charnes-Kortanek [2], [3], for infinite linear programs. Definitions related to those above were developed in [3] for convex programming problems.

3. <u>Linear Programming Equivalents for Programs</u> (P) <u>and</u> (P*)

We first present Program (P) and Program (P*) in slightly altered but equivalent forms, respectively

<u>Program</u> (P)

<u>Seek to</u> $\inf g(u,x)$

<u>subject to</u> $(u,x) \in 0 \times E \cap F \times E$ (15)

and

Program (P*)

Seek to $\sup\,[-g^*(-u^*,x^*)]$

subject to $(u^*,x^*) \in F^* \times 0 \oplus F^* \times E^*$ (16)

3.JL Program (P)

We seek homogenized convex programs of Programs (P) and (P*) respectively. These are to be called homogeneous derivant bi-functions, O^hese particular bifunctions are constructed from a linearization of the convex program (P) [(P*)] and are related to several linear homogeneous type programs in the literature [6], [14], [15].

To begin this construction we introduce a linear operator A as follows:

$$A : (F \times E) \times R \times R \longrightarrow (F \times E) \times R \times R$$

by

A((* x),?, 9) = ((£),(), 6). (17)

A is a continuous linear operator in the product topology on (FxE) x R x R. Using the pairing it follows that the adjoint A^T of A is given by:

$$A^T : (F^* \times E^*) \times R \times R \longrightarrow (F^* \times E^*) \times R \times R$$

$$A^T\left(\binom{u^*}{x^*},\zeta,\eta\right) = \left(\binom{u^*}{0},0,7]\right). \qquad (18)$$

A^T is also continuous in the product topology.

For any set $W \subset (F \times E) \times R$, let C(W) denote the homogenization of W ([28])*, also called the associated cone in [3], defined by

* See also [24], p*63.

$$C(W) = \{(^{Y\text{™}}_{Y}) \mid Y \wedge 0\}.$$

We shall also use the notation [3],

$$W^* = \{w^* \in (F^* \times E^*) \times R \mid w \in W = \wedge \langle w, w^* \rangle \geq 0\} \quad (19)$$

and

$$W^*_{x} = \{w^* \in (F^* \times E^*) \times R \mid w \in W =) \langle w, w^* \rangle \wedge -1\}. \quad (20)$$

Introduce Program (CP):

Seek the inf §

subject to $A((\pounds), ?, 6) = ((^{\circ}_{0}), 0, 1)$ (21)

and $((^{u}), §, 9) \in ct\{c(\text{epi } g)\} \subset (F \times E) \times R \times R$

where epi g denotes the epigraph of g, and

Program ((CP)*)

Seek the sup -yj

subject to $A^T((Jp, C>n> + ((^{0}_{Q}), 1, 0) \in ct\{C(\text{epi } g)\}^*,$ (22)

where $\{c\wedge\{c(\text{epi } g)\}\}^* \subset (F^* \times E^*) \times R \times R$

These programs form a dual pair of linear programming problems over closed convex cones as is well known. Ihis means that by introducing perturbations in the standard way for linear programs, see [24], p.311-312, a dual bifunction family can be associated with Programs (CP) and ((CP)*). Following this construction, let (CP) denote an associated closed convex bifunction for Program (CP). Then (CP)* denotes its adjoint bifunction also closed and convex.

In this terminology (CP)(O) denotes the value of the associated bifunction at the zero vector of perturbations, which as a vector lies in the space $(F \times E) \times R \times R$. Therefore (inf(CP))(O) is the value of Program (CP). Similarly, (sup(CP)*)(O) denotes the value of Program ((CP)*), where here $O \in (F^* \times E^*) \times R \times R$.

Using these identifications the following equivalences are more computational than conceptual.

<u>THEOREM</u> <u>1</u>. (a) Program (P) is CONS $\Longleftrightarrow$ Program (CP) is CONS in which case (inf G)(O) = (inf(CP))(O).

(b) Program (P) is AC $\Longleftrightarrow$ Program (CP) is AC in which case $(c\ell(\inf G))(O) = (c\ell(\inf(CP))(O)$.

(c) Program (P*) is CONS $\Longleftrightarrow$ Program ((CP)*) is CONS in which case (sup G*)(O) = (sup(CP)*)(O).

(d) Program (P*) is AC $\Longleftrightarrow$ Program ((CP)*) is AC in which case $(c\ell(\sup G^*))(O) = (c\ell(\sup(CP^*))(O)$.

<u>Proof</u>. The fact that Program (P) is CONS[AC] if and only if Program (CP) is CONS[AC] follows from the decomposition of $c\ell\{C(\text{epi } g)\}$ <u>i</u>.<u>e</u>.

$$c\ell\{C(\text{epi } g)\} = C(\text{epi } g) \cup \left\{\begin{matrix} O^+(\text{epi } g) \\ O \end{matrix}\right\}. \qquad (23)$$

See [24], p.63. Then by construction of the linear Program (CP) and its associated bifunction, denoted (CP), see [24], p.311-312; it follows that:

$$(\inf G)(O) = (\inf(CP))(O) \qquad (24)$$

and

$$(c\ell(\inf G))(0) = (c\ell(\inf(CP)))(0). \tag{25}$$

This proves parts (a) and (b).

Now Program (P*) is CONS[AC] if and only if Program ((CP)*) is CONS[AC] follows analogously from the decomposition of $c\ell\{C(\text{epi } g)\}^*$, i.e.

$$c\ell\{C(\text{epi } g)\}^* = C((\text{epi } g)^*_{-1}) \cup \left((\text{epi } g)^*_{0}\right), \tag{26}$$

see [3], p.681, where the proof there is valid for any closed convex set.

To prove the equalities in (c) and (d), it is easiest to use parts (a) and (b) together with Corollary 30.2.2 of Rockafellar [24].†

Applying this Corollary to the bifunctions associated with Programs P and P* we obtain:

$$(\sup G^*)(0) = (c\ell(\inf G))(0) \tag{27}$$

and

$$(c\ell(\sup G^*))(0) = (\inf G)(0). \tag{28}$$

Applying this Corollary to the bifunctions associated with Programs (CP) and ((CP)*) we obtain

$$(\sup(CP)^*)(0) = (c\ell(\inf(CP)))(0) \tag{29}$$

and

$$(c\ell(\sup(CP)^*))(0) = (\inf(CP))(0). \tag{30}$$

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† See Appendix 1, relation (*), for a statement of this result.

Now part (b) together with (27) and (29) shows

$$(\sup G^*)(0) = (\sup(CP)^*)(0) \tag{31}$$

which proves (c). Using part (a) with (28) and (30) shows

$$(c\ell(\sup G^*))(0) = (c\ell(\sup(CP)^*))(0) \tag{32}$$

which proves (d). This completes the proof of Theorem 1.

Program (CP) gives rise to a homogenized linear program to be introduced shortly. This program gives information about Program (P). We also need a homogenized linear program stemming from Program (P*). To construct this program one could work with Program ((CP)*). However, it is easier to construct another dual pair of linear programs whose primal is the associated cone problem of Program (P*). Following this route we obtain a parallel development of the homogeneous derivant bifunctions both positive and negative for Programs (P) and (P*) respectively.

3.2 Program (P*)

Analogous to section 3.1 define a linear operator B as follows

$$B : (F^*\times E^*) \times R \times R \longrightarrow (F^*\times E^*) \times R \times R$$

where

$$B(\binom{u^*}{x^*},\zeta,\eta) = (\binom{0}{x^*},0,\eta). \tag{33}$$

Thus its adjoint is given by

$$B^T : (F \times E) \times R \times R \rightarrow (F \times E) \times R \times R$$

where

$$B^T\left(\binom{u}{x}, \xi, \theta\right) = \left(\binom{0}{x}, 0, \theta\right). \tag{34}$$

Both B and B^- are continuous linear operators in their respective topologies.

Consider the following linear programs:

Program (CP*)

Seek the sup £

subject to $B\left(\binom{u^*}{x^*}, £, ?\right) = \left(\binom{0}{0}, 0, 1\right)$ (35)

and $\left(\binom{u^*}{x^*}, £, T\right) \in c£\{C(\text{epi } h^*)\} \subset (F^* \times E^*) \times R \times R$

where $h^*(u^*, x^*) = -g^\wedge(-u^\wedge, x^\wedge)$ on $F^* \times E^\wedge$,

and

Program ((CP*)*)

Seek the inf 9

subject to $B^T((£), ?, e) - \left(\binom{0}{0}, 1, 0\right) \in \{c\text{-}t\{C(\text{epi } h^*)\}\}^*$ (36)

where $\{c£\{C(\text{epi } h^*)\}\}^* \subset (F \times E) \times R \times R$.

Analogous to section 3.1 we denote by (CP*) an associated bifunction to Program (CP*). Ihe dual bifunction shall be denoted by (CP*)*. Then analogous to Theorem 1 we obtain the following theorem whose proof is completely symmetrical to the proof of Theorem 1, and therefore is omitted.

THEOREM 2. (a) Program (P*) is CONS $\Longleftrightarrow$ Program (CP*) is CONS in which case (sup G*)(0) = (sup(CP*))(0).

(b) Program (P*) is AC $\Longleftrightarrow$ Program (CP*) is AC in which case $(c\ell(\sup G^*))(0) = (c\ell(\sup(CP^*)))(0)$.

(c) Program (P) is CONS $\Longleftrightarrow$ Program ((CP*)*) is CONS in which case (inf G)(0) = (inf(CP*)*)(0).

(d) Program (P) is AC $\Longleftrightarrow$ Program ((CP*)*) is AC in which case $(c\ell(\inf G))(0) = (c\ell(\inf(CP^*)^*))(0)$.

4. The Homogenized Problems for (P) and (P*)

For linear programming problems over convex cones in linear topological spaces under minimization, the "positive homogeneous derivant" [15] or "modified homogeneous constraint set" of [14], is related to Duffin's homogenized program [6], p.163, developed 17 years ago. Duffin's homogenized program is always consistent and subconsistent and its value and subvalue is either 0 or $-\infty$. As a subsidiary linear extremum problem the positive homogeneous derivant may be consistent, inconsistent, asymptotically consistent, or strongly inconsistent, and thus the terminology CONS, INC, AC, and SINC has been used for these mutually exclusive and collectively exhaustive states. The following equivalences can then be verified.

Homogenized Program [6]	Positive Homogeneous Derivant [15]
(a) with subvalue 0	SINC
(b) with subvalue $-\infty$	AC
(c) with value 0	INC
(d) with value $-\infty$	CONS

Table 1

Each line in the table is an equivalence, e.g., for line (b), HP has subvalue $-\infty \Longleftrightarrow$ PHD is AC. Therefore the homogenized program when taken together with its 2 possible values and 2 possible subvalues is equivalent to the 4 duality states of the positive homogeneous derivant listed in Table 1.

Analogous to infinite linear programming we now develop both positive and negative homogeneous derivant bifunctions for closed convex bifunction dual families.

4.1 Homogeneous Derivant Bifunctions for Program (P)

Working on Program (CP) we obtain the positive homogeneous derivant.

Program (HD(CP))

$$\begin{array}{ll} \text{Seek} & \inf 0^T\left(\binom{u}{x},\xi,\theta\right) \\ \text{subject to} & \xi \leq -1 \\ \text{and} & A\left(\binom{u}{x},\xi,\theta\right) = \left(\binom{0}{0},0,0\right) \\ \text{where} & \left(\binom{u}{x},\xi,\theta\right) \in cl\{C(\text{epi } g)\}. \end{array} \tag{37}$$

Program (HD(CP)) gives rise to the following bifunction $(I\nu)(x)$ where $\nu = (u,\theta) \in F \times E$ and $x \in E$.

$$(I\nu)(x) = \begin{cases} 0 & \text{if} \quad (g0^+)\binom{u}{x} \leq -1 \quad \text{and} \quad \theta = 0 \\ 0 & \text{if} \quad \theta g\left(\frac{1}{\theta}\binom{u}{x}\right) \leq -1 \quad \text{and} \quad \theta > 0 \\ +\infty & \text{otherwise.} \end{cases} \tag{38}$$

As usual the bifunction (38) generates a collection of programs given by:

Program (iv)

Seek $\inf (Iv)(x)$

subject to $x \in E$ (39)

PROPOSITION 1. Program (10) is equivalent to Program (HD(CP)) in the sense of equivalence of all bifunction duality states of each program.

Proof. We use the fact that $0^+(\text{epi } g) = \text{epi}(g0^+)$, where $g0^+$ is the recession function of $g(u,x)$, see [24], p. 66. Now Program (10) is CONS $\Leftrightarrow \exists x \in E$ such that $(g0^+)\binom{0}{x} \le -1$. Upon setting $\xi = (g0^+)\binom{0}{x}$, this is equivalent to $\left(\binom{0}{x}, \xi, 0\right) \in 0^+(\text{epi } g)$ which is equivalent to Program (HD(CP)) being CONS. In this case the objective values agree because they are both 0 trivially.

Assume now that Program (10) is AC, Then there exists a net $\{((u^\gamma,\theta^\gamma),x^\gamma) \mid \gamma\}$ such that

$$\lim_\gamma (u_\gamma, \theta_\gamma) = 0$$

and $(g0^+)\binom{u_\gamma}{x_\gamma} \le -1$ if $\theta_\gamma = 0$ and $\theta_\gamma g\left(\theta_\gamma^{-1}\binom{u_\gamma}{x_\gamma}\right) \le -1$ if $\theta_\gamma > 0$.

Define

$$\xi_\gamma = \begin{cases} (g0^+)\binom{u_\gamma}{x_\gamma} & \text{if } \theta_\gamma = 0 \\ \theta_\gamma g\left(\theta_\gamma^{-1}\binom{u_\gamma}{x_\gamma}\right) & \text{if } \theta_\gamma > 0 \end{cases} \tag{40}$$

Now if (a) $\theta_\gamma = 0$, then $(\binom{u_\gamma}{x_\gamma}, \xi_\gamma, 0) \in \binom{\text{epi}(g0^+)}{0}$. If (b) $\theta_\gamma > 0$, $(\frac{1}{\theta_\gamma}\binom{u_\gamma}{x_\gamma}, \frac{1}{\theta_\gamma}\xi_\gamma) \in \text{epi } g$ by definition (40). This implies

$$(\binom{u_\gamma}{x_\gamma}, \xi_\gamma, \theta_\gamma) = \theta_\gamma(\frac{1}{\theta_\gamma}\binom{u_\gamma}{x_\gamma}, \frac{1}{\theta_\gamma}\xi_\gamma, 1) \in C(\text{epi } g).$$

Hence combining (a) and (b), it follows that for each γ

$$(\binom{u_\gamma}{x_\gamma}, \xi_\gamma, \theta_\gamma) \in C(\text{epi } g) + \binom{\text{epi}(g0^+)}{0} = c\ell\{C(\text{epi } g)\}.$$

Furthermore, $\lim_\gamma A(\binom{u_\gamma}{x_\gamma}, \xi_\gamma, \theta_\gamma) = \lim_\gamma(\binom{u_\gamma}{0}, 0, \theta_\gamma) = (\binom{0}{0}, 0, 0)$.

Finally $\limsup_\gamma \xi_\gamma \leq -1$ showing that Program (HD(CP)) is also AC. In this case the subvalues of IO and HD(CP) agree since they are both trivially zero.

On the other hand, assume that Program (HD(CP)) is AC. Then there exists a net $\{(\binom{u'_\gamma}{x'_\gamma}, \xi'_\gamma, \theta'_\gamma | \gamma\}$ such that

$$(\binom{u'_\gamma}{x'_\gamma}, \xi'_\gamma, \theta'_\gamma) \in c\ell\{C(\text{epi } g)\} \text{ for each } \gamma$$

and $\lim_\gamma(u'_\gamma, \theta'_\gamma) = 0$ and $\limsup_\gamma \xi'_\gamma \leq -1$. Since $c\ell\{C(\text{epi } g)\}$ is a convex cone, it follows that $(\binom{u_\gamma}{x_\gamma}, \xi_\gamma, \theta_\gamma) = 2(\binom{u'_\gamma}{x'_\gamma}, \xi'_\gamma, \theta'_\gamma) \in c\ell\{C(\text{epi } g)\}$. Thus $\limsup_\gamma \xi_\gamma \leq -2$.

Therefore, for infinitely many y,

$$(g_0^+)^{u}(x^\gamma) \geq \xi_\gamma \geq -1 \qquad \text{if} \quad \theta_\gamma = 0$$

and

$$\theta_\gamma \left(g_0^{u}\left(\frac{x^\gamma}{\theta_\gamma}\right)\right) \geq \xi_\gamma \geq -1 \qquad \text{if} \quad \theta_\gamma > 0.$$

Hence for infinitely many y, $(Iv^\gamma)(x^\gamma) < +\infty$. Since $\lim_\gamma v^\gamma = \lim_\gamma (u_\gamma, \theta_\gamma) = 0$, it follows that Program (10) is AC. This concludes the proof of Proposition 1.

Since it stems from a linear program, the bifunction (iv) (x) of (38) is closed convex. It is called the positive homogeneous derivant bifunction of Program (P). By applying definitions (9)-(14) to Program (10), additional states may be defined for Program (P). Because the value (and subvalue) of Program (10) is either 0 or $+\infty$, only 4 new states arise and are as follows:

We say that Program (P) is

(i)	HCONS	if Program (10) is	CONS
(ii)	HINC	if Program (10) is	INC
(iii)	HAC	if Program (10) is	AC
(iv)	HSINC	if Program (10) is	SINC.

(41)

Analogously the negative homogeneous derivant bifunction is determined by replacing the constraint $\xi \leq -1$ in Program (HD(CP)) by the constraint $\xi \geq 1$. This gives rise to another bifunction called a negative homogeneous bifunction.

$$(I^-v)(x) = \begin{cases} 0 & \text{if } (g_0^+)(x) \geq 1 \quad \text{and} \quad \theta = 0 \\ 0 & \text{if } \theta g_0\left(\frac{x}{\theta}\right) \geq 1 \quad \text{and} \quad \theta > 0 \\ +\infty & \text{otherwise.} \end{cases} \qquad (42)$$

Analogous to Proposition 1, we have the following result for the negative homogeneous derivant bifunction.

PROPOSITION 2.. Program (l"0) from (42) is equivalent to Program (HD~(CP)) where the latter is gotten from HD(CP) by replacing ? <1 -1 with % ^ 1, in the sense of equivalence of all bifunction duality states of each program. Analogously, Program (P) is said to be Ef CONS, H~INC, H~AC, or Ef SINC according to whether Program (I~0) is CONS, INC, AC, or SINC respectively.

4.2 Homogenized Programs for Program (P*)

Working with Program (CP*), (35), we obtain a positive homogenized linear program:

Program (HD(CP*))

Seek sup $0^T((\begin{smallmatrix}u^*\\x\end{smallmatrix})J£JT))$

subject to $B((£^*_*),£,7)) = ((\begin{smallmatrix}\circ\\0\end{smallmatrix}),0,0)$ (43)

and £ ^ 1

where $((\begin{smallmatrix}u^*\\x\end{smallmatrix})^*£J1?)$ e $cl\{$ C(epi h*) } = c(epi h*) U 0^+_n(epi h*) .

Let X* = (x*,7)) e E* x R and introduce the bifunction:

$$(JX^*)(u^*) = \begin{cases} 0 & \text{if } (h^*0^+)(\begin{smallmatrix}u^*\\x\end{smallmatrix}) \le 1 \text{ and } \eta = 0 \\ 0 & \text{if } \eta h^*(\wedge(\wedge)) \le 1 \text{ and } \eta > 0 \\ -\infty & \text{otherwise} \end{cases} \qquad (44)$$

and the related program:

Program (JO)

Seek sup(JO)(u*) (45)

subject to u* $\in$ F*.

Analogous to Proposition 1 (and its proof) we have the following result.

PROPOSITION 3. Program (JO) is equivalent to Program (HD(CP*)) in the sense of equivalence of all bifunction duality states of each program.

The bifunction (JX*)(u*), (44), is a positive homogeneous derivant for Program (P*). Analogous to introducing 4 new states for Program (P), we say that Program (P*) is

(i)	HCONS	if Program (JO) is	CONS	
(ii)	HINC	if Program (JO) is	INC	(46)
(iii)	HAC	if Program (JO) is	AC	
(iv)	HSINC	if Program (JO) is	SINC.	

Analogous to the development of (42), a negative homogenized program for Program (P*) is obtained by replacing the constraint $\zeta \geq 1$ in Program (HD(CP*)) by $\zeta \leq -1$. This new program is denoted HD^{-}(CP*). It gives rise to the negative derivant bifunction for Program (P*) as follows:

$$(J^{-}x^*)(u^*) = \begin{cases} 0 & \text{if } (h^*0^+)\binom{u^*}{x^*} \leq -1 \quad \text{and} \quad \eta = 0 \\ 0 & \text{if } \eta h^*\left(\frac{1}{\eta}\binom{u^*}{x^*}\right) \leq -1 \quad \text{and} \quad \eta > 0 \\ -\infty & \text{otherwise.} \end{cases} \qquad (47)$$

Analogous to the preceding propositions we have the following result.

PROPOSITION 4. Program (J^-O) from (47) is equivalent to Program ($HD^-(CP^*)$) where the latter is gotten from Program ($HD(CP^*)$) by replacing $\zeta \geq 1$ with $\zeta \leq -1$, in the sense of equivalence of all bifunction duality states of each program.

Program (P^*) is said to be H^-CONS, H^-INC, H^-AC, and H^-SINC according to whether the problem determined by $(J^-O)(u^*)$ is CONS, INC, AC, or SINC respectively.

5. Determining Permissible "Compound" Duality States of (P) and (P*)

Table 6 of Appendix 1 indicates that any convex extremum problem (P) (and its dual (P*)) has 7 permissible states itself. This is seen from examining the rims of the table. To each of these states, however, conceivably three new ones arise logically from its positive homogeneous derivant, namely, HSINC, HAC and HCONS, HAC and HINC. Thus, the mixing of permissible states of (P) with permissible states of its positive derivant gives rise to new states for (P) termed compound states, conceivably 21 in number. But Theorems 3, 4, and 5 applied to Table 6 restrict both the number of permissible compound states of (P) and (P*) individually and the number of permissible compound duality states between them jointly. The net result is Theorem 7 and Table 2 of the next section whose proof is given in Appendix 2.

An analogous approach is taken with respect to negative homogeneous derivants. However, Theorem 6 permits only a slight reduction in the number of permissible compound duality states, and hence the compound classification scheme here is much more combinatorial as set forth in Theorem 8 and Table 3.

Theorems 3 and 4 below are extensions to convex programming of Duffin's Corollaries 2 and 1, [6], respectively.

<u>THEOREM</u> <u>3</u>. Program (P) [(P*)] is AC $\Longleftrightarrow$ Program (P*) [(P)] is HINC.

<u>Proof</u>. Program (P) is AC $\Longleftrightarrow$ Program ((CP*)*) is AC by Theorem 2(d). By Corollary 2 [6], Program ((CP*)*) is AC $\Longleftrightarrow$ Program (HD(CP*)) is INC, when HINC is identified to the condition of

Corollary 2 by Table 1. Further Program (HD(CP*)) is INC $\Longleftrightarrow$ Program (P*) is HINC by Proposition 3.

Next, Program (P*) is AC $\Longleftrightarrow$ Program((CP)*) is AC by Theorem 1(d). By Corollary 2 [6] Program((CP)*)is AC $\Longleftrightarrow$ Program (HD(CP)) is INC using Table 1. Finally Program (HD(CP)) is INC $\Longleftrightarrow$ Program (P) is HINC by Proposition 1.

THEOREM 4. Program (P)[(P*)] is CONS $\Longleftrightarrow$ Program (P*)[(P)] is HSINC.

Proof. Program (P) is CONS $\Longleftrightarrow$ Program ((CP*)*) is CONS by Theorem 2(c). By Corollary 1 [6], Program ((CP*)*) is CONS $\Longleftrightarrow$ Program (HD(CP*)) is SINC, again using Table 1. Now Program (HD(CP*)) is SINC $\Longleftrightarrow$ Program (P*) is HSINC by Proposition 3.

Next, Program (P*) is CONS $\Longleftrightarrow$ Program((CP)*) is CONS by Theorem 1(c). By Corollary 1 [6], Program((CP)*) is CONS $\Longleftrightarrow$ Program (HD(CP)) is SINC. Therefore using Proposition 1, Program (HD(CP)) is SINC $\Longleftrightarrow$ Program (P) is HSINC, completing the proof of Theorem 6.

THEOREM 5.

(a) Program (P)[(P*)] is ABD $\Longrightarrow$ Program (P)[(P*)] is HSINC.

(b) Program (P)[(P*)] is IAC $\Longrightarrow$ Program (P)[(P*)] is HSINC.

(c) Program (P)[(P*)] is AUBD $\Longleftrightarrow$ Program (P)[(P*)] is AC and HAC.

Proof. Program (P)[(P*)] is ABD or IAC $\Longrightarrow$ Program (P*)[(P)] is CONS by Table 6 of Appendix 1. Hence Program (P)[(P*)] is HSINC by Theorem 4, which proves (a) and (b).

To prove (c) assume Program (P) [(P*)] is AUBD (hence AC), and assume to the contrary that Program (P) [(P*)] is HSINC. Then by Theorem 4, Program (P*) [(P)] is CONS which contradicts Program (P) [(P*)] being AUBD by Table 6. Therefore (P) [(P*)] is HAC, and therefore Program (P) [(P*)] is HAC and AC.

On the other hand, if Program (P) [(P*)] is AC and HAC, then adding an HAC solution net to an AC solution net yields an AUBD solution net to Program (P)[(P*)]. See Lemma 6 of [2] for this idea in the context of linear programming. This completes the proof of part (c) and hence Theorem 5.

THEOREM 6. program (P)[(P*)] is IAC => Program (P)[(P*)] is H~AC, see (42).

Proof. We work with Program (P) since the argument for Program (P*) is analogous. Assume that Program (P) is IAC and that to the contrary Program (P) is H~"SINC, (42). This means that Program (P~) defined by the bifunction -(Gu)(x) is HSINC* Therefore by Theorem 4, Program ((P~)*) is CONS, But Program (P) is IAC implying by definition that Program (P¯) is AUBD. But by Table 6, it is impossible for Program (P~) to be AUBD and Program ((P~)*) to be CONS. Therefore Program (P) must be H"AC. QED.

6. Compound Classification with Homogenized Bifunctions

6^JL Compound Classification Theorem with Positive Homogeneous Derivant Bifunctions

THEOREM *Tj.* Of the 121 mutually exclusive and collectively exhaustive compound duality states for Programs (P) and (P*) and their derivants (38) and

(44) respectively listed in Table 2, only 11 are possible and are those denoted by positive integers. A zero in Table 2 means the corresponding compound duality state is impossible, and the non-zero integer denotes the corresponding example in [2] of that state. [For those who are interested, 11 examples are also given in [13] in an infinite dimensional non-reflexive Banach space setting.]

The proof of this Theorem follows from repeated use of Theorems 3, 4, and 5 and Table 6 in an analogous way that the linear versions of these Theorems are used to derive the compound classification for the case of linear programming in topological vector spaces. See [14] and [15]. We include a complete proof of this compound classification theorem in Appendix 2.

6.2 Compound Classifications with Negative Homogeneous Derivant Bifunctions

The listing of compound characteristics for Program (P) and its negative homogeneous derivant is almost completely combinatorial. The Classification Theorem of Table 6 (see Appendix,1) involves 7 states for Program (P) and a *priori* there are 3 homogeneous states for each of these (H~SINC, H~AC and H'lNC, H~"AC and EfCONS) giving a possible total of 21. However, a slight reduction, to 20, is made possible by Theorem 6 which implies that Program (P) IAC and H^{-}SINC is impossible. The possible compound characteristics are set forth in Table 3.

Without the existence of a theorem which relates permissible states of a given problem to permissible states of the negative homogeneous derivant of its dual, we are

Compound Classification with

Positive Homogeneous Derivant Bifunctions

(P*) \ (P)	CONS				INC						
	AC								SINC		
	PAC							IAC			
	ABD	AUBD HAC			ABD	AUBD HAC		IAC	HSINC	HAC	
	HSINC	HINC		HCONS							
	BD		UBD		HSINC	HINC	HCONS	HSINC		HINC	HCONS
CONS AC PAC ABD HSINC BD	1	0	0	0	0	0	0	0	0	0	0
CONS AC PAC AUBD HAC HINC BD	0	0	0	0	2	0	0	0	0	0	0
CONS AC PAC AUBD HAC HINC UBD	0	0	0	0	0	0	0	3	0	0	0
CONS AC PAC AUBD HAC HCONS UBD	0	0	0	0	0	0	0	0	4	0	0
INC AC PAC ABD HSINC	0	5	0	0	0	0	0	0	0	0	0
INC AC PAC AUBD HAC HINC	0	0	0	0	0	11	0	0	0	0	0
INC AC PAC AUBD HAC HCONS	0	0	0	0	0	0	0	0	0	6	0
INC AC IAC HSINC	0	0	7	0	0	0	0	0	0	0	0
INC SINC HSINC	0	0	0	8	0	0	0	0	0	0	0
INC SINC HAC HINC	0	0	0	0	0	0	9	0	0	0	0
INC SINC HAC HCONS	0	0	0	0	0	0	0	0	0	0	10

Table 2

led to the following conjecture, which has been proved by Rom [25] for the linear case, which using the equivalences we have established extends to convex bifunctions.

Compound Classification with Negative Derivant Bifunctions

THEOREM 8. Of the 400 mutually exclusive and collectively exhaustive compound duality states for Programs (P) and (P*) and their negative homogeneous derivant bifunctions defined by Table 3, only 93 are possible.

7. Classification Schemes and Convex Analysis

•Hie concepts of value and subvalue of a program, see (13) and (14), are related to all of the duality states introduced in section 2, simply because every program has a value and subvalue, regardless of the duality state which it and its dual form. In this section we give this relationship by embedding each of the 11 possible duality states into a value and subvalue oriented classification scheme.

In order to apply the 11 examples [2] directly here and in the proof in Appendices 1 and 2, we alter the program formulation slightly. Replace Programs (I.C) and (II.C*) of [2] with Programs (P) and (P*) with the perturbations given below.

Program (P) $\inf(-c,x)$ subject to $Ax = b + u, \ x \in C$ (48)

Program (P*) $\sup(-b,u^*)$ subject to $A^T u^* - c - x^* \in C^*$, (49)

where C is the closed convex cone example in [2].

•then the 11 so-numbered examples are generated from the same data for c, A, b, and C in [2].

Compound States for (P)

with Negative Homogeneous Derivant Bifunctions

<table>
<tr><td colspan="9">CONS</td><td colspan="11">INC</td></tr>
<tr><td colspan="17">AC</td><td colspan="3" rowspan="4">SINC</td></tr>
<tr><td colspan="15">PAC</td><td colspan="2" rowspan="3">IAC</td></tr>
<tr><td colspan="3">ABD</td><td colspan="6">AUBD</td><td colspan="3" rowspan="2">ABD</td><td colspan="3" rowspan="2">AUBD</td></tr>
<tr><td colspan="6">BD</td><td colspan="3">UBD</td></tr>
<tr><td rowspan="2">$H\bar{S}INC$</td><td colspan="2">$H\bar{A}C$</td><td rowspan="2">$H\bar{S}INC$</td><td colspan="2">$H\bar{A}C$</td><td rowspan="2">$H\bar{S}INC$</td><td colspan="2">$H\bar{A}C$</td><td rowspan="2">$H\bar{S}INC$</td><td colspan="2">$H\bar{A}C$</td><td rowspan="2">$H\bar{S}INC$</td><td colspan="2">$H\bar{A}C$</td><td colspan="2">$H\bar{A}C$</td><td rowspan="2">$H\bar{S}INC$</td><td colspan="2">$H\bar{A}C$</td></tr>
<tr><td>$H\bar{I}NC$</td><td>$H\bar{C}ONS$</td><td>$H\bar{I}NC$</td><td>$H\bar{C}ONS$</td><td>$H\bar{I}NC$</td><td>$H\bar{C}ONS$</td><td>$H\bar{I}NC$</td><td>$H\bar{C}ONS$</td><td>$H\bar{I}NC$</td><td>$H\bar{C}ONS$</td><td>$H\bar{C}ONS$</td><td>$H\bar{I}NC$</td><td>$H\bar{I}NC$</td><td>$H\bar{C}ONS$</td></tr>
</table>

Table 3

In convex analysis, the seeds of a general classification result which is value and subvalue oriented were sown in 1965 in Rockafellar's Theorem 6 [23], p.179-180 and later firmed up in the bifunction terminology as Corollary 30.2.2 in Convex Analysis [24], p.315. If in addition the 11 possible states are demonstrated by examples, then this corollary can be used to prove the classification theorem of Table 6 for convex programming and is so used in Appendix 1.

In addition, Table 4 associates each of the 11 examples [2] to one of six cases determined by values and subvalues of $(\inf G)(0)$, $(c\ell(\inf G))(0)$, $(\sup G^*)(0)$, and $(c\ell(\sup G^*))(0)$. These examples and Corollary 30.2.2 then yield the following classification result.

Value-Subvalue Oriented Classification Theorem

THEOREM 9. Of the 36 mutually exclusive and collectively exhaustive cases for problems (P) and (P*) with respect to the values and subvalues $(\inf G)(0)$, $(\sup G^*)(0)$, $(c\ell(\inf G))(0)$, $(c\ell(\sup G^*))(0)$, only 6 are possible and are those denoted in Table 4 by positive integers. A zero in the table means the corresponding case is impossible and the non-zero integers denote the corresponding example of [2], possibly grouped together for a given value-subvalue case.

Proof. The proof follows from Rockafellar's Corollary 30.2.2 [24] and the Ben-Israel-Charnes-Kortanek linear programming example data, used for linear programs in 3 or 4 space of the form (48) and (49).

Value-Subvalue Oriented Classification Theorem

(P*) \ (P)			inf G < + ∞			inf G = + ∞		
			cl(inf G) finite	cl(inf G) = - ∞		cl(inf G) finite	cl(inf G) - ∞	cl(inf G) + ∞
			inf G	finite	inf G = -∞			
sup G* > - ∞	cl(sup G*) finite	sup G* finite	1	○	○	○	○	○
sup G* > - ∞	cl(sup G*) = + ∞	sup G* finite	○	○	○	2	○	○
sup G* > - ∞	cl(sup G*) = + ∞	sup G* = +∞	○	○	○	○	○	3,4
sup G* = - ∞	cl(sup G*) finite		○	5	○	○	○	○
sup G* = - ∞	cl(sup G*) + ∞		○	○	○	○	6,9, 10,11	○
sup G* = - ∞	cl(sup G*) - ∞		○	○	7,8	○	○	○

Table 4

Observe that the clustering of the duality states into the value-subvalue oriented scheme involves the forced mixing of different homogeneous characteristics either of the primal, dual, or of both problems. For example, duality states 6 and 9 are collected together for the value-subvalue case $(\inf G)(0) = +\infty$, $(c\ell(\inf G))(0) = -\infty$, $(\sup G^*)(0) = -\infty$, and $(c\ell(\sup G^*))(0) = +\infty$. Upon checking their positive homogeneous derivants we find:

Example \ Problem	(P)	(P*)
6	HINC	HCONS
9	HCONS	HINC

Table 5

Thus in the compound classification the 4 duality states 6, 9, 10, 11 are separated out, while they appear in one box in the value-subvalue oriented scheme. Similarly duality states 3 and 4 are lumped together, as well as duality states 7 and 8.

8. The Duality States of Proper or Improper Convex Functions

Let f be a closed convex function on E. We embed f into a closed convex proper bifunction $g(u,x)$ on $F \times E$ such that

$$g(0,x) = f(x), \quad x \in E. \tag{50}$$

The existence of many convex functions g satisfying (50) is clear geometrically, see for example Rockafellar's "New Applications of

Duality in Nonlinear Programming", presented at the 7th International Symposium on Mathematical Programming, the Hague, 1970.

In this section we have two objectives. First, we generalize Rockafellar's Theorem 27.1(i) [24] characterizing $0 \in c\ell(\text{dom } f^*)$. Second, we show which duality states are possible when f is proper and which are possible when f is improper, where the dual convex programs stem from the closed convex bifunction $g(u,x)$.

THEOREM 10. Let $g(u,x)$ be a closed convex proper bifunction on $F \times E$ and $f(x) = g(0,x)\, \forall x$. Then

(1) Program (P^*) AC $\Longrightarrow 0 \in c\ell(\text{dom } f^*)$

(2) If $f(x)$ is proper, then $0 \in c\ell(\text{dom } f^*) \Longrightarrow$ Program (P^*) AC.

Proof. (1) Assume Program (P^*) is AC. Then there exists a net $\{(x^*_\gamma, u^*_\gamma) \mid \gamma\}$ such that $-g^*(-u^*_\gamma, x^*_\gamma) > -\infty$ and $\lim_\gamma x^*_\gamma = 0$. Therefore for each γ,

$$\inf_{(0,x)} \{-(x,x^*_\gamma) + f(x)\} \geq \inf_{(u,x)} \{<(u,x),(u^*_\gamma,-x^*_\gamma)> + g(u,x)\}$$

$= -g^*(-u^*_\gamma, x^*_\gamma) > -\infty$. This implies that $x^*_\gamma \in \text{dom } f^*$ and hence $0 \in c\ell(\text{dom } f^*)$.

(2) $f(x)$ proper implies that Program (P) is CONS. Assume $0 \in c\ell(\text{dom } f^*)$. Then by Theorem 27.1(i) of [24], $(f0^+)(x) \geq 0$. But by an elementary calculation,

$$(g0^+)(0,x) \geq (f0^+)(x), \quad x \in E. \tag{51}$$

Therefore by (41), see also (38), Program (P) is HINC. But only duality states 1, 5, and 7 are possible for Program (P) and Program (P*) since Program (P) is CONS and HINC. And in these states Program (P*) itself is AC. This completes the proof of (2) and hence the proof of Theorem 10.

COROLLARY 1. Let $g(u,x)$ be a closed convex proper bifunction on $F \times E$ and $f(x) = g(0,x)$, $x \in E$. Then $0 \in cl(\text{dom } f^*)$ except when program pair (P)-(P*) is in duality state 8. Moreover, in duality state 8, $0 \notin cl(\text{dom } f^*)$.

Proof. We consider two cases.

Case 1. Program (P) is CONS. Only 4 states (1,5,7,8) are possible. Except for state 8, Program (P*) is AC and hence by Theorem 10(1), $0 \in cl(\text{dom } f^*)$.

Case 2. Program (P) is INC. This means $f(x) = (G0)(x) = +\infty$, $x \in E$ and trivially $0 \in cl(\text{dom } f^*)$ since $\text{dom } f^* = E^*$ in this case.

It remains to show that $0 \notin cl(\text{dom } f^*)$ for duality state 8. Assume (P)-(P*) is in state 8. Then if f is proper, it follows that $0 \notin cl(\text{dom } f^*)$ by Theorem 10(2), since Program (P*) is SINC. If f is improper, then since Program (P) is CONS, there exists $\bar{x}$ such that $f(\bar{x}) = -\infty$. This means $\text{dom } f^* = \emptyset$, and hence $0 \notin cl(\text{dom } f^*)$.

Therefore when (P)-(P*) is in duality state 8, $0 \notin cl(\text{dom } f^*)$.

COROLLARY 2. Let $g(u,x)$ be a closed convex proper bifunction on $F \times E$ and let $f(x) = g(0,x)$, $x \in E$. Consider the duality states of the programs (p)-(p*): $\inf (G0)(x)$, $x \in E$ and $\sup (G^*0)(u^*)$, $u^* \in F^*$.

(i) In duality states 1, 5, and 7, f(x) <u>is proper</u>;

(ii) In duality state 8, f(x) <u>may or may not be proper and both cases are realizable</u>.

(iii) In duality states 2, 3, 4, 6, 9, 10, 11, f(x) <u>is improper</u>.

<u>Proof</u>. (i) and (iii) readily follow from Theorem 10. In (ii) a proper function f is given by example 8 [2] taken in the form (48) and (49). The function identical to $-\infty$ gives an example of state 8 having improper f.

<u>Acknowledgement</u>.

The writer is indebted to Richard J. Duffin for many discussions on the concepts of this paper and his suggestions for clarifying their presentation.

Appendix 1^

A Convex Conjugate Transform Classification Theorem

THEOREM 11. Of the 49 mutually inclusive and collectively exhaustive duality states of Programs (P) and (P*) only 11 are possible and are those denoted in Table 6 by positive integers. A zero in Table 6 means that the corresponding state is impossible. The possible duality states are numbered according to examples 1-11 of Ben-Israel-diarnes-Kortanek [2], when the dual programs of each are taken in the form of (48) and (49).

Proof. We shall use Rockafellar*s Corollary 3o<>2.2, page 315 [24] which reads:

(inf G)(0) = (c£(sup G*)(0) ^ (sup G*)(0) = (c£(inf G))(0) (*)

where G is any closed convex bifunction from F to E. We shall use (*) working row by row, starting with the state of (P*0) for each row.

Row 1 (P*) is CONS, BD, ABD. This means (inf G))(0) is bounded and also (c£(inf G))(0) is bounded. Hence (P) is also CONS, BD, ABD, i.e., only state 1 occurs.

Row 2L (P*) CONS, BD, and AUBD. This means (inf G) (0) = +oo ^ (PO) INC and also (c£(inf G)) (0) is bounded => (P) is ABD. Hence only state 2 occurs in row 2.

(P*) \ 0?)			AC						SINC
			PAC					IAC	SINC
			CONS			INC			
			ABD	AUBD		ABD	AUBD	IAC	5INC
			BD		UBD				
U < P:	CONS	0 W	I	0	0	0	0	0	0
		AUBD Q tt	0	o	0	2	o -	0	0
		P D	0	0	0	0	0	3	4'
u A >	U S	P <	0	5	0	0	0	0	0
		Q PQ P A	0	0	0	0	* u	0	6
u IA		U IA	0	0	7	0	0	V 0-	o ;
SINC		SINC	0	0	8	•° 1	*9	. 0	10

Table 6

Row 3 (P*) CONS, UBD. This means $(c\ell(\inf G))(0) = +\infty$. If $0 \in c\ell(\text{dom } G)$, i.e., AC, then (P) is IAC. If $0 \notin c\ell(\text{dom } G)$, then (P) is SINC. Hence only states 3 and 4 are possible in row 3.

Row 4 (P*) INC, ABD. Since $(c\ell(\sup G^*))(0)$ is bounded so is $(\inf G)(0)$, i.e., (P) is CONS, BD (hence AC). But $(\sup G^*)(0) = -\infty \Longrightarrow (c\ell(\inf G))(0) = -\infty \Longrightarrow$ (P) is AUBD since it is AC. Hence only state 5 is possible in row 4.

Row 5 (P*) INC, AUBD. Hence $(\inf G)(0) = +\infty \Longrightarrow$ (P) INC. Also $(\sup G^*)(0) = -\infty \Longrightarrow (c\ell(\inf F))(0) = -\infty$. Hence if (P) is AC, then it is AUBD. Otherwise it is SINC. Hence only states 11 and 6 are possible in row 5.

Row 6 (P*) IAC. This means $(c\ell(\sup G^*))(0) = -\infty \Longrightarrow \inf F0 = -\infty \Longrightarrow$ (P) is CONS, UBD. Hence only state 7 is possible.

Row 7 (P*) SINC. This means $(\sup G^*)(0) = -\infty \Longrightarrow (c\ell(\inf G))(0) = -\infty$ also. However $(c\ell(\sup G^*))(0)$ cannot be finite and hence we consider two cases.

(1) $(c\ell(\sup G^*))(0) = +\infty$. This means $(\inf G)(0) = +\infty \Longrightarrow$ (P) INC. If (P) AC then it is AUBD, since $(c\ell(\inf G))(0) = -\infty$. Otherwise it is SINC.

(2) $(c\ell(\sup G^*))(0) = -\infty$. This means $(\inf G)(0) = -\infty \Longrightarrow$ (P) is CONS, UBD. Hence only states 8, 9, and 10 are possible in row 7.

The proof is completed by remarking that the 11 states designated as the only possible ones than can occur, do in fact occur as demonstrated by the 11 examples in [2]. QED.

The following table gives the values of the extrema (bounded or unbounded) for problems (P) and (P*) for each of the 11 states.

DUALITY STATES	Program (P)		Program (P*)	
	(inf G) (O)	($c\ell$(inf G)) (O)	(sup G*O)	($c\ell$(sup G*)) (O)
1	finite	finite	finite	finite
2	$+\infty$	finite	finite	$+\infty$
3,4	$+\infty$	$+\infty$	$+\infty$	$+\infty$
5	finite	$-\infty$	$-\infty$	finite
6,9,10,11	$+\infty$	$-\infty$	$-\infty$	$+\infty$
7,8	$-\infty$	$-\infty$	$-\infty$	$-\infty$

<u>Table 7</u>

Appendix 2

Proof of the Compound Classification Theorem with PHD Closed Convex Bifunctions

The Impossible States

Row 1 The impossible states follow from row 1 of Table 6. For the only possible state, state 1, (P) and (P*) are both HSINC from Theorem 4.

Row 2 The impossible states follow from row 2 of Table 6. For the only possible state, state 2, (P) is HSINC from Theorem 5(a). Since (P*) is AUBD it follows from Theorem 5(c) that (P*) is HAC. Theorem 3 implies that (P*) is also HINC.

Rows 3,4 Row 3 of Table 6 splits into 2 rows according to whether (P*) is HINC or HCONS since it necessarily is HAC from Theorem 5(c). In these two rows the only possible states involve (P) being IAC, state 3, or (P) being SINC, state 4. If (P) is IAC, then by Theorem 3, (P*) is HINC, and of course (P) is HSINC by Theorem 5(b). If (P) is SINC, then by Theorem 3, (P*) is HCONS. Further since (P*) is CONS, Theorem 4 shows that (P) is HSINC.

Row 5 The impossible states follow from row 4 of Table 6. For the only possible state, state 5, (P) is HAC by Theorem 5(c) and HINC by Theorem 3. Theorem 5(a) shows that (P*) is HSINC.

Row 6,7 Row 5 of Table 6 splits into two rows according to whether (P*) is HINC or HCONS since by Theorem 5(c) it is HAC. In these two cases the only possible states involve (P) being AUBD or (P) being SINC. If (P) is AUBD, then (P) is HAC by Theorem 5(c) and HINC by Theorem 3. Similarly (P*) is also HAC and HINC. If, on the other hand, (P) is SINC, then (P*) is HCONS by Theorem 3, and by Theorem 3 (P) is also HINC.

Row 8 The impossible states follow from row 6 of Table 6. For the only possible state, state 7, (P*) is HSINC by Theorem 5(b) and (P) is HAC and HINC by Theorem 5(c) and Theorem 3 respectively.

Rows 9,10,11 Row 7 of Table 6 splits into 3 rows by the mutually exclusive states: (a) HSINC, (b) HAC and HINC, and (c) HAC and HCONS. Thus one must determine how the possible states 8, 9, and 10 fall in these subclasses.

State 8 By Theorem 5(c), (P) is HAC. Hence it is either HINC or HCONS, and Theorem 3 shows it is HCONS. (P*) is HSINC by Theorem 4.

State 9 Theorem 3 shows that (P) is HCONS. By Theorem 4, (P*) is HAC. By Theorem 3, (P*) is HINC.

State 10 By Theorem 3, (P) and (P*) are both HCONS and hence HAC automatically.

The Possible States The 11 examples of [2] show that the 11 states are indeed realizable, when the dual programs of each are taken in the form of (48) and (49).

Appendix 3

An Example of a Positive Homogeneous Derivant Bifunction

Let $x \in R^1$ and $u \in R^1$ and define

$$g(u,x) = \begin{cases} -\log(x+u) & \text{if } x > -u \\ +\infty & \text{if } x \leq -u. \end{cases} \tag{52}$$

Then

$$(Gx^*)(u^*) = \begin{cases} 1 + \log(-x^*) & \text{if } x^* < 0 \text{ and } -u^* - x^* = 0 \\ -\infty & \text{otherwise.} \end{cases} \tag{53}$$

It follows that

$$(g0^+)(u,x) = \begin{cases} 0 & \text{if } x + u \geq 0 \\ +\infty & \text{otherwise.} \end{cases} \tag{54}$$

Upon setting $v = (u,0)$, relation (38) becomes

$$(IV)(x) = \begin{cases} 0 & \text{if } (g0^+)(\;\;) \;\; -1 \\ 0 & \text{if } \theta g(\frac{1}{\theta}(\frac{u}{x})) \leq -1 \\ +\infty & \text{otherwise.} \end{cases} \tag{55}$$

Therefore according to (41), Program (P) is HINC because of (54). To see, however, that Program (P) is HAC, take

$$(u_k, \theta_k) = (0, \tfrac{1}{k}) \quad \text{and} \quad x_k = \frac{e^k}{k},$$

Then $\lim_k (u_k, \theta_k) = (0,0)$ and $(u_k, \theta_k) \in \text{dom } I$ for each k.

Furthermore for each k, $(I(u_k, \theta_k))(x_k) = 0$ since

$\theta_k g(\frac{1}{\theta_k}(\begin{smallmatrix} u_k \\ x_k \end{smallmatrix})) = \frac{1}{k}(-\log e^k) = -1$. Hence Program (P) is HAC.

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