AN EXTENSION OP THE NUCLEOLUS
TO NON-ATOMIC SCALAR MEASURE GAMES

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Introduction

In [3] Schmiedler defined a solution concept for n-person characteristic function games called the nucleolus. He derived the nucleolus by considering a vector of coalition excesses for a particular imputation and then arranging the excesses in descending order. Then, by ordering these rearranged vectors lexicographically and taking the minimum under this ordering, the nucleolus, a unique imputation, was deduced.

Because of its uniqueness and other desirable properties the nucleolus is an important bargaining concept in n-person game theory. It is one plausible way to resolve a characteristic function game. For these reasons it is of interest to try to extend the concept to infinite player games. The main difficulty is generalizing the lexicographical ordering to the case where there are an uncountable number of coalitions.

However, in [2], Kohlberg proved an equivalent characterization of the nucleolus for finite player games. This characterization will be used to extend the nucleolus to a class of infinite player games, the non-atomic scalar measure games. The definition of the nucleolus can be easily extended to much larger classes of games.

Non-atomic scalar measure games were treated by Aumann and Shapley in [1], although they were concerned primarily with the value. They form an important class of infinite player games and allow us to take a first step in applying this bargaining concept to infinite player games.

Extension of the Nucleolus

This paper deals exclusively with non-atomic scalar measure games. By this we mean the characteristic function V is given by $V(S) = f(\mu(S))$, where $f(0) = 0$, μ is a non-atomic measure on X and $S \in B$, the Borel field of X. For simplicity, we shall assume that $X = [0,1]$, $\mu(X) = 1$ and $f(1) = 1$. The imputations will be all positive non-atomic measures with $\nu(X) = 1$.

Kohlberg's characterization of the nucleolus for finite player games [2] will be used to extend the nucleolus to scalar measure games.

Definition 1. For any non-atomic measure, $\nu(X) = 1$, the coalition array that belongs to (V,ν) is the collection $\{b_\alpha\}_{\alpha \in R}$ such that $b_\alpha = \{S | V(S) - \nu(S) = \alpha\}$.

Definition 2. A coalition array has property I if for every real α, for each non-atomic signed measure η defined on B, $\eta(S) \geq 0 \;\; \forall\, S \in \bigcup_{\beta \geq \alpha} b_\beta$ and $\eta(X) = 0$ implies $\eta(S) = 0$ $\forall\, S \in \bigcup_{\beta \geq \alpha} b_\beta$.

Definition 3. A non-atomic measure ν is in the nucleolus of (X,B,V) if the coalition array that belongs to (V,ν) has property I.

Theorem 1. Every non-atomic scalar measure game has a non-empty nucleolus.

Proof. $V(S) = f(\mu(S))$ where μ is a non-atomic measure. We will show that the coalition array that belongs to (V,μ) has property I. Partition the Borel field into an array $\{C_y\}_{0\leq y\leq 1}$ where $C_y = \{S|\mu(S) = y\}$. Clearly, since V is a scalar measure game, for all α either $b_\alpha = \emptyset$ or $b_\alpha = \cup C_y$ for some values of y between 0 and 1.

Now, suppose that the coalition array that belongs to (V,μ) does not have property I. Then $\exists\, \alpha\ \exists\, \eta$ with $\eta(X) = 0$ $\eta(S) \geq 0$ for all $S \in \bigcup_{\beta\geq\alpha} b_\beta$ and $\eta(S_o) > 0$ for some $S_o \in \bigcup_{\beta\geq\alpha} b_\beta$. In particular this implies that $\eta(S) \geq 0$ for every $S \in C_{\mu(S_o)}$, $\eta(X) = 0$ and $\eta(S_o) > 0$. This is a contradiction.

To see this, consider the vector measure $(\mu,\mu + \eta)$. If $1 > \mu(S_o) > 0$ then $(\mu + \eta)(S_o) > \mu(S_o)$. Note that (1,1) and (0,0) are in the range of $(\mu,\mu + \eta)$. Since this is a non-atomic vector measure its range is convex and compact. Thus there is an S' such that $\mu(S') = 1 - \mu(S_o)$ and $(\mu + \eta)(S') > \mu(S')$. Therefore $\eta(S') > 0$ and since $\mu(X-S') = \mu(S_o)$ then $\eta(X-S') \geq 0$. As a result $\eta(X) > 0$ a contradiction. If instead $\mu(S_o) = 0$ or $\mu(S_o) = 1$ then $S_o \in b_o$ since $f(0) = 0$ and $f(1) = 1$. Note that $C_1 \cup C_o \subset b_o$ for this reason. So if $\eta(X) = 0$, $\eta(S_o) > 0$ and $\eta(S) \geq 0$ $\forall\, S \in b_o$ then $\eta(X-S_o) \geq 0$ since $\mu(X-S_o) = 0$ or 1. But this implies that $\eta(X) > 0$ again a contradiction.

Therefore it has been shown that the coalition array that belongs to (V,μ) has property I and so μ is in the nucleolus of (X,B,V).

In [3] Schmiedler proved the uniqueness of the nucleolus for finite games. Unfortunately in the extension to infinite games unicity is not preserved. The following example points this out. Let $V(S) = f(\mu(S))$ where

$$f(x) = \begin{cases} 0 & x = 0 \\ 1/2 & 0 < x < 1 \\ 1 & x = 1 \end{cases}$$

and μ is a non-atomic measure. Clearly for any non-atomic measure ν such that $\nu(X) = \mu(X)$ and ν is absolutely continuous with respect to μ, then $V(S) = f(\nu(S))$ also. Therefore by Theorem 1, both ν and μ are in the nucleolus of (X,B,V).

In the formulation of the problem we ruled out consideration of imputations which contained atomic measures. By allowing only non-atomic measures this encompasses the second condition in Kohlberg's Property I, i.e. $\eta(\{x_o\}) > 0 \ \forall \ x_o$ such that $V(\{x_o\}) = 0$. In this case, every $x_o \in [0,1]$. If instead we use Kohlberg's original conditions for property I, some undesirable elements can appear in the nucleolus. Let $x_o \in [0,1]$ be arbitrary then

$$\nu(S) = \begin{cases} 1 & \text{if } x_o \in S \\ 0 & \text{if } x_o \notin S \end{cases}$$

is in the nucleolus of (X,B,V) for any non-atomic scalar measure game. Note that for any α, $\{S | x_o \notin S \ \ f(\mu(S)) = \alpha\} \subset b_\alpha$. Then for $\eta(X) = 0$, $\eta(\{y\}) \geq 0 \ \forall \ y \in X$ and $\eta(S) \geq 0$ $\forall \ S \in \bigcup_{\beta \geq \alpha} b_\beta$. This implies that $\forall \ S$ with $x_o \notin S \ \ \eta(S) = 0$ and because $\eta(\{x_o\}) \geq 0$ then $\eta\{x\} = 0$ since $\eta(X) = 0$.

That this imputation should be allowed in the nucleolus is quite unreasonable as it is as far from a conciliatory solution as is possible. That this would otherwise be allowed can be attributed to the measure theoretic pathologies of the problem. The inapplicability of this condition to this type of game is due to the fact that it is possible in games with a continuum of players for each individual element to receive zero, but for an aggregate of elements to receive a positive quantity. Thus one should proceed with caution when attempting to extend these results to scalar measure games whose measures may have atoms.

We now prove a lemma which will be useful in characterizing the nucleolus.

<u>Lemma</u>. If μ, ν are non-atomic measures such that $\mu(X) = \nu(X)$ and there is $0 < \alpha < 1$ there is an S with $\mu(S) = \alpha$ $\nu(S) > \alpha$ then for every $0 < \alpha < 1$ there are S_1, S_2 with $\mu(S_1) = \mu(S_2) = \alpha$ and $\nu(S_1) > \alpha > \nu(S_2)$. Conversely, if there is $0 < \alpha < 1$ such that for all S with $\mu(S) = \alpha = \nu(S)$, then $\nu = \mu$.

<u>Proof</u>. Since μ, ν are non-atomic then (ν, μ) is a non-atomic vector measure hence its range is convex and compact.

If $\exists$ S with $\nu(S) > \mu(S) = \alpha$, $0 < \alpha < 1$ then because $\nu(X) = \mu(X)$ by convexity for each β $1 > \beta \geq \alpha$ $\exists$ S_β with $\mu(S_\beta) = \beta < \nu(S_\beta)$. Similarly (0,0) is in the range of (ν, μ) so that for any $\delta < \alpha$ $\exists$ S_δ with $\mu(S_\delta) = \delta < \nu(S_\delta)$. Also because $\nu(S_\alpha) > \mu(S_\alpha) = \alpha$ then $\nu(X-S_\alpha) < 1-\alpha = \mu(X-S_\alpha)$. Applying the same argument yields $\forall$ β $1 > \beta > 0$ $\exists$ S'_β with $\mu(S'_\beta) = \beta > \nu(S'_\beta)$.

If on the other hand $\exists$ α such that for all S with $\mu(S) = \alpha$ $\nu(S) = \mu(S)$, and $\nu(X) = \mu(X)$ then $\nu = \mu$. To see this suppose there is an S_o $\mu(S_o) \neq \nu(S_o)$ then by the first part of this lemma there is S_α with $\mu(S_\alpha) = \alpha \neq \nu(S_\alpha)$ a contradiction.

<u>Theorem 2</u>. If f is continuous at 0,1 then the nucleolus of a scalar measure game is $\{\mu\} \cup \{\nu | \nu \in \text{int core of } V\}$ where int core of $V = \{\nu | \nu \in$ core of V and if $\nu(S) = V(S)$ then $\mu(S)$ or $\mu(X-S) = 0\}$.

<u>Proof</u>. By Theorem 1 μ is in the nucleolus. If $\nu \in$ int core then $b_o = \{S | \mu(S) = 0 \text{ or } \mu(X-S) = 0\}$. Otherwise if $\nu(S) > 0$ when $\mu(S) = 0$ then $\nu(X-S) < V(X-S) = V(X) = \nu(X)$ so that ν is not in the core. Also $b_\alpha = \emptyset$ for $\alpha > 0$, so b_o is the first non-empty element in the coalition array. To show that the coalition array that belongs to (V, ν) has property I it is sufficient to show that for any $\epsilon < 0$ there is a y_o such that $\{S | \mu(S) = y_o\} \subset \bigcup_{0 \leq \alpha \leq \epsilon} b_\alpha$. If this property holds then property I follows from the previous lemma.

Since ν, μ are non-atomic and ν is absolutely continuous with respect to μ, $\lim_{x \to +0} \sup_{\mu(S)=x} v(S) = 0$. Therefore, given $\varepsilon > 0$ there is a y_1 with $\sup_{\mu(S)=x} v(S) < \varepsilon/2$ for all $x < y_1$. Because f is continuous at 0 there is a y_2 such that $|f(x)| < \varepsilon/2$ for all $x < y_2$. Therefore there is an y_0 with $|f(y_0) - \sup_{\mu(S)=y_0} v(S)| < \varepsilon$. As a result $\{S \mid \mu(S) = y_0\} \subset \bigcup_{0\le\alpha\le\varepsilon} b_\alpha$. Now, to show (V,v) has property I. If $\eta(S) \ge 0$ for all $S \in \bigcup_{0\le\alpha\le\varepsilon} b_\alpha$ and $\eta(X) = 0$ then $\eta(S) \ge 0$ for all S with $\mu(S) = y_0$. Therefore as in the proof of Theorem 1 $\eta(S) = 0$ for all S with $\mu(S) = y_0$. By the previous lemma, $v(S) = \mu + \eta(S)$ for all S with $\mu(S) = y_0$ therefore $v = \mu + \eta$ and $\eta(S) = 0$ for all S including $S \in \bigcup_{0\le\alpha\le\varepsilon} b_\alpha$. Since this holds for any $\varepsilon > 0$ v is in the nucleolus.

Now, to show that if $v \ne \mu$ is not in the int core of V then v is not in the nucleolus.

If v is in the core but not in the int core then there is an S_0 with $1 > \mu(S_0) > 0$ and $v(S_0) = f(\mu(S_0))$. Since $\mu \ne v$, by Lemma 1 there is an S' with $\mu(S') = \mu(S_0)$ and $v(S') < \mu(S')$. Since v is in the core of V, then $v(S_0) = f(\mu(S_0)) = f(\mu(S')) \le v(S') < \mu(S')$ so that $\mu(S_0) > v(S_0)$. Consider the coalition array that belongs to (V,v), $b_0 = \{S \mid f(\mu(S)) = v(S)\}$. Let $\eta = \mu - v$. Then for all $S \in b_0$ $\eta(S) \ge 0$ and $\eta(X) = 0$, but $\eta(S_0) > 0$.

Thus the coalition array cannot have property I so v is not in the nucleolus.

If $v \ne \mu$ is not in the core then $\sup_{S \in B} V(S) - v(S) = \Delta > 0$. It will be sufficient to show that

$$\sup_{S\in B} V(S) - v(S) > \sup_{V(S) 2 M(S)} \dot{V}(S) - v(S) = \sup_{x\in[O,l]} f(x)-x = D.$$

If D = O then trivially this inequality holds. If D > 0 then because f is continuous at 0,1, f(0) = 0 and f(l) = 1 then there is some $e > 0$ such that

$$D = \sup_{XG [e, 1-e]=Q} f(x)-x, \text{ and } D > \sup_{xeX-Q} f(x)-x.$$

By the lemma & S-^S-$_2$ such that v(S_1) < e = M(S-$_1$) and $^V(^S2$^ ^ l-€-6. Because (v,/i) is a non-atomic vector measure, for each xeQ there is an S with f-i(S) = x and v(S) < x-6. Now let $\{x_i\}_{i=1}^{\infty}$ $x_i \in Q$ be a sequence such that $\lim_{i\to\infty} f(x_i)-x_i = D$. For each x_i there is a S_i with $v(S_i)$ ^ fi(S_i)-£ = x_i~6 so that as i oo there will be some S_i with

$$f(x_i) - v(S_i) = V(S_i) - v(S_i) 2 ° + 1 -$$

Therefore A > D.

To see that v cannot be in the nucleolus consider the coalition array that belongs to (V,v). Since A > D, for each $S \in \bigcup_{A \geq a > D} b_a$, jLt(S) > v(S) or /i-v > 0 and yet /i-v(x) =0 so this coalition array cannot have property I.

Examining the proof one can deduce even more. It is not necessary that f be continuous at 0 and 1, merely that $\sup_{x\in[0,1]} f(x)-x$ does not occur there. With this in mind we have the following corollary.

REFERENCES

[1] Aumann, R. J. and L. S. Shapley, Values of Non-Atomic Games, Part jE: The Axiomatic Approach, The RAND Corporation, RM-5468-PR, November 1968•

[2] Kohlberg, E., "On the nucleolus of a characteristic function game", SIAM J. Applied Math, .20(1971), 62-66.

[3] Schmeidler, D., "The nucleolus of a characteristic function game", SIAM J. Applied Math. .17(1969), 1163-1170.

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Corollary 1. The nucleolus of a scalar measure game is unique if $\sup_{x[0,1]} f(x)-x > \limsup_{x \to 0,1} f(x)-x$.

Proof. μ is the nucleolus. There are no elements in the core of V otherwise $\sup f(x)-x = 0$ which contradicts the hypothesis. By Theorem 2 if $\limsup_{x \to 0,1} f(x)-x < \sup_{x \in [0,1]} f(x)-x$ then by repeating the argument one can easily show that for $\nu \neq \mu$ the coalition array that belongs to (V,ν) does not have property I.

Conclusions

The conclusions of this paper are somewhat mixed. The main task has been accomplished; the nucleolus has been extended to infinite player games and its existence has been proven for scalar measure games. Furthermore it is suspected that with a few alterations of the definition, the nucleolus will exist for a much larger class of games including non-atomic vector measure games.

Nevertheless, the nucleolus is of less value in the infinite case than the finite. It would appear to be most useful when there is no core at least in the case of scalar measure games. But the nucleolus is only slightly smaller than the core when the core exists and this tends to diminish its importance. It could be of use in further reducing the acceptable imputations on a game especially if it were used with another preference ordering bargaining concept.