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DERIVABILITY CONDITIONS FOR GODEL'S
SECOND INCOMPLETENESS THEOREM

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by R. G. Jeroslow[1,2]

This paper presents three generalizations of the Second Incompleteness Theorem of Godel (see [2]) which apply to a broader class of formal systems than previous generalizations (as, e.g., the generalization in [1]).

The content of all three of our results is that it is versions of the <u>third</u> derivability condition of Hilbert and Bernays (see page 286 of [3]) which are crucial to Godel's theorem. The second derivability condition plays some role in certain logics, but even there, only in the far weaker form of a definability condition (see Theorem 2).

The elimination of the first derivability condition allows the application of the Consistency Theorem to cut-free logics which cannot <u>prove</u> that they are closed under cut.

It is Theorem 1 which will probably have primary interest for readers who are not concerned with technical proof theory or with foundations, for it treats logics with quantifiers, and in that case one can dispose entirely of the first and second derivability conditions. The derivation of this result requires a "new twist" on old arguments. Specifically, we use a new variation on the standard self-referential lemma (this is Lemma 5.1 of [1]) to obtain a

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[2] Theorem 2 of this paper was presented by the author at the AMS 1971 Summer Meeting at State College, Pennsylvania, on Sept. 3, 1971. The three results of this paper were reported in an earlier manuscript, "On Godel's Consistency Theorem," in October, 1971. The present paper constitutes a revision of the earlier manuscript.

somewhat different self-referential construction than has previously been employed (this is our Lemma in Section 2 below). With this new construction, we consider the sentence φ which "expresses" the fact: "My negation is provable." Then, using hypotheses associated with Godel's First Incompleteness Theorem, one shows that $\vdash \neg \varphi$ is impossible in a consistent logic. Finally, <u>using only</u> a version of <u>the third</u> Hilbert-Bernays derivability condition, one shows that the provability of the consistency statement implies $\vdash \neg \varphi$, and hence that consistency is unprovable.

In the standard kinds of arguments for versions of Godel's Second Underivability Theorem, as exemplified, e.g., in the proof of Theorem 5.6 of [1], the following different strategy is employed. Via the standard self-referential lemma, a sentence λ is constructed which "expresses" the fact: "I am not provable." Then with hypotheses for the First Incompleteness Theorem, one shows that $\vdash \lambda$ is impossible in a consistent logic. Finally, one shows that the provability of the consistency statement implies $\vdash \lambda$, and then concludes that consistency is unprovable; <u>but</u> this latter argument requires, in the standard proofs, <u>all three</u> of the Hilbert-Bernays derivability conditions.

In view of the preceding discussion, the "new twist" on existing arguments which gives Theorem 1 consists for the most part in a different self-referential construction and a different choice of the self-referential sentence. The distinction between various kinds of self-referential constructions was first discussed in [7].

Theorems 2 and 3 treat Godel's Second Incompleteness Theorem in the setting of quantifier-free logics. These results alone deal with the kind of consistency statements which are relevant to Hilbert's original finitist program [4]. The devices required to derive Theorems 2 and 3 are complex, new self-referential sentences.

We originally derived Theorems 1 and 3 as simplifications of the argument for Theorem 2, which we discovered first. The referee's careful study of our original proof of Theorem 1 isolated the new self-referential lemma we mentioned above and also the precise differences between our derivation of Theorem 1 and standard derivations of analogous results in logics with quantifiers. We have restructured our original proof of Theorem 1 following the referee's suggestions.

Kreisel has informed us that a result in his recent joint work with Takeuti implies that our hypotheses in Theorem 1, while sufficient for Godel's Second Underivability Theorem, are not strong enough to obtain Lob's result.

Our results are not sensitive to the connotation of the formulae used to express proof or provability, and they hold so long as extensional criteria are met -- e.g., that the formula does <u>in fact</u> define a proof or provability predicate, etc. The supposedly "vague" issue of connotation and meaning in this technical setting will be discussed in a later paper, where it will be given a precise, mathematical treatment, based on the ideas of G. Kreisel in [6].

S. Feferman has independently removed any form of the second derivability condition as a hypothesis in the setting of Theorem 1. For history, the reader is referred to Section 5 below.

1. We now state certain preliminary hypotheses on a theory T, which are met commonly in logics for which merely some version of the <u>First</u> Incompleteness Theorem is known. Then we shall add further conditions explicitly in our theorems in order to derive versions of the Second Incompleteness Theorem.

Let L be the language of T. It is assumed that there is a 1-1 correspondence between some set D of closed terms and pseudo-terms (see below) of L and the set of all terms, formulae, and sequences of formulae. Let the closed term or pseudo-term $\overline{\alpha}$ correspond to the syntactic object α; we also say that $\overline{\alpha}$ is the godel numeral of α.

It is assumed that there is a class F of functions, closed under composition, such that the domain of each $f \in F$ is some (finite) Cartesian product of D, and f maps into D.

It is assumed that, to every $f \in F$, there is an encoding $A_f(x_1,\ldots,x_n, x_{n+1})$ of the graph of f, i.e., of the relation $f(x_1,\ldots,x_n) = x_{n+1}$; and we shall abuse notation and write this encoding as $f(x_1,\ldots,x_n) = x_{n+1}$, as if the latter statement were a formula of L (which is the case in free variable systems, at least up to the issue of use vs. mention).

In further elaborating this abuse of notation, we shall use, e.g., $k(g(x,y), h(y)) = z$ to abbreviate the encoding $m(x,y) = z$ of the function $m(x,y) = k(g(x,y),h(y))$, and this latter function is in F, by closure of F under composition. In logics with quantifiers we shall also call the expressions "$f(x_1,\ldots,x_n)$" (which often do not actually exist in L) pseudo-terms. For a wff $\alpha(x)$ and $f \in F$, $\alpha(f(x))$ denotes $(\exists y)(\alpha(y) \wedge f(x) = y)$ in quantifier logics. The meaning of $\alpha(f(x))$ in quantifier-free languages is clear, since in this case only terms are used. In quantifier-free languages, $\overline{f}$ is the godel numeral of the term for $f \in F$; in systems with quantifiers, $\overline{f}$ is the godel numeral for A_f.

In the usual contexts, F consists of the primitive recursive functions.

The following is assumed regarding F:

F1) There is a function $sub \in F$ such that for all wffs $\alpha = \alpha(x)$ with one free variable x and for all $f \in F$, $sub(\overline{\alpha},\overline{f})$ is $\overline{\alpha(f(\overline{f}))}$.

F2) There is a function $ng \in F$ such that, for all wffs α, $ng(\overline{\alpha})$ is $\overline{\neg \alpha}$.

Let $\vdash \alpha$ denote the fact that α is provable in T.

We assume the following regarding the logic T:

T1) For any $f \in F$, if $f(a) = b$, then $\vdash f(a) = b$.

T2) If $\vdash f(a) = b$, then for all wffs $\lambda(x)$ of L,

$$\vdash \lambda(f(a)) \Longleftrightarrow \vdash \lambda(b)$$

T3) The theorems of T are closed under the application of the following rules (where we by no means assume that these rules are in any way explicitly rules of T);

$$\text{R1} \ \frac{\text{nothing}}{\varphi \rightarrow \varphi}$$

$$\text{R2} \ \frac{\varphi(x)}{\varphi(t)}, \quad \begin{array}{ll} x & \text{a free variable} \\ t & \text{any closed pseudo-term or term} \end{array}$$

$$\text{R3} \ \frac{\varphi \rightarrow \lambda, \ \varphi \rightarrow \phi}{\varphi \rightarrow \lambda \wedge \phi}$$

$$\text{R4} \ \frac{\varphi \rightarrow \alpha, \ \neg \alpha}{\neg \varphi}$$

2. The following lemma, which provides a new means of obtaining self-referential sentences, was discovered by the referee, after an examination of our original proof of Theorem 1 below which contained a particular application of this lemma.

Lemma: With every formula $\psi(x)$ one can effectively associate a formula φ and a closed term (or pseudo-term) t such that the following holds:

(1) φ is $\psi(t)$

(2) $\vdash t = \overline{\varphi}$.

Proof. Define the function f by

$$f(d) = \text{sub}(\overline{\psi(x)}, d)$$

for $d \in D$. Clearly, $f \in F$, and hence $f(y)$ is a term (or pseudo-term). Set $t = f(\overline{f})$, and $\varphi = \psi(t)$. Then (1) is trivial and using F1 and T1 we have $\vdash f(\overline{f}) = \overline{\psi(f(\overline{f}))}$, which gives (2). Q.E.D.

We now can state our first version of the Godel-Hilbert-Bernays Theorem whose intended application is to logics with quantifiers.

Theorem 1: Suppose that pr(u) is any wff of T such that the following conditions hold:

(1) If [~\, then (~prd).

(2) For all closed pseudo-terms t,

[-pr(t) - pr($\overline{pr(t)}$))

Let Q-Con be the formula ~i (pr(u)/\pr (neg(u))). Then if T is consistent, Q-Con is not provable in T_0

Proof: Take \|((x) = pr(neg(x)) in the lemma and let ^ and t be as in that lemma.

By Rl of T3, [- cp -• cp. By the lemma, {-1 = $\overline{cp}$. By T2, these results give j-cp-> pr(neg($\overline{cp}$)). By hypothesis 2, since neg(t) is a closed pseudo-term, we also have |– cp –^ pr($\overline{cp}$). Thus by R3, we have |– cp –• pr(neg($\overline{cp}$))/\pr($\overline{cp}$) .

Noting the last result in the paragraph above, let us suppose, for the sake of contradiction, that [-Q-Con. By R2 of T3, this gives |-~f(pr($\overline{cp}$) A pr(neg($\overline{cp}$))), and hence by R4 of T3, we have |–j cp.

We derive a contradiction from [--7 cp as follows. By hypothesis (1), f-pr($\overline{-7\ cp}$), and so, for the contradiction, it suffices to show j--| pr($\overline{\sim 7\ cp}$), since T is assumed consistent.

Since neg(t) = $\overline{-7\ cp}$, T2 and the fact that f--| cp gives |__jpr($\overline{-|\ cp}$), as desired. Q.E.D.

Remark 1: If one wishes, one can introduce the abstract concept of a "E^{*0}_{1} sentence[11] and then restate (2) in the more natural form:

(2)' for every Σ_1 sentence X_9

$\vdash \lambda \to pr(\overline{X})$.

All the preliminary hypotheses in 1. can then be restated taking into account the form of the formulae. We shall actually carry out such a detailed examination of hypotheses in 3. below when we deal with a consistency statement of the free variable form, which is more directly related to Hilbert's ideas [4] than is Q-Con.

Remark 2. Taking the hypotheses in 1. and taking hypothesis (1) for granted, hypothesis (2), which is of course a form of the third derivability condition of Hilbert and Bernays [3], gives a "best possible" result in terms of the general class of logics treated in Theorem 1.

To see this is the case, one need only consider Kreisel's example of a logic P* on page 154 of [6]. Here, a proof in P = Peano arithmetic is accepted as a proof in P* provided only that the addition of its end formula to the set of theorems of P with shorter proofs (in terms of godel numbers) does not lead to an inconsistency in propositional logic above.

To be specific, let prf (x,y) be the usual formula used to express the fact that x is (the g.n. of) a proof of (the formula with g.n.) y in Peano arithmetic P, and let prop(y,z) express the fact that y is proposionally equivalent to the negation of z. Then the condition prf*(x,y) that x is a proof of y in P* is described by the formula

$$\mathrm{prf}(x,y) \wedge (\forall u < x)(\forall z)(\mathrm{prop}(y,z) \supset \neg\, \mathrm{prf}(u,z))$$

and the condition pr*(y) that y is provable in P* is taken as $(\exists x)\mathrm{prf}^*(x,y)$.

Since there are no inconsistencies of any kind in P, evidently P* and P have the same theorems, in fact, the same proofs. Thus, all the hypothesis of 1. and the hypothesis (1) of Theorem 1 are met by P*, with pr(x) taken as the provability predicate for P* (not P). However, since inconsistencies in P* are excluded on trivial grounds, the consistency statement Q-Con (constructed from this pr(x)) is provable.

Evidently, then, the hypothesis (2) fails for P*, and hence that hypothesis generally is needed.

Remark 3: Kreisel and Takeuti will shortly be reporting their proof-theoretical results regarding Takeuti[1]s system GLC augmented by a rule corresponding to the axiom of infinity. Kreisel informs us that Q-Con is provable in this logic, and hence, as weak as our hypothesis (2) for Theorem 1 appear to be, it fails to hold in this natural logic.

3. In free variable systems, formulae pr(x) expressing provability are simply not available, so the hypotheses of Theorem 1 are not relevant. And if one merely wishes a result regarding the free variable part[3] of a logic T, evidently hypotheses for the theorem must be phrased more delicately than in Theorem 1.

In this Section, we give a result for the free variable part of logics, which permits control over the form of the consistency statement so that it is stated entirely in terms of free variable formulae.

We recall the definition of an encoding in 1. The class <S of formulae consists of all prepositional combinations of formulae which arise by instantiating into any free variable position of an encoding any closed terms or pseudo-terms.

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[3] The term "free variable part" of a logic with quantifiers is made precise by specifying a class of functions and encodings of those functions. Then the cited part of the logic consists of those formulae which can be construed as representing equality between functions and propositional combinations of such formulae, such that the formulae are provable in the logic (perhaps by deductions involving other formulae). Usually, the primitive recursive functions with their "natural" encodings are chosen. With this understanding, the free variable part of Peano arithmetic is stronger than Primitive Recursive Arithmetic because it contains the encoding of that formula of Primitive Recursive Arithmetic which asserts the consistency of the latter theory.

In quantifier-free formalisms, the notion of a pseudo-term is vacuous: there are only terms.

To the F-list of hypotheses we add:

F3) There is a function $sb \epsilon F$ such that, for all $\varphi = \varphi(x) \epsilon \mathcal{E}$ and closed pseudo-terms t, $sb(\overline{\varphi},t)$ is $\overline{\varphi(t)}$.

F4) There is a function $h \epsilon F$ such that $h(\overline{\alpha}) = 0$ precisely if $\alpha \epsilon \mathcal{E}$ (here 0 is some constant of L).

The hypothesis F4 is needed merely to <u>state</u> the form of the consistency statement which we shall use.

We retain the assumptions T1-T3 on the theory T, though for functions of two variables and formulae $\alpha \epsilon \mathcal{E}$ only. However, we permit t in R2 to be any pseudo-term. We also add, under T3, that T shall be closed under the application of the following rule (for α,β numerical formulae):

$$R5 \ \frac{\alpha}{\beta \rightarrow \alpha}$$

The above hypotheses and the hypotheses of Theorem 2 below are met by many quantifier-free systems, e.g., Primitive Recursive Arithmetic.

<u>Theorem 2</u>: Let $prf \epsilon F$ be such that

$$(\exists \alpha)(prf(\overline{\alpha},\overline{\beta}) = 0) \Longleftrightarrow \vdash \beta$$

and let us define Con to be

$$\neg (h(u) = 0 \wedge prf(v,sb(u,z)) = 0 \wedge prf(w,sb(ng(u),z)) = 0).$$

We suppose also that the following conditions hold:

(1) There is a function $c \epsilon F$ such that, if $prf(\overline{\alpha},\overline{\phi(v)}) = 0$ with v as free variable, and t is any closed pseudo-term, we have $prf(c(\overline{\alpha},\overline{t}), \overline{\phi(t)}) = 0$.

(2) For all $\rho(z) \in \mathcal{E}$ there exists $L \in F$ such that $\vdash \rho(z) \rightarrow prf(L(z), sb(\overline{\rho},z)) = 0$.

Then if T is consistent, Con is not provable in T.

Proof. Apply the lemma of Section 2 to the formula $\psi(x)$ = '$\neg prf(c(z,z),sb(x,z)) = 0$' and let t and φ be as given in that Lemma, so that $\varphi(z)$ is

$$\neg (prf(c(z,z), sb(t,z)) = 0)$$

and let $\rho(z)$ be $\varphi(z)$ without the negation sign.

Since the lemma gives $\vdash t = \overline{\varphi}$ and $\vdash ng(\overline{\rho}) = \overline{\varphi}$, two applications of T2 and one application of R1 under T3 gives

$$\vdash \rho(z) \rightarrow prf(c(z,z), sb(ng(\overline{\rho}),z)) = 0.$$

However, hypothesis (2) gives

$$\vdash \rho(z) \rightarrow prf(L(z), sb(\overline{\rho},z)) = 0$$

and hence, upon combining the last two results and using $h(\overline{\rho}) = 0$ from T1 plus R5 and then R3 used twice, we obtain

$$\vdash \rho(z) \rightarrow (prf(c(z,z), sb(ng(\overline{\rho}),z)) = 0 \wedge prf(L(z), sb(\overline{\rho},z)) = 0 \wedge h(\overline{\rho}) = 0).$$

Now suppose $\vdash$Con. Then rule R2 under T2 used three times gives

$$\vdash \neg (h(\overline{\rho}) = 0 \wedge prf(L(z),sb(\overline{\rho},z))= 0 \wedge prf(c(z,z), sb(ng(\overline{\rho}),z)) = 0).$$

If we combine this result with the last result of the previous paragraph, and use R4 under T3, we obtain $\vdash \neg \rho(z)$.

Let α be such that $prf(\overline{\alpha}, \overline{\neg \rho(z)}) = 0$. Then by (1), together with T1, we have

$$\vdash prf(c(\overline{\alpha},\overline{\alpha}), \overline{\neg \rho(\alpha)}) = 0.$$

On the other hand, since $\vdash \neg \rho(z)$, the definition of $\rho(z)$ shows that

$$\vdash \neg prf(c(\overline{\alpha},\overline{\alpha}), sb(t,\overline{\alpha})) = 0.$$

The last two results, together with $\vdash sb(t, \bar{\alpha}) = \overline{\neg \rho(\alpha)}$ (as is implied by T1) show that T is inconsistent. This is a contradiction Q.E.D.

Remark 1: Let $c\ell \varepsilon F$ be such that $c\ell(\bar{\alpha}) = 0$ precisely if $\bar{\alpha}$ is a closed numerical formula. The most natural choice of a consistency statement, in terms of Hilbert's ideas evidently is

$$\neg (c\ell(u) \wedge prf(v,u) = 0 \wedge prf(w,ng(u)) = 0).$$

However, if the formula $sb(x,y) = z$ expressing the substitution operation is chosen in the usual manner, we will have

$$\vdash c\ell(sb(\bar{\rho},z)) = 0.$$

Thus, in the common logics, the consistency statement Con of Theorem 2 would be provable if the natural consistency statement were provable.

Remark 2: Again, we have a "best possible" result. To see that hypothesis (2) is needed generally, one need only take the free variable part of P*, for which hypothesis (1) clearly holds (for P*, see Remark 2 of 2). Moreover, hypothesis (2) by itself (even with the general hypotheses from 2.) is not enough, since it is satisfied by the free variable part of the logic discussed in Remark 3 of Section 1.

Remark 3: One easily sees how to adapt the proof of Theorem 2 to establish essentially the same result if hypothesis (2) is replaced by

$$\vdash \rho(z) \rightarrow pr(sb(\bar{\rho},z))$$

and the result is phrased in terms of provability ($pr(x)$) rather than proof ($prf(x,y)$). This observation allows us to answer a question of Kreisel, which he posed us in private correspondence, and which was the entire motivation for the research undertaken for this paper, specifically: since at least one of the derivability conditions of Hilbert and Bernays must fail for P*, can one indicate a <u>single</u> derivability condition which <u>must</u> fail? Evidently, the third derivability condition fails, since the others are not needed to insure the unprovability of Con.

4. We now give a third and last version of the Consistency Theorem. The general "background hypotheses" are the same as in 3., except that under T3 the rules R3, R4, and R5 are to be dropped and replaced by the rule

R6 -7 i a - P> V» i (VAC* Ag)
-7ot

where all formulae involved are in £. The change to R6 is motivated purely by convenience in proving Theorem 3 below; in practice, it is as easy to verify as the other rules.

<u>Theorem 3</u>: Let prfgF be such that

$$(\exists\alpha)(\mathrm{prf}(\bar{\alpha},\bar{\beta}) = 0) <-> \vdash\beta$$

and let us define Con^1 to be

-i (h(u) = 0/\prf(v,u) = 0 Aprf(w,sb(ng(u),z)) = 0).

We suppose that, for all formulae p(z)g£ , there is a function LgF such that

[-p(z) - prf(L(z), sb(^,z)) = 0.

Then if T is consistent, Con^1 is not provable.

<u>Proof</u>: Apply the lemma of Section 2 to the formula \|r(x) = ' -| prf(z,x) = 0^f and let cp and t be as given in that lemma. By the main hypothesis of Theorem 3 (i.e., the third derivability condition) applied to -? cp(z) as p(z), we have

|__ in prf(z,t) = 0 - prf(L(z), sbCTcp,*)) = 0.

Since $\vdash t = \bar{\varphi}$ by T1, T2 gives

$$\vdash \neg\neg\, \mathrm{prf}(z,\bar{\varphi}) = 0 \rightarrow \mathrm{prf}(L(z),\ \mathrm{sb}(\overline{\neg\varphi},z)) = 0.$$

Note that, by T1, we also have $\vdash h(\bar{\varphi}) = 0$.

Now suppose that $\vdash$Con'. By R2 of T3, we have

$\vdash \neg\, (h(\bar{\varphi}) = 0 \wedge \mathrm{prf}(z,\bar{\varphi}) = 0 \wedge \mathrm{prf}(L(z),\ \mathrm{sb}(\overline{\neg\varphi},z)) = 0)$.

If we then combine the last three results and apply R6, we obtain

$$\vdash \neg\, \mathrm{prf}(z,\bar{\varphi}) = 0.$$

Using T2, this immediately gives $\vdash \neg\, \mathrm{prf}(z,t) = 0$, i.e., $\vdash \varphi(z)$. Now let α be such that $\mathrm{prf}(\bar{\alpha},\bar{\varphi}) = 0$. Hence, by T1, $\vdash \mathrm{prf}(\bar{\alpha},\bar{\varphi}) = 0$; but since $\vdash \neg\, \mathrm{prf}(z,\bar{\varphi}) = 0$, R2 shows that $\vdash \neg\, \mathrm{prf}(\bar{\alpha},\bar{\varphi}) = 0$. Thus, T is inconsistent, a contradiction. Q.E.D.

Remark 1: The consistency statement Con' expresses consistency in the following form: if a numerical formula $\varphi(x)$ is provable, then for no term t is $\neg\, \varphi(t)$ provable. Unless T can prove that it is closed under substitution for terms -- essentially, a version of the second derivability condition -- Con' need not be provably equivalent in T (nor even coprovable in T) with Con.

Remark 2: This result is related to a question of Kreisel, who established (with Takeuti) that Con' was not provable in Takeuti's GLC augmented by a form of the axiom of infinity, although Con is provable there. Their arguments, which establish the non-derivability of Con', involve, I am told, certain properties particular to the GLC. Kreisel conjectured that the phenomenon occurred in a general setting analogous to that of Theorem 3, a setting in which hypothesis (1) of Theorem 2 is removable, as is indeed the case.

5. As is evident from previous remarks, these results have been established in the course of answering some questions of Professor Kreisel. He read an earlier draft of a manuscript [5] containing a form of Theorem 2 and suggested that the result be stated in its present form, so that it is available for application to the free variable part of cut-free systems of intuitionist logic. It was also his suggestion to separate the purely technical results discussed here from the issue of consistency statements which correspond to a well-determined intended meaning, an issue which is also treated in [5].

At Professor Feferman's suggestion, Theorem 1 was included. We corresponded with him after having Theorem 2, and this correspondence raised the issue of logics with quantifiers possessing a provability predicate pr(x). We noted that, if the proof of Theorem 2 was adapted to a quantifier logic, and a few easy simplifications made, Theorem 1 resulted.

While this paper seems, at first glance, to show that the first and second derivability conditions are almost entirely irrelevant to the Godel-Hilbert-Bernays Theorem, there is a certain interrelation which is worth mentioning. Specifically, if one is interested in consistency statements with the intended meaning, such as those treated in [5], it can be shown that the first and second derivability conditions, together with commonly-occurring logic, implies the third derivability condition.

We are most grateful to Professor Kreisel for stimulating correspondence on the matters discussed in this paper; our reliance on this correspondence is manifest.

We also wish to thank Professor Erwin Engeler for encouraging our research.

Carnegie-Mellon University
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