MARTINGALES IN BANACH *-ALGEBRAS

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Abstract

Utilizing an abstract Radon-Nikodym theorem that we have obtained for positive linear functionals on a B*-algebra, we are able to demonstrate the equivalence of two possible conditions for defining a martingale in a B*-algebra. One condition is in terms of the positive linear functionals on the algebra. The other is in terms of expectation-like mappings obtained from our Radon-Nikodym theorem. We then define appropriate norms on the algebra which are analogous to the classical L^2 and L^1 norms. Two convergence theorems are then obtained. In these proofs neither functional diagonalization of the algebra nor utilization of expectation-like mappings is required. The essential assumptions are compactness-like conditions on the topology.

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1. Introduction. Let (X, Σ, μ) be a probability measure space. For every real number t suppose that $\mathfrak{F}_t$ is a sub-σ-field of Σ such that $\mathfrak{F}_s \subseteq \mathfrak{F}_t$ whenever $s \leq t$. In addition, for every t let x_t be a μ-integrable random variable such that x_t is measurable with respect to $\mathfrak{F}_t$. Then $\{x_t\}_{t \in \mathbb{R}}$ is called a martingale if $E(x_t | \mathfrak{F}_s) = x_s$ whenever $s \leq t$, where $E(x_t | \mathfrak{F}_s)$ denotes the conditional expectation of x_t with respect to $\mathfrak{F}_s$. In [3], it is shown that $\{x_t\}_{t \in \mathbb{R}}$ is a martingale if and only if

$$\int_A x_t d\mu = \int_A x_s d\mu$$

for all A in $\mathfrak{F}_s$. Thus a martingale can be defined in terms of either the conditional expectation or the integral. For more information on martingales, see J. Doob [3] or P. Meyer [4].

In [7], H. Umegaki extended this concept to the non-commutative integration theory of W*-algebras, as developed by I. Segal [6]. In this theory the integral is replaced by a positive linear functional on a W*-algebra and the sub-σ-fields are replaced by sub-W*-algebras. Also, it is possible to construct on the W*-algebra a linear mapping which is very similar to the classical expectation mapping. In this context Umegaki defined a martingale (or M-net, in his terminology) and obtained generalizations of several classical L^1 and L^2 convergence theorems for martingales. (For completeness, we mention in passing that in the appendix of [2], W. Arveson has considered a slightly different generalization of martingales to W*-algebras.)

In this work we extend some of Umegaki's results to Banach *-algebras. In [1], we obtained an abstract Radon-Nikodym theorem for positive linear functionals on a B*-algebra. The resulting Radon-Nikodym "derivatives" are linear mappings similar to the classical expectation mappings. In Section 3 we then prove under certain restrictions the equivalence of two possible conditions for defining a martingale in a B*-algebra. One condition is in terms of the positive linear functional on the algebra and the other is in terms of the expectation-like mappings obtained from the abstract Radon-Nikodym theorem.

In order to make the most general definition, we define a martingale in terms of the positive linear functional since the existence of the expectation-like mappings involves placing additional restrictions on the functional.

In Sections 4 and 5 we obtain two convergence theorems for martingales. We restrict our attention to the case when the martingale is a sequence. First we define on the algebra appropriate norms which are analogous to the classical L^2 and L^1 norms. We then consider when the martingale converges in each of these norms. A feature of the proofs in these sections is that we do not require that the functional diagonalizes any part of the algebra. The property of diagonalization (or centralization) is important in the non-commutative integration theory of W*-algebras and in Umegaki's proofs. Furthermore, we never use expectation-like mappings in these proofs, as opposed to Umegaki's proofs which rely heavily on them. On the other hand, we have not been able to obtain all of Umegaki's results in the more general setting of Banach *-algebras.

2. *Definitions and Preliminary Results*. For the general theory of Banach algebras, see C. Rickart [5]. For this section and Section 3 we assume that N is a sub-B*-algebra of a B*-algebra M with identity, assuming also that N has the identity. Throughout the paper $\mathbb{N}$ will denote the positive integers.

A positive linear functional ρ on M is said to be *faithful* on a subset S of M if whenever $h \in S$ and $\rho(h^*h) = 0$, then $h = 0$.

The functional ρ is said to *diagonalize* the subset S if $\rho(nm) = \rho(mn)$ for all $m \in M$ and $n \in S$.

The *center* of N is the set $\{x \in N \mid xn = nx \text{ for all } n \in N\}$.

If q is a projection in the center of N, then the set $N_q = \{qnq \mid n \in N\}$ is a sub-*-algebra of N.

For reference we give a lemma and the main theorem from [1]. The proofs are omitted.

LEMMA. *Let q be a projection in the center of N. Suppose that ρ is a positive linear functional on M which satisfies the following conditions*:

(a) ρ *diagonalizes* N

(b) ρ *is faithful on* N_q

(c) *There is a constant* $k > 0$ *such that*

$\rho(x^*xy^*y) \leq k\rho(x^*x)\rho(y^*y)$ *for all* $x \in N$ *and* $y \in N$.

Then N_q *forms a Hilbert algebra under the inner product* $(a,b) = \rho(b^*a)$.

We denote the completion of N_q in this inner product by $\overline{N_q}$. Note that $\overline{N_q}$ is not necessarily contained in M. But the multiplication and involution operator of M can be extended to $\overline{N_q}$. Part (c) of the above lemma allows multiplication to be extended and the involution is an isometry on N_q and so extends to an isometry on $\overline{N_q}$.

THEOREM 1. *Suppose that* ρ *and* q *are as in the lemma. Let* σ *be another positive linear functional on* M *such that there is a constant* $K > 0$ *for which* $\sigma(x^*x) \leq K\rho(x^*x)$ *for all* $x \in M$. *Then there is a mapping* Φ *from* M *to* $\overline{N_q}$ *such that*

(a) $\sigma(ax) = \rho(a\Phi(x))$ *for all* $a \in \overline{N_q}$ *and* $x \in M$,

(b) Φ *is linear*,

(c) $\Phi(bx) = b\Phi(x)$ *for all* $b \in N_q$ *and* $x \in M$.

If σ *also diagonalizes* N, *then*

(d) Φ *is adjoint preserving*,

(e) *the mapping* Φ *is unique in the sense that any other mapping from* M *to* $\overline{N_q}$ *with the above four properties is equal to* Φ.

This is an abstract Radon-Nikodym theorem. The mapping Φ can be considered as a Radon-Nikodym derivative or abstract conditional expectation.

3. Abstract Martingales. Since the classical definition of a martingale can be given equivalently in terms of either the conditional expectation or the integral, we consider two similar possible defining conditions for an abstract martingale and then show that under suitable assumptions both conditions are equivalent.

Let D be a directed set and P a positive linear functional on M. Suppose that for every $\alpha \in D$, there is a positive linear functional ρ_α on M and a projection q_α in the center of N such that each triple $(\rho, \rho_\alpha, q_\alpha)$ satisfies the hypotheses of Theorem 1. We thus have a collection of algebras $\overline{N}_{q_\alpha}$ and mappings $\$_\alpha$. For convenience we denote $\overline{N}_{q_\alpha}$ by $\overline{N}_\alpha$. We shall assume that $\overline{N}_\alpha \subset \overline{N}_\beta$ whenever $\alpha < \beta$. Now let $\{x_\alpha\}_{\alpha \in D}$ be a set of elements in M such that $x_\alpha \in \overline{N}_\alpha$. Then conditions (a) and (b) in the following theorem could be thought of as defining $\{x_\alpha\}$ to be a martingale.

THEOREM 2. Suppose that whenever $\alpha < \beta$, then

(1) $\$_\alpha(x_\alpha) = x_\alpha$ and

(2) $\rho_\beta(a x_\alpha) = \rho_\alpha(a x_\alpha)$ for all $a \in N_\alpha$.

Then for $\alpha < \beta$, the following statements are equivalent;

(a) $\rho(a x_\alpha) = \rho(a x_\beta)$ for all $a \in N_\alpha$

(b) $W = V$

Proof; We prove (a) implies (b). Suppose $\alpha \leq \beta$. By (1) we have $\$_\beta(x_\beta) = x_\beta$ and $\$_\beta(x_\alpha) = x_\alpha$. From Theorem 1 we have

$$\sigma_\alpha(ax_\alpha) = \rho(a\Phi_\alpha(x_\alpha))$$

for all $a \in N_\alpha$. Since $\Phi_\alpha(x_\alpha) = x_\alpha$, we have by (a)

$$\rho(a\Phi_\alpha(x_\alpha)) = \rho(a\Phi_\beta(x_\beta)).$$

But $\rho(a\Phi_\beta(x_\beta)) = \sigma_\beta(ax_\beta)$ for all a in N_α since $N_\alpha \subseteq N_\beta$. From the property of the expectation-like mappings and from (1) we have

$$\sigma_\alpha(ax_\alpha) = \sigma_\beta(ax_\alpha).$$

Thus we have

$$\sigma_\beta(ax_\beta) = \alpha_\beta(ax_\alpha) = \sigma_\alpha(ax_\alpha).$$

By (2), it now follows that $\sigma_\alpha(ax_\beta) = \sigma_\alpha(ax_\alpha)$. We also have

$$\sigma_\alpha(ax_\beta) = \rho(a\Phi_\alpha(x_\beta)) \qquad \text{and}$$

$$\sigma_\alpha(ax_\alpha) = \rho(a\Phi_\alpha(x_\alpha)) = \rho(ax_\alpha)$$

for all a in N_α. Thus $\rho(ax_\alpha) = \rho(a\Phi_\alpha(x_\beta))$ for all a in N_α. So $\rho(a[x_\alpha - \Phi_\alpha(x_\beta)]) = 0$ for all a in N_α. Since x_α and $\Phi_\alpha(x_\beta)$ are both in N_α, if $a = [x_\alpha - \Phi_\alpha(x_\beta)]^*$ then $\rho(aa^*) = 0$. Since ρ is faithful on N_α, $a = 0$ and so $x_\alpha = \Phi_\alpha(x_\beta)$. Thus (a) <u>implies</u> (b).

It is easily seen that the above argument can be reversed to obtain (b) <u>implies</u> (a). ∎

We now give our definition of martingale using condition (a) of the previous theorem.

DEFINITION. Let M be a Banach *-algebra and ρ a positive linear functional on M and D a directed set. For every $\alpha \in D$ let N_α be a subspace of M. Then $\{x_\alpha, N_\alpha\}$ is called a *martingale on* M *with respect to* ρ *if*

(1) $N_\alpha \subseteq N_\beta$ whenever $\alpha \leq \beta$,

(2) $x_\alpha \in N_\alpha$ for every α,

(3) $\rho(a x_\alpha) = \rho(a x_\beta)$ whenever $\alpha \leq \beta$ and $a \in N_\alpha$.

Since M and ρ will be fixed, we shall call $\{x_\alpha, N_\alpha\}$ just a *martingale*.

The following proposition gives some properties of abstract martingales analogous to those of classical martingales.

PROPOSITION. *Let* $\{N_i\}_{i=1}^{\infty}$ *be an increasing sequence of* *-*subalgebra of* M *and let* $\{x_i\}$ *be a sequence with* $x_i \in N_i$ *for every* i. *Let us define*

$$y_1 = x_1$$

and

$$y_n = x_n - x_{n-1}$$

for all $n \geq 2$.

(a) If $\{x_i, N_i\}$ *is a martingale, then for every* $n \geq 2$, $\rho(a y_n) = 0$ *if* $a \in N_{n-1}$. *Also,*

$$\rho(x_i^* x_i) = \sum_{j=1}^{i} \rho(y_j^* y_j)$$

for all $i \geq 1$.

(b) *If* $\{y'_n\}_{n\in\mathbb{N}}$ *is a sequence such that* $y'_n \in L^1(N_n)$ *for every* n *and if* $\rho(a y'_n) = 0$ *for all* $a \in N_{n-1}$ *whenever* $n \geq 2$, *then the sequence*,

$$\{x'_n = \sum_{i=1}^{n} y'_i, N_n\}$$

is a martingale.

The proofs of (b) and the first part of (a) follow immediately from the definition of martingales. The summation formula in (a) is given by a straightforward induction proof. We shall therefore omit these proofs.

3. *Convergence in* L^2. Throughout the rest of the paper M will denote a Banach *-algebra with identity such that $\|x\| = \|x^*\|$ for all $x \in M$. The letter ρ will denote a (continuous) positive linear functional on M which is also faithful on M.

The functional ρ then defines a natural inner-product on M by $(x,y) = \rho(y^*x)$. The associated norm $\|x\|_2 = \rho(x^*x)^{1/2}$ is then analogous to the L^2 norm from classical integration theory. We shall denote the Hilbert space completion of M in this inner-product by $L^2(M)$ or usually just by L^2.

Since $\|x\|_2 = \|x^*\|_2$, the involution operation extends to an isometry on L^2. Also note that although the multiplication in M may not extend to L^2, it is possible to define $\rho(xy)$ for x and y in L^2 by $\rho(xy) = (y,x^*)$ since the inner-product extends to L^2.

For the following theorem we assume that $\{x_n, N_n\}$ is a martingale on L^2 such that each subspace N_n is also closed under the involution operation.

THEOREM 3. *Suppose that there exists a* $K > 0$ *such that* $\|x_n\|_2 < K$ *for all* n. *Then there exists a* $x_\infty \in L^2$ *such that* $\{x_n\}$ *converges in* L^2 *norm to* x_∞.

Furthermore, if $N_\infty = \overline{\cup N_n}$, *then* $\{\{x_n\} \cup \{x_\infty\}, \{\overline{N_n}\} \cup \{N_\infty\}\}$ *is a martingale on* L^2.

Proof: We first establish the existence of a weak limit x_∞ of the sequence $\{x_n\}$. We then show that $\{\{x_n\} \cup \{x_\infty\}, \{\overline{N_n}\} \cup \{N_\infty\}\}$ is a martingale, which will then imply that $\{x_n\}$ converges in L^2 norm to x_∞.

For every i, let P_i be the orthogonal projection on the closed subspace $\overline{N_i}$. Then $\{P_i\}$ is an increasing sequence of projections which converges strongly to an orthogonal projection P. (P is just the orthogonal projection on the subspace $N_\infty = \overline{\cup N_n}$). Also, for every i, $P_j - P_i$ is an orthogonal projection for every $j > i$.

We now show that for $j > i$, $P_i x_j = x_i$. First, we observe that it is easy to show that $\{x_n, \overline{N_n}\}$ is a martingale. We then have that

$$(P_i x_j - x_i, P_i x_j - x_i) = (P_i x_j, P_i x_j) - (x_i, P_i x_j) + (x_i, x_i) - (P_i x_j, x_i).$$

But the first summand is just

$$(x_j, P_i x_j) - (x_i, P_i x_j) = 0$$

since $P_i x_j \in \overline{N_i}$ and $\{x_i, \overline{N_i}\}$ is a martingale. Similarly the last summand is

$$(x_i, x_i) - (x_j, P_i x_i) = (x_i, x_i) - (x_j, x_i) = 0.$$

Thus $\|P_i x_j - x_i\|_2 = 0$ implies $P_i x_j = x_i$.

Now for $a \in L^2$, we have for $j > i$

$$|(x_j - x_i, a)| = |((P_j - P_i)x_j, (P_j - P_i a)| \leq \|x_j\|_2 \|(P_j - P_i)a\|_2$$
$$\leq K\|(P_j - P_i)a\|_2.$$

Then as $j \to \infty$, the last expression converges to zero since $P_j a$ and $P_i a$ converge to Pa. Thus $\{x_i\}$ is a weak Cauchy sequence in the Hilbert space L^2. But Hilbert space is weakly sequentially complete and so $\{x_i\}$ converges weakly to some x_∞ in L^2.

Now for every $a \in L^2$

$$|(Px_\infty - x_\infty, a)| \leq |(Px_\infty - Px_n, a)| + |(x_n - x_\infty, a)|$$

where $Px_n = x_n$. Again the last expression converges to zero as $n \to \infty$. Thus $(Px_\infty - x_\infty, a) = 0$ and so $Px_\infty = x_\infty$ and so x_∞ is in N_∞.

To show that $\{\{x_n\} \cup \{x_\infty\}, \{\overline{N_n}\} \cup \{N_\infty\}\}$ is a martingale, it is sufficient to show that $\rho(ax_n) = \rho(ax_\infty)$ for all $a \in N_n$, that is, we have to show $(x_n, a^*) = (x_\infty, a^*)$ for all $a \in N_n$. But since N_n is closed under involution, it is sufficient just to show $(x_n, a) = (x_\infty, a)$ for all $a \in N_n$.

Holding n and a fixed, we have

$$|(x_n - x_\infty, a)| \leq |(x_n - x_m, a)| + |(x_m - x_\infty, a)|.$$

For $m > n$, $(x_n - x_m, a) = 0$ and as $m \to \infty$, $\{(x_m - x_\infty, a)\}$ converges to zero. Thus $(x_n - x_\infty, a) = 0$.

Finally, we show that $\{\|x_n - x_\infty\|_2\}$ converges to zero as $n \to \infty$. For this, we have

$$(x_n - x_\infty, x_n - x_\infty) = (x_n, x_n) - (x_\infty, x_n) + (x_\infty, x_\infty) - (x_n, x_\infty).$$

By the martingale property just proved, we have $(x_n, x_n) = (x_\infty, x_n)$ for all n. Since $\{x_n\}$ converges to x_∞ weakly, $\{(x_n, x_\infty)\}$ converges to (x_∞, x_∞) as $n \to \infty$. Thus $\{\|x_n - x_\infty\|_2\}$ converges to zero as $n > \infty$. ∎

5. Convergence in L^1. The proof of the convergence theorem in this section is more difficult than in the L^2 case since we do not have a Hilbert space structure at our disposal. In order to get a satisfactory theorem we assume that a compactness condition on the unit ball of M and two uniformity conditions on the martingale are satisfied.

First, we define a norm on M similar to the L^1 norm from classical integration theory. For x in M, let

$$\|x\|_1 = \sup_{\|y\| \leq 1} \{|\rho(xy)| + |\rho(yx)|\}.$$

Then $\|\cdot\|_1$ is a norm on M. (Recall that $\|\cdot\|$ denotes the norm in the Banach *-algebra M.) We use both $\rho(xy)$ and $\rho(yx)$ to compensate for the fact that ρ may not diagonalize the algebra.

We take the completion of M in this new norm and denote it by $L^1(M)$ or usually just by L^1. As in the L^2 case, the involution is easily seen to satisfy $\|x\|_1 = \|x^*\|_1$ and so the involution extends to an isometry on L^1.

As in the L^2 case, multiplication in M may not extend to L^1 but we have the following result. Suppose $x \in M$ is fixed. Define $F_x(y) = \rho(yx)$. Then it is routine to verify that F_x is a $\|\cdot\|_1$-continuous linear functional on M and so extends to a unique $\|\cdot\|_1$-continuous linear functional on L^1. We denote this extension still by F_x. Thus $x \in M$ and $y \in L^1$ we can define $\rho(yx)$ to be $F_x(y)$. Similarly considering the functional $G_x(y) = \rho(xy)$, we can define# $\rho(xy)$ when $x \in M$ and $y \in L^1$. Henceforth, these values will be the definitions of $\rho(yx)$ and $\rho(xy)$ when $x \in M$ and $y \in L^1$.

We now suppose that for every $n \in \mathbb{N}$, N_n is a subspace of L^1 which is also closed under the involution and that $\cup N_n$ is dense in L^1. Assume also that for every n, $x_n \in M \cap N_n$ and $\{x_n, N_n\}$ and $\{x_n^*, N_n\}$ are both martingales on L^1. This means that for $m \geq n$, $\rho(ax_n) = \rho(ax_m)$ and $\rho(ax_n^*) = \rho(ax_m^*)$ for all $a \in N_n$. The second equation is equivalent to $\rho(x_n a) = \rho(x_m a)$ for all $a \in N_n$ since N_n is closed under involution.

In the following, F_n and G_n will denote respectively the linear functionals F_{x_n} and G_{x_n}, whereas $\|F_n\|_1$ and $\|G_n\|_1$ will denote the norms of these functionals as continuous linear functionals on the normed space L^1.

One could define $\rho(xy)$ as $\overline{\rho(y^*x^*)}$, but it is easily seen that both definitions give the same result.

We now consider the following conditions:

(A) For every sequence $\{c_n\} \subseteq \{x \mid \|x\| \leq 1\} \subseteq M$, there is a subsequence $\{c_{n_k}\}$ and a point $c \in L^1$ such that $\{\rho(c_{n_k})\}$ converges to $\rho(c)$;

(B) Whenever $\{\rho(y_n)\}$ converges to zero, then both $\{\rho(y_n x_n)\}$ and $\{\rho(x_n y_n)\}$ converge to zero. Moreover there is a $K > 0$ such that $\|F_n\|_1 < K$ and $\|G_n\|_1 < K$ for all n.

Condition (A) is a type of sequential compactness condition on the unit ball in M (which is a subset of a multiple of the unit ball $\{x \mid \|x\|_1 \leq 1\}$ in L^1). The statements in (B) can be thought of as uniform convergence and boundedness conditions on the martingale.

For the following theorem, let $M_n = N_n \cap M$ and $M_\infty = (\bigcup_n N_n) \cap M$.

THEOREM 4. *Suppose that* (A) *and* (B) *are satisfied. Then there exists an element* x_∞ *in* L^1 *such that* $\{x_n\}$ *converges to* x_∞ *in the* L^1 *norm.*

Except for the requirement that $x_\infty \in M_\infty$, $\{\{x_n\}\cup\{x_\infty\}, \{M_n\}\cup\{M_\infty\}\}$ *satisfies all other requirements for being a martingale on* M.

Proof: Let $N = \bigcup N_n$. Then by assumption N is dense in L^1. By the martingale property we have that for every $y \in N$, $\lim_n \rho(yx_n)$ and $\lim_n \rho(x_n y)$ both exist. For if $y \in N_k$, then $\rho(yx_k) = \rho(yx_n)$ and $\rho(x_k y) = \rho(x_n y)$ for all $n > k$. We now show that both of these limits exist when $y \in L^1$.

Let y be in L^1 and $\{y_n\} \subseteq N$ such that $\{y_n\}$ converges to y in L^1. Then for $\varepsilon > 0$, choose p such that $\|y_p - y\|_1 < \varepsilon$. Then

$$|\rho(y(x_n - x_m))| \leq |\rho((y_p - y)(x_n - x_m))| + |\rho(y_p(x_n - x_m))|.$$

For $m,n > k$, where $y_p \in N_k$, the second term on the right hand side is 0 by the martingale property. The first term is bounded by $2K\|y_p - y\|_1 < 2K\varepsilon$. Thus $\{\rho(yx_n)\}$ is a Cauchy sequence and converges. Similarly $\lim\limits_n \rho(x_n y)$ exists.

To show that $\{x_n\}$ is a L^1 convergent sequence, we assume that it is not a L^1 Cauchy sequence. Thus there is an $\varepsilon > 0$ such that for every $n \in \mathbb{N}$, there are $p(n)$ and $k(n)$ such that

$$\|x_{p(n)} - x_{k(n)}\|_1 > \varepsilon$$

for all n, that is

$$\sup_{\substack{\|c\| \leq 1 \\ c \in M}} \{|\rho(c(x_{p(n)} - x_{k(n)}))| + |\rho((x_{p(n)} - x_{k(n)})c)|\} > \varepsilon$$

It is easy to see that the above sup is bounded from above uniformly in n using the uniform boundedness of the maps F_n and G_n and the fact that the sup is taken over a subset of a multiple of the unit ball $\{x \mid \|x\|_1 \leq 1\}$ in L^1. Denote this bound by R. We also note that $p(n) \to \infty$ and $k(n) \to \infty$ as $n \to \infty$.

By the sup property, for every n there is a $c_n \in M$ such that $\|c_n\| \leq 1$ and

$$R > |\rho(c_n(x_{p(n)} - x_{k(n)}))| + |\rho((x_{p(n)} - x_{k(n)})c_n)| > \varepsilon.$$

Now one of the terms in the middle expression must contain a subsequence bounded away from 0. Since $(x_n, \mathcal{N}_n)$ and $(x_n, \mathcal{N}_n)$ are both martingales, without loss of generality we can assume that the first terms are all bounded away from 0:

$$R > |\rho(c_n(x_{p(n)} - x_{k(n)}))| > \delta > 0$$

for all n. Thus the set of numbers in the above inequality contains a convergent subsequence and again without loss of generality we can assume that the entire sequence converges:

$$|\rho(c_n(x_{p(n)} - x_{k(n)}))| \to a \neq 0 \qquad (1)$$

as $n \to \infty$.

By (A) there is a subsequence $\{c_{j(n)}^*\}$ of $\{c_n\}$ and an element $c \in L^1$ such that $\{\rho(c_{j(n)}^* - c)\}$ converges to zero and $j(n) \to \infty$ as $n \to \infty$.

But by (B) with $y_n = c_{j(i)}^* - c$ if $n = k(j(i))$ for some i and $y_n = 0$ otherwise, we see that $\{\rho((c_{j(n)}^* - c)x_{k(j(n))})\}$ converges to zero as $n \to \infty$. Similarly $\{\rho((c_{j(n)}^* - c)x_{p(j(n))})\}$ converges to zero as $n \to \infty$. Since as $n \to \infty$, $\{\rho(c(x_{p(j(n))} - x_{k(j(n))}))\}$ converges to zero, we have also $\{\rho(c_{j(n)}^*(x_{p(j(n))} - x_{k(j(n))}))\}$ converging to zero which contradicts (1).

Thus (x_n) is a Cuachy sequence in L^1 and so converges to an element in L^1, say, x_∞. ∎

Ihe proof of the second statement in the theorem is very similar to the proof for the analogous statement in Theorem 3 and so we omit it. We just remark that x_J may not be in M, but just somewhere in the completion L^1_{co} of M.

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