CONVEX ANALYSIS TREATED BY

LINEAR PROGRAMMING

by

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## Abstract

The theme of this paper is the application of linear analysis to simplify and extend convex analysis. The central problem treated is the standard convex program -- minimize a convex function subject to inequality constraints on other convex functions. The present approach uses the support planes of the constraint region to transform the convex program into an equivalent linear program. Then the duality theory of infinite linear programming shows how to construct a new dual program of bilinear type. When this dual program is transformed back into the convex function formulation it concerns the minimax of an unconstrained Lagrange function. This result is somewhat similar to the Kuhn-Tucker theorem. However no constraint qualifications are needed and yet perfect duality maintains between the primal and dual programs.

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## 1. *Introduction*,

The point of departure of this paper is the following standard extremal problem of convex analysis.

*Program P. Let* $g_0(z), g_1(z), \ldots, g_p(z)$ *be convex functions for* $z \in R^n$. *Seek the value*

$$M_P = \inf g_0(z)$$

*for all* $z$ *subject to the constraints*

$$g_j(z) \le 0 \qquad j = 1, \ldots, p.$$

This is, of course, a constrained extremal problem. Far reaching consequences, both for theory and for computation, develop when one attempts to convert to an unconstrained program by way of Lagrange multipliers. The best known and most important result in that direction is the Kuhn-Tucker theorem. However, the Kuhn-Tucker theorem applies only if a further proviso termed a "constraint qualification" is added. For example, the superconsistency condition of Slater postulates the existence of a point $z^*$ such that

$$g_j(z^*) < 0 \qquad j = 1, \ldots, p.$$

Here we wish to avoid such ad hoc assumptions.

To develop a corresponding Lagrange program the following new Lagrange function is introduced

$$L(z,\lambda) = g_0(z) + \sum_{j=1}^{p} \lambda_j g_j(z) + \sum_{j=1}^{n} \lambda_{-j} z_j .$$

Here the $\lambda_j$ are Lagrange multipliers. By use of this function the constraints, $g_j \leq 0$, are eliminated in the following program.

<u>Program Q</u>. <u>Seek the value</u>

$$M_Q = \lim_{\nu \to \infty} \sup_{\lambda} \inf_{z} L(z,\lambda)$$

<u>for all</u> $\lambda_j$ <u>subject to the constraints</u>

$$\inf_{z} L(z,\lambda) > -\infty$$

$$\lambda_j \geq 0 \qquad j = 1,2,\ldots,p,$$

$$\nu^{-1} \geq \lambda_{-j} \geq -\nu^{-1} \qquad j = 1,2,\ldots,n,$$

<u>and for</u> $\nu = 1,2,\ldots$ .

This program, of minimax type, is termed a <u>Lagrange dual</u> of P.

We shall say Program P is consistent if its constraints can be satisfied for some $z$. We shall say Program Q is consistent if its constraints can be satisfied for arbitrarily large integers $\nu$.

A central result of this paper is that P and Q are dual in the following sense.

DEFINITION, Two extremal programs are said to be in perfect' duality when the following properties hold:

(a) jEf one program is consistent and has a. finite value then the other program is consistent,

(b) If both programs are consistent they have equal finite values.

The method of proof is to view Program P as a question about a closed convex set. But such a set can be defined by support planes. In this way Program P is reformulated as a linear program termed Program A. Then linear programming theory is employed to show that Program A is in perfect duality with a bilinear program termed Program E. But Program E is simply a reformulation of Program Q.

The main thrust of the proof is furnished by a perfect duality theorem for infinite linear programs given in a previous paper [1]. This approach is appealing because linear analysis has a much better developed structure than convex analysis. Thus proofs are simplified and new concepts are suggested. The linear approach was also employe^ in a previous paper, "Linearizing geometric programs'[1] [3], That paper gave a new proof of the perfect duality theorem of geometric programming. The work "convex" was not needed or used.

Application of linear programming to convex analysis has also been made by Kallina and Williams [11] and by Kortanek [6]. Their applications relate to results of Rockafellar [10] and of Peterson [9] .

## 2. Ordinary and subordinary programs.

A program O is said to be subconsistent if the constraints can be satisfied to an arbitrary close degree of approximation [1]. Alternatively the program can be restated in a form S in which the relaxation of the constraints is made explicit. In this paper O is termed an ordinary program and S is termed a subordinary program. Thus saying a program is subconsistent is equivalent to saying its subordinary form is consistent. The value of the subordinary program is termed the subvalue of the ordinary program.

The subordinary form of Program P is termed P* and is defined as follows.

Program P*. Seek the value

$$M_{P*} = \lim_{\nu \to \infty} \inf_{z} g_0(z)$$

for all z subject to the constraints

$$g_j(z) \leq 1/\nu \qquad j = 1,\ldots,p,$$

and for $\nu = 1,2,\ldots$ .

Program P*, as well as Program P, can be transformed to an extremal problem about a closed convex set. Then the support planes to this set lead to a linear program. When this is done the linear theory suggests a corresponding Lagrange dual program of the following form.

Program Q*. Seek the value

$$M_{Q^*} = \sup_{A} \inf_{z} [g_0(z) + \sum_{j=1}^{p} A_j g_j(z)]$$

for all $A_j$ subject to the constraints;

$$\inf_{z} [g_0(z) + \sum_{j=1}^{p} A_j g_j(z)] > -\infty,$$

$$A_j \geq 0 \qquad j = 1, \ldots, p.$$

We shall say that Program P* is consistent if the constraints can be satisfied for arbitrarily large integers v and $M_{P^*} \neq +\infty$. Program Q* is consistent if the constraints can be satisfied for some A. The linear programming analysis which shows that P and Q are in perfect duality also shows that P* and Q* are in perfect duality.

The Lagrange program Q* is a variant of a program studied by John, Kuhn, Tucker, Dorn, Wolfe, Stoer and very many others. For detailed references see the book of Mangasarian [8] and the survey by Geoffrion [5].

## 3. The value gap.

An ordinary program P can be inconsistent and yet the subordinary program P* may be consistent and have a finite value. An example of this state is the minimization of the function $g_0 = e^{-x}$ subject to the single constraint $g_1 = e^{-x} \leq 0$. This program is inconsistent but allowing $x \longrightarrow +\infty$ shows that the subordinary program has value zero.

If the ordinary program is consistent then the subordinary program is necessarily consistent. It might be supposed that these programs have the same value. However consider the counterexample.

Program R. Seek the infimum of $e^{-y}$ subject to the constraint $(x^2+y^2)^{1/2} - x \leq 0$.

The function $(x^2+y^2)^{1/2}$ is a cone standing on its vertex and so is convex. The sum of convex functions is convex so $g_1 = (x^2+y^2)^{1/2} - x$ is convex. However the constraint can only be satisfied if $y = 0$ and $x \geq 0$. This means that $M_R = 1$.

Next consider the corresponding subordinary program. Given any fixed value of y it is easily checked that $g_1 \longrightarrow 0$ as $x \longrightarrow \infty$. This shows that $M_{R*} = 0$. Thus $M_R > M_{R*}$ and the demonstration is complete that there is a finite gap between the value and the subvalue of Program R.

## 4. Infinite linear programs.

We wish to phrase a linear program which is analogous to the convex program P. Thus let $a_{ij}$ and $c_i$ be real constants for $i \in I$ and $j \in J$ where I and J are index sets. The set I is arbitrary but J is specified (at first) to be the set of all integers. Then $J_+$ will denote the positive integers and $J_-$ will denote the negative integers. In this terminology a linear program is now defined.

*Program A. Seek the value*

$$M_A = \inf_x x_0$$

*for all* x *subject to the constraints:*

$$\Sigma_J a_{ij} x_j \geq c_i \qquad i \in I,$$

$$x_j \leq 0 \qquad j \in J_+.$$

There is a formal procedure for expressing the dual of a linear program. Thus the formal dual of Program A has the following statement.

*Program B'. Seek the value*

$$M_{B'} = \sup_y \Sigma_I y_i c_i$$

*for all* y *subject to the constraints:*

$$\Sigma_I y_i a_{ij} = 0 \qquad j \in J_-,$$

$$\Sigma_I y_i a_{ij} = 1 \qquad j = 0,$$

$$\Sigma_I y_i a_{ij} \geq 0 \qquad j \in J_+ ,$$

$$y_i \geq 0 \qquad i \in I.$$

If I and J are finite sets then both A and B' are finite linear programs. Then, as a consequence of the duality theory for finite linear programming, it is well-known that the formal dual programs A and B' are also in perfect duality.

When I or J are infinite sets then A and B' are infinite linear programs. However counterexamples are known in which A and B' are formal duals but are not in perfect duality [2,4,7]. In that case we say there is a _duality gap_. In a paper titled "Infinite Programs" [1] it was shown that perfect duality can be insured by relaxing the constraints of the ordinary dual B'. This follows directly from the first theorem of that paper which we term the _perfect duality theorem_.

To apply the PD theorem it is first necessary to introduce a vector space U for the rows of the matrix $a_{ij}$. Moreover U must be assigned a locally convex topology. The PD theorem is a very general theorem so this selection is in no way unique. It is convenient to let U have the ordinary product topology $R \times R \times \ldots$ . Then, as is well known, the conjugate space U* of U consists of vectors x with components $x_j$ for $j \in J$ such that $x_j = 0$ except for a finite set of indices j. Because of this property U* has been termed a _finite sequence space_. The bilinear form relating U and U* is then

$$(u,x) = \Sigma_J u_j x_j .$$

Next a vector space V must be introduced for the columns of the matrix $a_{ij}$. Let this space also be assigned the product topology. Then the conjugate space V* is a finite sequence space. It consists of vectors y with components $y_i$ for ie I such that $y_i = 0$ except for a finite set of indices i. Hie bilinear form relating V* and V is

$$(y,v) = \Sigma_I y_i v_i .$$

Next the PD theorem requires "positive" cones P, X, Q, and Y in the spaces U, U*, V and V* respectively. The natural choice here is to take Q to be the positive orthant in V. Then Y must be taken to be the polar cone of Q. Thus Y consists of vectors y whose components satisfy

$$y_\pm \wedge 0, \quad i \in I.$$

The cone P in U is defined by the relations:

$$U_j \wedge 0 , \quad j \in J_+,$$

$$Uj = 0, \quad j\ ft\ J_+.$$

Then X must be taken to be the polar cone of P so

$$Xj\ 1\ 0 \qquad j\ e\ J_+,$$

$$x_j \text{ arbitrary} \qquad j / J_+ .$$

This completes the requirements of the PD theorem. The conclusion of the PD theorem is that an ordinary program and its subordinary dual are in perfect duality.

(Before proceeding, the following extension of the PD theorem is worth noting. The PD theorem as stated requires that $U^*$ be the conjugate space of $U$. However an inspection of the proof shows that it is only necessary that the spaces $U$ and $U^*$ be "in duality" as defined by Bourbaki. This observation is due to Kretschmer [7]).

It is now understood that in the statements of Programs A and B' the vectors $x$ and $y$ are in finite sequence spaces. As a consequence the summations appearing in Programs A and B' are automatically convergent. The finite sequence space was introduced in programming by Charnes, Cooper, and Kortanek, see [4].

To proceed it is necessary to formulate the subordinary program to B'. We term this program B and the specifications of the PD theorem give it the following form.

<u>Program B</u>. <u>Seek the value</u>

$$M_B = \lim_{\nu \to \infty} \sup_{y} \Sigma_I y_i c_i$$

<u>for all</u> $y$ <u>subject to the constraints</u>:

$$\nu^{-1} \geq \Sigma_I y_i a_{ij} \geq -\nu^{-1} \qquad j = -1,-2,\ldots,-\nu,$$

$$\nu^{-1} \geq \Sigma_I y_i a_{ij} - 1 \geq -\nu^{-1} \qquad j = 0,$$

$$\Sigma_I y_i a_{ij} \geq -\nu^{-1} \qquad j = 1,2,\ldots,\nu,$$

$$y_i \geq 0 \qquad i \in I.$$

<u>and for</u> $\nu = 1,2,\ldots$ .

Comparing Programs B' and B we see the following difference.

In Program B' there are equalities to the zero vector of U space. But in Program B the zero vector is replaced by an $\varepsilon$-neighborhood of the origin where $\varepsilon = \nu^{-1}$. In accord with the formulation of the PD theorem the equality in the space V* is not relaxed in the subordinary program.

The conclusions of the PD theorem now insures that Program A and Program B are in perfect duality. To check this it is necessary to translate the new terminology of this paper into the language of [1] and to note that condition (b) is then obvious.

It is convenient to employ the following terminology. A sequence y(1),y(2),... which satisfies the constraints of Program B corresponding to $\nu = 1,2,\ldots$ is termed a feasible sequence. Moreover, if

$$\lim_{\nu \to \infty} \Sigma_I y_i c_i = M_B$$

then y(1),y(2),... is termed an optimal feasible sequence.

To relate Program A to Programs P and Q we now introduce a chain of Programs A, B, C, D, and E.

## 5. Infinite parametric programs.

Given an arbitrary set of parameters

$$\lambda_j \quad \text{for} \quad j \in J$$

a linear program C, related to B, is now defined.

Program C. Seek the value function

$$M_C(\lambda) = \lim_{\nu \to \infty} \sup_{y} \Sigma_I y_i c_i$$

for all y subject to the constraints:

$$\nu^{-1} \geq \Sigma_I y_i a_{ij} - \lambda_j \geq \nu^{-1} \qquad j = 0, \pm 1, \ldots, \pm \nu,$$

$$y_i \geq 0 \qquad i \in I.$$

and for $\nu = 1, 2, \ldots$ .

Clearly the following program is a formal dual of C'.

Program D. Seek the value function

$$M_D(\lambda) = \inf \Sigma_J \lambda_j x_j$$

for all x subject to the constraints

$$\Sigma_J a_{ij} x_j \geq c_i \qquad i \in I.$$

Program D is in ordinary form and Program C is in subordinary form. Thus by virtue of the PD theorem the programs C and D are in perfect duality. As a consequence they have the same value functions. It is worth noting with Kallina and Williams [11] that this value function is concave.

## 6. The Lagrange bilinear program

In order to eliminate the constraints

$$x_j \geq 0 \quad \text{for} \quad j \in J_+$$

from Program A the Lagrange function

$$L(x,\lambda) = \Sigma_j \lambda_j x_j$$

is introduced in a bilinear program termed E. This program stems from Program D and is defined as follows.

Program E. Seek the value

$$M_E = \lim_{v \to \infty} \sup_{\lambda} \inf_{x} \Sigma_j \lambda_j x_j$$

for all $x$ subject to the constraints;

$$\Sigma_j a_{ij} x_j \geq c_i \qquad i \in I$$

and for all $\lambda$ subject to the constraints;

$$\inf_x \Sigma_j \lambda_j x_j > -\infty,$$

$$v^{-1} \geq \lambda_j \geq -v^{-1} \qquad j = -1,-2,\ldots,-v,$$

$$v^{-1} \geq \lambda_j - 1 \geq -v^{-1} \qquad j = 0,$$

$$\lambda_j \geq -v^{-1} \qquad j = 1,2,\ldots,v.$$

and for $v = 1,2,\ldots$ .

A sequence $\{\lambda(v)\}$ which satisfies these conditions for $v = 1,2,\ldots$ is said to be a feasible multiplier sequence. If, in addition,

$$M_E = \lim_{v \to \infty} \inf_x L(x,\lambda(v)) \quad (\text{possibly } +\infty)$$

then {A(v)} is said to be an <u>optimal multiplier sequence</u>.
We term the bilinear program E the Lagrange dual of Program A, the following basic lemma reveals a close relationship between Program A and Program E.

<u>Duality Inequality Lemma</u>, <u>Suppose that Programs</u> A <u>and</u> E <u>are both consistent. Then</u>

$$M_A \geq M_E$$

<u>and both these values are finite</u>.

<u>Proof</u>. Let x* be any point which satisfies the constraints of Program A. Let {A} be a feasible multiplier sequence. Then recall that the sum £_A.x. has only a finite number of non-zero
J 3 3
terms so as v —> oo the following limits hold.

lim 5;_J» A.x* = 0,
3 3

lim $A_Q$x£ = $x^*_O$, and

$\overline{\text{lim}}$ Sj A.x! ^ 0 .
+ j j

The last inequality follows because $x^*_j$ £ 0 for j e $J_+$ if x* is feasible for Program A. Thus given a 6 > 0 there exists $v_Q$ such that for v ]≥ $v_O$

L(x*,A) 1 ^ + 5 .

Hence

$\inf_x$ L(x,A) £ L(x*,A) £ $x^*_O$ + 5.

Now allow $\nu \longrightarrow \infty$ so

$$M_E = \lim_{\nu \to \infty} \sup_{\lambda} \inf_{x} L(x,\lambda) \leq x_0^* + \delta.$$

Taking the infimum with respect to $x^*$ gives $M_E \leq M_A + \delta$. Since $\delta$ is arbitrary this proves the stated inequality of the lemma.

We now state the key theorem of this paper.

<u>THEOREM 1</u>. <u>Programs A and E are in perfect duality</u>.

<u>Proof</u>. First suppose that the constraints of Program A are consistent and that the value $M_A$ is finite. Then since A and B are in perfect duality it follows that $M_B = M_A$. Also there is an optimal feasible sequence for Program B, say $y^*(1), y^*(2), \ldots$ . Let $x$ satisfy the constraint

$$\Sigma_J a_{ij} x_j \geq c_i \qquad i \in I.$$

Then since $y_i^* \geq 0$ we see that

$$\Sigma_I y_i^* c_i \leq \Sigma_J \Sigma_I y_i^* a_{ij} x_j.$$

This relation can be written as

$$\text{(i)} \qquad \Sigma_I y_i^* c_i \leq \Sigma_J \lambda_j^* x_j$$

where $\lambda_j^*$ is defined by the equations

$$\lambda_j^* = \Sigma_I y_i^* a_{ij} \qquad j \in J.$$

Thus the sequence $\{y^*(\nu)\}$ leads to the associated sequence $\{\lambda^*(\nu)\}$.

Moreover we see that $|\lambda_j^*(\nu)| \leq \nu^{-1}$ for $J_-$, $|\lambda_0^*(\nu)-1| \leq \nu^{-1}$, and $\lambda_j^*(\nu) \geq -\nu^{-1}$ for $J_+$ but $|j| \leq \nu$.

In the inequality (i) takes the infimum for $x$ satisfying the constraints of Program D. This gives

$$\text{(ii)} \qquad \Sigma_I y_i^* c_i \leq M_D(\lambda^*).$$

Thus $\{\lambda^*(\nu)\}$ is a feasible multiplier sequence for Program E. Since both A and E are consistent the lemma shows that $M_E$ is finite. This proves half of part (a) of the definition of perfect duality.

Next suppose that Program E is consistent and has finite value. Let $\{\lambda'(\mu)\}$ be a feasible multiplier sequence for $\mu = 1,2,\ldots$ . Then for $\lambda = \lambda'(\mu)$ the Program D is consistent and has a finite value. Then since C and D are in perfect duality it follows that Program C is consistent. Thus for each $\mu$ there is a feasible sequence $\{y_i'(\nu,\mu)\}$ for $\nu = 1,2,\ldots$ such that

$$|\Sigma_I y_i'(\nu,\mu) a_{ij} - \lambda'(\mu)| \leq \nu^{-1} \quad \text{for} \quad j = 0,\pm 1,\ldots,\pm \nu.$$

Taking $\mu = \nu$ gives:

$$2/\nu \geq \Sigma_I y_i'(\nu,\nu) a_{ij} \geq -2/\nu \qquad j = -1,-2,\ldots,-\nu$$

$$2/\nu \geq \Sigma_I y_i'(\nu,\nu) a_{ij} - 1 \geq -2/\nu \qquad j = 0,$$

$$\Sigma_I y_i'(\nu,\nu) a_{ij} \geq -2/\nu \qquad j = 1,2,\ldots,\nu.$$

Thus $y''(\nu) = y'(2\nu,2\nu)$ is a feasible sequence for B.

Since Program B has a feasible sequence it also has an optimal feasible sequence, say $y^*(1), y^*(2), \ldots,$ . Conceivably

$$M_B = \lim_{\nu \to \infty} \Sigma_I y_i^* c_i = \infty .$$

Nevertheless we can still define $A^*$, as in the first part, of the proof and again obtain relations (i) and (ii). Then take the limit superior of relation (ii) as $\nu \to \infty$. So

(iii) $$M_B = \lim \Sigma_I y_i^* c_i \leq \overline{\lim}\, M_D(\lambda^*) \leq M_E .$$

Thus $M_B$ is finite and since A and B are in perfect duality it follows that A is consistent. This completes the proof of condition (a) of the definition of perfect duality.

It remains to prove part (b) of the definition. Thus suppose that both A and E are consistent then the lemma states that $M_A$ and $M_B$ are finite and $M_A \geq M_E$. However relation (iii) holds so

$$M_A = M_B \leq M_E .$$

Hence $M_A = M_E$ and the proof is complete.

There are various modifications of the theorem just proved. For example we could make $J_+$ or $J_-$ finite index sets. For application to convex programming in $R^n$ we take both $J_+$ and $J_-$ to be finite index sets. Moreover we take $A_0 = 1$. This is no loss of generality because if $A(\nu)$ is a feasible sequence so also is $A'(\nu) = A(2\nu)/A_0(2\nu)$.

7. <u>Programs defined by a convex set.</u>

In Program A specialize $J$ to be the integers $-n,\ldots,-1,0,1,\ldots,p$ where $n \geq 0$ and $p \geq 1$. Then the first constraint on $x$ stated in Program A is

$$\Sigma_J a_{ij} x_j \geq c_i \qquad i \in I.$$

In other words $x$ is restricted to a region in $R^{n+p+1}$ formed by the intersection of half-spaces. But such a region is a convex set. Moreover any given closed convex set $\Omega$ can be represented as such an intersection of half-spaces. This is accomplished by choosing

$$\Sigma_J a_{ij} x_j = c_i \qquad i \in I$$

to be a suitable set of support planes at the boundary points of $\Omega$.

Let Program A so specialized be termed Program $A^1$. It may be stated in the following equivalent form.

<u>Program</u> $A^1$. <u>Let</u> $\Omega$ <u>be a closed convex set in</u> $R^{n+p+1}$ <u>for</u> $n \geq 0$ <u>and</u> $p \geq 1$. <u>Let</u> $(x_{-n},\ldots,x_o,\ldots,x_p)$ <u>be the coordinates of a point</u> $x$ <u>in</u> $R^{n+p+1}$. <u>Then seek the value</u>

$$M_{A^1} = \inf_x x_o$$

<u>for all</u> $x$ <u>subject to the constraints</u>

$$x \in \Omega,$$

$$x_1 \leq 0, \quad x_2 \leq 0,\ldots,x_p \leq 0.$$

Likewise the Lagrange program E specialized to the same closed convex set has the form.

_Program $E^1$. Seek the value_

$$M_{E^1} = \lim_{\nu \to \infty} \sup_{\lambda} \inf_{x} \Sigma_J \lambda_j x_j$$

_for all_ x _subject to the constraint_

$$x \in \Omega$$

_and for all_ $\lambda$ _subject to the constraints:_

$$\inf_{x} \Sigma_J \lambda_j x_j > -\infty ,$$

$$\nu^{-1} \geq \lambda_j \geq -\nu^{-1} \qquad j = -1,\ldots,-n,$$

$$\lambda_0 = 1$$

$$\lambda_j \geq -\nu^{-1} \qquad j = 1,\ldots,p.$$

_and for_ $\nu = 1,2,\ldots$ .

Then Theorem 1 specializes to give the following result.

COROLLARY 1. _Programs $A^1$ and $E^1$ are in perfect duality._

By making an additional hypothesis on the set $\Omega$ we are led to an important further specialization.

COROLLARY 2. _Let Program $A^2$ be Program $A^1$ when $\Omega$ has the property that if $x \in \Omega$ then $x^+ \in \Omega$ where $x_j^+ \geq x_j$ for $j > 0$ and $x_j^+ = x_j$ otherwise. Then Programs $A^2$ and $E^2$ are in perfect duality where Program $E^2$ is the same as Program $E^1$ except that the last constraint is replaced by_

$$\lambda_j \geq 0 \qquad j = 1,\ldots,p.$$

Proof. Let $x_k^+ = x_k + m$ for one $k > 0$ and $m > 0$. Let $x_j^+ = x_j$ if $j \neq k$. Thus

$$L(x^+,\lambda) = L(x,\lambda) + m\lambda_k.$$

Allowing $m \to +\infty$ shows that if $\lambda_k < 0$ then $\inf L(x^+,\lambda) = -\infty$. This is a violation of the constraints so it is necessary that $\lambda_k \geq 0$ for $k = 1,\ldots,p$. This proves the corollary.

Next consider a more radical modification in which $J$ is the set of integers $0,1,\ldots,p$. Then the set $\Omega$ is in $R^{p+1}$.

COROLLARY 3. *Let Programs $A^3$ and $E^3$ be Programs $A^2$ and $E^2$ when $n = 0$. Then Programs $A^3$ and $E^3$ are in perfect duality.*

Proof. This is a direct consequence of Corollary 2. Note that Program $E^3$ can be written in the following simplified form.

Program $E^3$. *Seek the value*

$$M_E = \sup_{\lambda} \inf_{x} (x_0+\lambda_1 x_1 +\ldots+ \lambda_p x_p)$$

*for all* $x$ *subject to the constraint*

$$x \in \Omega,$$

*and for all* $\lambda$ *subject to the constraints*

$$\lambda_j \geq 0 \qquad j = 1,\ldots,p.$$

8. The perfect duality of P and Q.

To relate Programs A and P consider the convex function $g_0(z),\dots,g_p(z)$ for $z \in R^n$. These functions were introduced in Program P. Then define functions $G_0(x),\dots,G_p(x)$ for $x \in R^{n+p+1}$ by the relations

$$G_j(x) = g_j(z) - x_j \qquad j = 0,\dots,p,$$

$$x_{-j} = z_j \qquad j = 1,\dots,n.$$

Clearly the functions $G_j$ are also convex. A set $\Omega$ in $R^{n+p+1}$ is defined by the inequalities

$$G_0(x) \leq 0, \quad G_1(x) \leq 0,\dots,G_p(x) \leq 0.$$

As is well-known this implies that $\Omega$ is a closed convex set. Moreover if $x \in \Omega$ then $x^+ \in \Omega$ if $x_j^+ \geq x_j$ for $j > 0$ and $x_j^+ = x_j$ otherwise. Thus the set $\Omega$ satisfies the conditions of Corollary 2.

Consider Program $A^2$ for the set $\Omega$ just defined. If $x \in \Omega$ we have $G_j(x) \leq 0$ and this is equivalent to $g_j(z) \leq x_j$. Thus the constraint $x_j \leq 0$ implies $g_j(z) \leq 0$. No further constraints are imposed on z. This shows that if $A^2$ is consistent so also is P.

Conversely if P is given to be consistent let $x_j = g_j(z)$ for $j = 0,1,\dots,p$ and we see that $A^1$ is consistent. Clearly $\inf g_0(z) = \inf x_0$ so $M_P = M_A$. Thus Programs P and $A^2$ are equivalent.

COROLLARY 4. Programs P and Q are in perfect duality.

Proof. Consider Programs Q and $E^2$. Then if $x \in Q$ and A satisfies the constraints $A_j \geq 0$ for $j = 1,\ldots,p$ we see

$$g_0(z) + \sum_{1}^{p} A_j g_j(z) + \sum_{1}^{n} A_{-j} z_j \geq \sum_{-n}^{p} A_j x_j .$$

This will be an equality if for a given $z$ we put $x_j = g_j(z)$ for $j = 0,\ldots,p$. It follows that inf $L(z,A)$ in $Q$ is the same as inf $L(x,A)$ in $E^2$. Consequently $M_Q = M_{E^2}$ and it is seen that $Q$ and $E^2$ are equivalent. But it was seen above that P and $A^2$ are equivalent so the proof is complete.

9. <u>The perfect duality of</u> P* <u>and</u> Q*.

Consider Program P*. Let $(x_0, x_1, \ldots, x_p)$ be the coordinates of a point $x$ in $R^{p+1}$. Let $\Omega^o$ be the set in $R^{p+1}$ such that $x \in \Omega^o$ if

$$g_j(z) \leq x_j \qquad j = 0, 1, \ldots, p$$

for some $z$ in $R^n$. If $x'$ and $x''$ are both in $\Omega^o$ then

$$ax_j' + b_j x_j'' \geq ag_j(z') + bg_j(z'') \geq g_j(az'+bz'')$$

provided $a \geq 0$, $b \geq 0$ and $a + b = 1$. Thus the set $\Omega^o$ is convex.

The set $\Omega^o$ may not be a closed set so let $\Omega$ be its closure. Then $\Omega$ is a closed convex set. It is clear from the definition that if $x \in \Omega$ then $x^+ \in \Omega$ provided $x_j^+ \geq x_j$ for $j = 0, \ldots, p$. Thus Corollary 3 applies to $\Omega$ and we first compare Programs Q* and $E^3$.

For a given $\lambda$ the function $L(x,\lambda)$ is continuous in $x$ so $\inf L(x,\lambda)$ is the same whether $x$ is allowed to vary over $\Omega^o$ or over $\Omega$. But for $x \in \Omega^o$ we have $g_j(z) \leq x_j$ for some $z$ so

$$g_0(z) + \sum_{1}^{p} \lambda_j g_j(z) \leq x_0 + \sum_{1}^{p} \lambda_j x_j,$$

For a given $z$ this can be made into an equality by taking $x_j = g_j(z)$. This shows that

$$(1) \qquad \inf_z L_Q(z,\lambda) = \inf_x L_E(x,\lambda).$$

Thus Program Q* is consistent if and only if Program $E^3$ is consistent. Moreover it follows from (1) that 14 = M o. Thus Programs Q* and $E^3$ are equivalent.

<u>COROLLARY 5^</u> <u>Programs P* and Q* are in perfect duality</u>.

<u>Proof</u>. Suppose Program P* is consistent and has a finite value $M_p$^. Then it is clear that the point (Mp^+v~ 1,v~ 1, ...,v~ 1) is in flP for an arbitrary positive integer v. Forming the closure as v —> oo shows that the point (tfL^ , 0, . . . ,0) is in £2. Thus M 3 <^ M^*« On the other hand if (x 0, ...,0) is in *SI* then (x0+v~, \T ,.•.9 v ) is in 0 for an arbitrary positive integer v so x0 ^ Mp** Hence M 3 = M^^..

Conversely suppose Program A is consistent and has a finite value M ^. Then it is clear that (M ^,0, • • • ,0) is in *Q,* and that again M A = M^*.

This shows that Program P* is consistent and has a finite value if and only if Program $A^3$ is consistent and has a finite value. Moreover M 3 = M^*. Previously we had seen that Programs Q* and $E^3$ are equivalent. Thus Corollary 3 now proves that P* and Q* satisfy part (a) of the definition of perfect duality. Moreover M^^. = Mn*#

Next consider part (b) of the PD definition. We are given that P* and Q* are both consistent. If P* has a finite value then (b) follows from the above argument. If p* does not have a finite value then we may have $M_p$^. = -oo but not M^ = +oo by the definition of consistency* But if M^ = -oo then it is clear that (0,0,..,0) is in *Q,* so Program $A^3$ is consistent. But by the equivalence of Q* and $E^3$ we know $E^3$ is consistent. Thus Corollary 3 states that $A^3$ and $E^3$ have equal finite values. So P* must also have a finite value. QED.

The proofs in this paper all rest on the general perfect duality theorem of linear programming given in reference [1]. In that paper arguments were based on abstract topological considerations. However the special case of the PD theorem needed for this paper can be proved directly by elementary limit theorems.

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