REAL FUNCTIONS ON COMPLETE SPACES HAVE CONTINUOUS RESTRICTIONS TO DENSE SUBSETS

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§1. Introduction.

In this paper we show that every function of a complete metric space into a separable metric space has a continuous restriction to some dense subset of its domain. The original objective was to show that this theorem is true for functions from the real line into the real line, and for functions of the Euclidean m-space into the Euclidean n-space.

The proof of this theorem depends on the Baire category theorem [3], and for the sake of convenience some corollaries of this theorem are listed in section 2.

The main part of the proof will be found in section 3, where we construct dense subsets for arbitrary functions from complete metric spaces into the Cantor set C.

In the last section we prove the main theorem and list some corollaries and extensions.

The question discussed in this paper originated from Problem 9 in the problem book of the Department of Mathematics of Carnegie-Mellon University.

§2. Subsets of the first category.

2.1. Proposition. *Let A be a subset of a space X. Then the collection of all points of A for which every neighborhood contains interior points of X\A is nowhere dense in X.*

(In formula: $A \cap A^{co-}$ is nowhere dense in X.)

Proof. From the calculus of open sets in a topological space it follows that

$$(A \cap A^{co-})^{-o} \subset (A^{-} \cap A^{co-})^{o} = A^{-o} \cap A^{co-o} = A^{-o} \cap A^{-oco} = \emptyset$$

Thus, by definition, $A \cap A^{co-}$ is nowhere dense.

2.2. Notation Conventions. Let A be a subset of a space X. We define $t(A)$ to be the set of all points of A which have a neighborhood which intersects A in a first category collection of X. We define $p(A)$ to be the set of all points q of A such that in every neighborhood U_q of q there exist an open subset O such that $O \cap A$ is a first category collection in X.

The set $A\setminus p(A)$ will be denoted by $s(A)$.

If $\mathcal{A}$ is a collection of subsets of X, then $P(\mathcal{A}) = \cup\{p(A) \mid A \in \mathcal{A}\}$ and $S(\mathcal{A}) = X \setminus P(\mathcal{A})$.

Remark. If $q \in s(A)$ then there exist a neighborhood U_{qA} of q such that every open subset of U_{qA} has a second category intersection with A.

2.3. Proposition. *Let X be a metrizable space. Then $t(A)$ is a first category subset of X for each subset A of X.*

Proof. From the Bing metrization theorem [1] we can find a σ-discrete base $\mathcal{B}$ for the space X. Let $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$, where each $\mathcal{B}_n$ is a discrete collection. Let $\mathcal{B}_n^*$ be the subcollection of all members of $\mathcal{B}_n$ which have a first category intersection with A. Let $B \in \mathcal{B}_n^*$. Then $A \cap B = \bigcup_{k=1}^{\infty} N_k(B)$ in which every N_k is nowhere dense. But now we can prove that for every $k \in \mathbb{N}$ the collection $\bigcup \{N_k(B) \mid B \in \mathcal{B}_n^*\}$ is nowhere dense. Since the collection $\mathcal{B}_n^*$ is discrete and $N_k(B) \subset B$ we have

$$(\bigcup\{N_k(B) \mid B \in \mathcal{B}_n^*)^{-o} = \bigcup\{(N_k(B))^{-o}\} = \emptyset$$

since every N_k is nowhere dense.

Now $t(A) = (\bigcup_{n=1}^{\infty} \mathcal{B}_n^*) \cap A = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} (\bigcup\{N_k(B) \mid B \in \mathcal{B}_n^*\})$ and thus $t(A)$ is a countable union of nowhere dense subsets, which was required.

2.4. Proposition. *Let X be a metrizable space. Then for every subset A of X the subset $p(A)$ is of the first category in X.*

Proof. A point q of A is a member of p(A) either when q e t(A) or when every neighborhood of q contains an open subset whose intersection with A is contained in t(A). Proposition 2.1 applied to A\t(A) implies that p(A)\t(A) is nowhere dense. Now it follows from 2.3 that p(A) is a first category subset of X.

2.5. Proposition. If G j-js ja countable family of subsets of ci complete metric space X, then P(G) *is^* a_ subset of the first category in X.

Proof. Follows immediately from 2.4.

2.6. Proposition. Let X be a. complete metric space and let $G = \bigcup_{n=1}^{\infty} G_n$ be a, countable collection of finite covers of X. Let q *e* S(G). Then for every n and for every A such that q e A e G_n there exist a_ neighborhood U(qJ,A) oj: q such that S(G) 0 A has ji second category intersection with every non empty open subset 0 o£ U(q,A).

Proof. Let q e S(G). Since for every n the collection G_n covers X we have that qeA for some A e. G_n. It is clear that q G s(A), and from remark 2.2 it follows that q has a neighborhood U(q^A) such that every open subset 0 *cz* U(q^A) has a second category intersection with A. Since G is countable, P(G) is a subset of the first category and hence (OHA)\P(G) is a second category subset of X. This finishes the proof.

§3. Mappings into the Cantor Set.

3.1. Lemma. *Let* X *be a complete metric space and let* f *be a mapping from* X *into the Cantor set* C. *Then there exists a subset* S_o *of* X *such that*

(i) S_o *has a second category intersection with every non-empty open set in* X.

(ii) *If* $q \in S_o$ *and* $\{W_n\}$ *is a neighborhood base at* $f(q)$ *then there exists a neighborhood base* $\{U_n\}$ *of* q *such that for each* n, $S_o \cap f^{-1}[W_n]$ *has a second category intersection with every open subset of* U_n.

Proof. Consider C as the product of countably many copies of the discrete pair $\{0,1\}$. Let $\pi_i: C \longrightarrow \{0,1\}^i$ be the canonical projection of C onto the product of the first i coordinate spaces. Define $f_i = \pi_i \circ f$. Then f_i is a function from X into $\{0,1\}^i$ and defines a natural finite partition of X:

$$\mathcal{G}_i = \{f_i^{-1}(y) \mid y \in \{0,1\}^i\}.$$

For the sake of completeness we define the cover $\mathcal{G}_o = \{A_o\} = \{X\}$, and we define $A_n(q) = f_n^{-1}(f_n(q)) \in \mathcal{G}_n$ for all $q \in X$; $n \in \mathbb{N}$. Now let $\mathcal{G} = \bigcup_{n=0}^{\infty} \mathcal{G}_n$ and let S_o be $S(\mathcal{G})$. Then it follows from 2.5 and the Baire category theorem that S_o satisfies (i). In order to prove (ii), we define $W_n = \pi_n^{-1} \circ f_n(q)$ for an arbitrary $q \in S_Q$. (i.e. W_n consists of all points of C which have the first n coordinates equal to the first n coordinates of f(q)). When n runs through the natural numbers this is a local base of f(q).

Since $A_n(q) = f^{-1}[W_n] \in G_n \subset a$ it follows from 2.6 that there exists a $U(q,A_n(q))$ such that $S_Q \cap A_n(q)$ has a second category intersection with every open subset of $U(q,A_n(q))$. When we choose an open local base $\{U_n\}$ at q such that $U_n \subset U(q,A_n(q))$ then the collection $\{U_n\}$ satisfies (ii). It is obvious that (ii) holds for every local base at f(q) as soon as it is true for the particular local base $\{W_n\}$.

3.2. Lemma. Let X be a complete metric space, f a function from X into C, and S_o a subset of X which satisfies (i) and (ii) of Lemma 3.1. Let B be open in X, and let $q \in B \cap S_Q$. Then there exist a closed neighborhood B' of q and a set $S_q \subset B \cap S_o$ such that

(i) $q \in S_q$ and $f|S_q$ is continuous at q.

(ii) S_q has a second category intersection with every non-empty open subset of B.

(iii) S_q satisfies condition (ii) of Lemma 3.1.

(iv) $B' \subset B$ and $S_Q \cap (B \setminus B') = S_q \cap (B \setminus B')$. (i.e. no point of the boundary of B can be a cluster point of the set $(S_o \cap B) \setminus S_q$.).

Proof: Let B' be an arbitrary closed neighborhood of q which is contained in B. Let $\{W_n\}$ be a neighborhood base at $f(q)$, and let $\{U_n\}$ be a neighborhood base of q such that for each n the set $S_o \cap f^{-1}(W_n)$ has a second category intersection with every open subset of U_n. (cf. Lemma 3.1). Since X is regular and first countable we can find a neighborhood base $\{V_n\}$ of q such that for every n we have: $V_n = \bar{V}_n^{o}$; $\bar{V}_{n+1} \subset V_n$ and $V_n \subset U_n \cap B'$. Next we are going to replace S_o by $S_o \cap f^{-1}[W_n]$ on $\bar{V}_n$; but we must do it carefully in order to meet all requirements. We define:

$$S_q = B \cap S_o \cap [\bigcup_{n=0}^{\infty} \{A_n(q) \setminus \bar{V}_{n+1}\}] \cup \{q\}. \qquad (\text{N.B.} A_o(q) = X).$$

Now S_q is $S_o \cap f^{-1}(W_n)$ if we restrict our attention to $\bar{V}_n \setminus \bar{V}_{n+1}$. Outside $\bar{V}_1$ we have S_o, and inside all V_n we only have q.

In order to check (i) we take an arbitrary neighborhood of $f(q)$. This neighborhood contains some W_n, and now $V_n \cap S_q$ is a neighborhood of q with respect to S_q, which is mapped entirely into the required neighborhood of $f(q)$.

In order to check condition (ii) we suppose that O is a non-empty open subset of B. If $O = \{q\}$ then $O \cap S_q$ is clearly second category. If $O \neq \{q\}$ then there exists a $p \in O$ such that $p \neq q$. Let m be the least number such that $p \notin \bar{V}_m$. Then $O \setminus \bar{V}_m$ is a neighborhood of p which intersects

V_{m-1}. (N.B. We define $V_o = X$). It follows from 3.1 (ii) that $O \cap (V_{m-1} \setminus V_m^-)$ has a second category intersection with $A_{m-1}(q) \cap S_o$. We conclude that O has a second category intersection with S_q.

In order to check (iii), let $p \in S_q$. If $p = q$ then the preceding part shows that the collection $\{V_n\}$ meets all requirements. If $p \neq q$ then there exists an m such that $p \in V_m^- \setminus V_{m+1}^-$. Now $f_m(p) = f_m(q)$. We use as a neighborhood base for p the collection $\{U_n(p) \cap (U_m(p) \setminus V_{m+1}^-)\}$. This is a subcollection of the $\{U_n\}$ at p in Lemma 3.1, and it follows from this lemma that it inherits the required properties from 3.1. (ii).

Condition (iv) is an immediate consequence of the definitions.

3.3. Lemma. Let X be complete metric, $f: X \longrightarrow C$ a mapping from X into the Cantor set; let S_o satisfy (i) and (ii) of Lemma 3.1. Let $\mathfrak{B}$ be a discrete collection of non-empty open subsets of X and let $\varphi: \mathfrak{B} \longrightarrow S_o$ be a choice function which assigns to every $B \in \mathfrak{B}$ a point $\varphi(B) \in S_o \cap B$. Then there exists a subset S of S_o such that:

(i) $\varphi[\mathfrak{B}] \subset S$.

(ii) $f|S$ is continuous at every point of $\varphi[B]$.

(iii) S satisifes (i) and (ii) of Lemma 3.1.

(iv) $S_o \setminus \cup\mathfrak{B} = S \setminus \cup\mathfrak{B}$.

Proof: We define S from the S_q of the previous lemma:

$$S = \cup \{S_q | q = \varphi(B) : B \in \mathcal{B}\} \cup (S_o \setminus \cup\mathcal{B}).$$

Next we will check the conditions.

(i) For every q we have $q \in S_q \in S_o$ and hence $\varphi[B] \subset S \subset S_o$.

(ii) The continuity on $\varphi[\mathcal{B}]$ follows from the continuity in every q.

(iii) Let O be open in X. If O intersects some B then 3.1. (i) follows from 3.2 (ii). If O is non-empty and intersects no B then 3.1. (i) follows from 3.1. (i) for S_o. If $p \in B$ for some $B \in \mathcal{B}$ then 3.1 (ii) follows from 3.2 (iii). If $p \notin \cup\mathcal{B}$ then there exists an open neighborhood O of p which intersects only one B. If we take $O \setminus B'$ then we have an open neighborhood of p in which S coincides with S_o and now 3.1 (ii) follows immediately.

(iv) Obvious since $S_q \subset B$ for every $B \subset \mathcal{B}$.

3.4. Theorem. *Let X be a complete metric space and let f be an arbitrary mapping from X into the Cantor set C. Then there exists a dense subset D of X such that $f|D$ is continuous.*

Proof: Let $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$ be a σ-discrete collection of open sets of X which constitute a base for the topology. We define our subset D by induction.

Start. Let S_o be the subset defined in Lemma 3.1. Let $\mathcal{B}_1^*$ be the collection of all non-empty members of $\mathcal{B}_1$. Let $\varphi_1: \mathcal{B}_1^* \longrightarrow S_o$ be any choice function on $\mathcal{B}_1^*$ and let S_1 be a subset which satisfies 3.3 (i), (ii), (iii) and (iv). Define $D_1 = \varphi_1[\mathcal{B}_1]$.

Step. If for every $m \in \mathbb{N}$, $m < n$ the sets $\mathcal{B}_m^*$, S_m, and D_m are defined, then we define the collection $\mathcal{B}_n^*$ by:

$$\mathcal{B}_n^* = \{B \mid B \in \mathcal{B}_n;\ B \neq \emptyset;\ B \cap D_m = \emptyset \quad \text{for all} \quad m < n\}.$$

$\varphi_n: \mathcal{B}_n^* \longrightarrow S_{n-1}$ is just a choice function and S_n is a subset of S_{n-1} which can be defined by Lemma 3.3 and which satisfies 3.3 (i), (ii), (iii) and (iv) with respect to $\mathcal{B}_n^*$, S_{n-1} and φ_n. We define $D_n = \varphi_n[\mathcal{B}_n^*]$.

When we define $D = \bigcup_{n=1}^{\infty} D_n$, then it follows from 3.3(i) and (iv) that $D \subset S_n$ for every n. Now it follows from 3.3(ii) that, whenever $p \in D_n$, the function $f|D$ is continuous at p, and therefore f is continuous on D. Since every non-empty member of $\mathcal{B}$ contains at least one member of D it follows that D is dense. This finishes the proof.

§4. Conclusions.

4.1. *Theorem*. *If* f *is an arbitrary mapping from a complete metric space* X *into a separable metric space* Y *then there exists a dense subset* D *of* X *such that* $f|D$ *is continuous*.

Proof; The closed unit interval is a continuous image of the Cantor set, and hence the Hilbert cube is also a continuous image of the countable product of Cantor sets, and this is again the Cantor set. Let φ be a continuous function of the Cantor set C onto the Hilbert cube H. Let ψ be an arbitrary function from H into C such that $\varphi \circ \psi$ is the identity on H. (In fact, ψ can be seen as the composition of φ^{-1} with a choice function on the subsets of φ^{-1}.). Let g be an arbitrary function from X into H. Then $\psi \circ g$ is a function from X into C. Let D_g be the dense subset of X constructed in 3.4. Then $\psi \circ g|D$ is continuous. Since φ is continuous, $\varphi \circ \psi \circ g|D = g|D$ is continuous. Since Y is separable metric, it is homeomorphic with a subset of H. Let θ be the homeomorphism: $\theta : Y \longrightarrow H$. If we put $g = \theta \circ f$ then since θ^{-1} is 1-1 and continuous we find that $\theta^{-1} \circ g$ is continuous on D_g and hence $\theta^{-1} \circ \theta \circ f = f$ is continuous on D_g.

4.2. *Corollaries*. Every function from the real line into the real line can be restricted to some dense subset such that the restriction is continuous.

Every function from the Euclidean n-space into the Euclidean m-space can be restricted to some dense subset such that the restriction is continuous.

4.3. Proposition. (a) *If a space Z can be mapped one to one onto a complete metric space such that the inverse images of dense subsets are dense, then every mapping from Z into a separable metric space Y can be restricted to some dense subset of Z such that the restriction is continuous.*

(b) *If X is a dense subspace of a space X* and every function from X into Y can be restricted to some dense subset of X such that the restriction is continuous, then the same is true for mappings from X* into Y.*

Proof: Both parts are obvious.

4.4. Corollaries: If f is a function from X into Y then there exist a dense subset D of X such that $f|D$ is continuous when Y is separable metric and X meets one of the following requirements:

(i) X is the Sorgenfrei half-open interval space.

(ii) X is any compactification of a complete metric space.

(iii) X is some set of ordinals.

(iv) X is the Urysohn space. (i.e. the lexicographical product of the closed interval [0,1] with the two point set {0,1}.

(v) Every point of X has a completely metrizable neighborhood

(vi) X is a topological manifold.

Proof: (v) The proof is carried out by transfinite induction. Well-order the points of X. To the first point we assign a complete metric neighborhood. For every other point which is not in the closure of all earlier-assigned neighborhoods we assign a neighborhood which is complete metric and which does not intersect any of the previously assigned neighborhoods. The union of these neighborhoods is a dense complete metric subspace of X, and we can apply 4.3(b).

The parts (i), (ii), (iii), (iv) and (vi) are left to the reader.

4.5. Proposition. (a) A topological space X is a Baire-space if and only if every function f from X into a countable T_2-space can be restricted to a dense subset of X, such that the restriction is continuous.

(b) A topological space X is scattered if and only if every function f defined on some subset Y of X with arbitrary range, can be restricted to some dense subset of Y such that the restriction is continuous.

Remark. During the preparation of this paper I discovered that some of the main results had been published in [2].

References

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[2] H. Blumberg, "New Properties of all Real Functions", Trans. Amer. Math. Soc. 2± (1922), 113-128.

[3] J. L. Kelley, General Topology, Van Nostrand, 1955.

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