A MINIMAL-LENGTH PROBLEM IN R^n

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Abstract

Let $\|u\|_p$ $(p \geq 1)$ denote the Hölder norm $[\sum_{k=1}^{n} |u_k|^p]^{\frac{1}{p}}$ of the vector $u = (u_1, \ldots, u_n)$ in the real n-dimensional space $\mathbb{R}^n$, and let C be a differentiable curve in $\mathbb{R}^n$ with the parametric representation $u = u(t)$, $t_1 \leq t \leq t_2$, which passes through each of the coordinate planes in $\mathbb{R}^n$. If

$$L_p(C) = \int_C \frac{\|du\|_p}{\|u\|_p} ,$$

the exact lower bounds of $L_p(C)$ are determined in the following two cases:

(a) C does not pass through the origin;

(b) C is restricted to the unit sphere $\|u\|_p = 1$. In the euclidean case $(p=2)$, both of these bounds are known to have the value $\frac{\pi}{2}$[1,2,4,8].

In addition to their geometric interest, these results have various applications in the theory of differential equations.

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Let $P_k = (p_{k1}, p_{k2}, \ldots, p_{kn})$, $k = 1, \ldots, n$ be n points in the real n-dimensional space $\mathfrak{R}^n$ which are subject to the restriction $p_{kk} = 0$, $k = 1, \ldots, n$ (i.e., geometrically speaking, let there be such a point on each of the coordinate planes), and let C be a differentiable curve which passes through all these points but not through the origin. If $u = u(t)$ is a parametric representation of C and if $\|v\|$ denotes the Euclidean norm of the vector v, then

$$\int_C \frac{\|du\|}{\|u\|} \geq \frac{\pi}{2}, \tag{1}$$

where the constant $\frac{\pi}{2}$ is the best possible. This inequality first appeared in connection with problems concerning the differential equation $u' = Au$ (where A is a continuous $n \times n$-matrix) [2,4] (for related results in infinite-dimensional spaces see [3,6]). A purely geometric proof was subsequently given by B. Schwarz [8], and the argument has been further simplified in a recent paper by L. M. Kelly and J. Zaks [1]. Actually, in the two latter papers, the result is formulated for curves C on the unit sphere $\|u\| = 1$, in which case (1) reduces to

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$$(2) \qquad \int_C \|du\| \ge \frac{\pi}{2}, \qquad (\|u\|=1).$$

It is not difficult to show that, in spite of the seemingly greater generality of (1), the estimates (1) and (2) are equivalent, indeed, if we set $\|u\| = \rho$ and v is the unit vector defined by $u = \rho v$, we have

$$\left(\frac{\|du\|}{\|u\|}\right)^2 = \left\|dv + \frac{d\rho}{\rho}v\right\|^2 = \|dv\|^2 + \left(\frac{d\rho}{\rho}\right)^2 \ge \|dv\|^2$$

(where we have made use of the fact that $2(v,dv) = d(\|v\|^2) = 0$), and this shows that (2) implies (1).

In the present paper we shall consider the analogous length problems which are obtained if the Euclidean norm $\|u\|$ is replaced by the Hölder norms $\|u\|_p = [\sum_{i=1}^{n} |u_i|^p]^{1/p}$ $(p \ge 1)$ of the vector $u = (u_1, \ldots, u_n)$. We shall establish the following three results.

THEOREM I. *Let* C *be a differentiable curve in* R^n *which has a point in common with each of the coordinate planes but does not contain the origin. Then*

$$(3) \qquad \int_C \frac{\|du\|_p}{\|u\|_p} \ge c_p,$$

where

$$(4) \qquad c_p = \int_0^\infty \frac{ds}{(1+s^p)^{\frac{1}{p}}(1+s^q)^{\frac{1}{q}}}, \qquad \frac{1}{p} + \frac{1}{q} = 1.$$

The constant c_p *is the best possible,* THEOREM II. Let C be a differentiable curve on the "unit sphere" $\|u\|_p = 1$ which has a point in common with each of the coordinate planes in $\Re^n$. Then

(5) $$\int_C \|du\|_p \geq \gamma_p,$$

where

(6) $$\gamma_p = \frac{1}{p}\int_0^1 [s^{1-p} + (1-s)^{1-p}]^{\frac{1}{p}} ds.$$

The constant γ_p is the best possible.

THEOREM III. If C has the properties stated in Theorem II, except that it is restricted to the surface $\|u\|_q = 1$ rather than to $\|u\|_p = 1$ $(p^{-1}+q^{-1}=1)$, and if $1 \leq p \leq 2$, then

(7) $$\int_C \|du\|_p \geq \delta_p,$$

where

(8) $$\delta_p = \int_0^\infty \frac{ds}{(1+s^q)^{\frac{2}{q}}} = \frac{\Gamma^2(\frac{1}{q})}{q\Gamma(\frac{2}{q})}$$

Again, the constant δ_p is the best possible.

If $p = q = 2$, the assertions of Theorems II and III reduce to (2), while the assertion of Theorem I reduces to (1).

A result essentially equivalent to Theorem I can be found in [5]. However, the rather elaborate proof of this result in [5] makes extensive use of the properties of the differential equation $x' = Ax$, and it would appear desirable to have a proof which is relatively short and, at the same time, does not obscure the essentially geometric character of the result.

The proofs of Theorems I - II are based on the following lemma which shows that, in all cases, it is sufficient to consider the space $\mathbb{R}^2$.

LEMMA I. *Let C_0 be a differentiable arc in $\mathbb{R}^n$ which has a point in common with each of the coordinate planes, and denote by C the smallest subarc of C_0 which has these properties. If $u = u(t)$, $t \in [t_1, t_2]$, is a parametric representation of C, then there exists a continuous, piecewise differentiable arc C' in $\mathbb{R}^2$ with a parametric representation $v = v(t) = [v_1(t), v_2(t)]$, $t \in [t_1, t_2]$ such that $v_1(t_1) = v_2(t_2) = 0$ and*

$$\|v(t)\|_p = \|u(t)\|_p, \quad \|v'(t)\|_p \leq \|u'(t)\|_p, \tag{9}$$

wherever $v'(t)$ exists. If $1 \leq p \leq 2$, there also exists a piecewise differentiable arc C'' in $\mathbb{R}^2$ with a representation $v = v(t) = [v_1(t), v_2(t)]$, $t \in t[t_1, t_2]$ for which $v_1(t_1) = v_2(t_2) = 0$ and

$$\|v(t)\|_q = \|u(t)\|_q, \quad \|v'(t)\|_p \leq \|u'(t)\|_p, \tag{10}$$

$(p^{-1} + q^{-1} = 1)$, *wherever $v'(t)$ exists.*

PROOF OF LEMMA I. By assumption, each of the components $u_k(t)$ $(k=1,\ldots,n)$ of $u(t)$ vanishes at some point of $[t_1, t_2]$. Two of the components vanish at t_1 and t_2, respectively. By suitable re-numbering, we may assume that $u_1(t_1) = u_2(t_2) = 0$. We now show that there exists a continuous, piecewise differentiable vector function which, in addition to (9), has the following properties: $v_1(t_1) = v_2(t_2) = 0$, $v_3(t) \equiv 0$, $v_k(t) = u_k(t)$ for $k = 3,\ldots,n$. To construct $v(t)$ we observe that, by assumption, there exists a $t^* \in [t_1,t_2]$ such that $u_3(t^*) = 0$. Accordingly, the functions $v_1(t), v_2(t)$ defined by

$$v_1 = u_1, \ t \in [t_1,t^*], \ v_1 = (|u_1|^p+|u_3|^p)^{\frac{1}{p}} \operatorname{sgn} u_1(t^*), \ t \in [t^*,t_2]$$

(11)

$$v_2 = (|u_2|^p+|u_3|^p)^{\frac{1}{p}} \operatorname{sgn} u_1(t^*), \ t \in [t_1,t^*], \ v_2 = u_2, \ t \in [t^*,t_2]$$

(where $\operatorname{sgn} u_1(t^*)$ can be taken to be either 1 or -1 if $u_1(t^*) = 0$) is continuous and piecewise differentiable on $[t_1,t_2]$, and the vector $v = (v_1,v_2,0,u_3,\ldots,u_n)$ clearly satisfies $\|v\|_p = \|u\|_p$ for all $t \in [t_1,t_2]$. To show that $v'(t)$ is subject to the inequality (9), it is sufficient to verify that

$$(12) \qquad |v_1'|^p + |v_2'|^p \leq |u_1'|^p + |u_2'|^p + |u_3'|^p$$

wherever v_1' and v_2' exist. By (11), this will follow from

$$(13) \qquad |v_1'|^p \leq |u_1'|^p + |u_3'|^p, \qquad t \in [t^*,t_2],$$

and

$$(14) \qquad |v_2'|^p \leq |u_2'|^p + |u_3'|^p, \qquad t \in [t_1,t^*].$$

Since, by (11),

$$(15) \qquad |v_1'||v_1|^{p-1} \leq |u_1'||u_1|^{p-1} + |u_3'||u_3|^{p-1}, \quad t \in [t^*,t_2]$$

and

$$(16) \qquad |v_2'||v_2|^{p-1} \leq |u_2'||u_2|^{p-1} + |u_3'||u_3|^{p-1}, \quad t \in [t_1,t^*],$$

it follows from the Hölder inequality (and the fact that $(p-1)q = p$ if $p^{-1} + q^{-1} = 1$) that

$$|v_1'||v_1|^{\frac{p}{q}} \leq [|u_1'|^p + |u_3'|^p]^{\frac{1}{p}}[|u_1|^p + |u_3|^p]^{\frac{1}{q}}, \quad t \in [t^*, t_2],$$

and

$$|v_2'||v_2|^{\frac{p}{q}} \leq [|u_2'|^p + |u_3'|^p]^{\frac{1}{p}}[|u_2|^p + |u_3|^p]^{\frac{1}{q}}, \quad t \in [t^*, t_2].$$

Since we have $|v_1|^p = |u_1|^p + |u_3|^p$ and $|v_2|^p = |u_2|^p + |u_3|^p$ in $[t^*, t_2]$ and $[t_1, t^*]$, respectively, this proves (13) and (14). As pointed out above, this implies (12) and thus establishes the inequality (9), where $v(t)$ is a vector such that $v_3(t) \equiv 0$ throughout $[t_1, t_2]$.

The process just described "removed" the component $u_3(t)$ of the original vector $u(t)$. If we apply this procedure successively to $u_4(t), \ldots, u_n(t)$, we ultimately obtain a two-dimensional vector function $v(t) = [v_1(t), v_2(t), 0, \ldots, 0] = [v_1(t), v_2(t)]$ which satisfies all the conditions imposed on the parametric representation of the arc C' in the statement of Lemma I.

To prove the existence of the arc C'', we again apply the Hölder inequality to (15) and (16) (reversing, however, the roles of p and q). This leads to

$$|v_1'||v_1|^{p-1} \leq [|u_1'|^q + |u_3'|^q]^{\frac{1}{q}}[|u_1|^{(p-1)p} + |u_3|^{(p-1)p}]^{\frac{1}{p}}, \quad t \in [t^*, t_2]$$

and

$$|v_2'||v_2|^{p-1} \leq [|u_2'|^q + |u_3'|^q]^{\frac{1}{q}}[|u_2|^{(p-1)p} + |u_3|^{(p-1)p}]^{\frac{1}{p}}. \quad t \in [t_1, t^*],$$

respectively. Applying the inequality $a^s + b^s \geq (a+b)^s$ $(a, b \geq 0,\ s \geq 1)$ to the cases $a = |u_1|^p$, $b = |u_3|^p$, $s = p - 1$ and $a = |u_2|^p$, $b = |u_3|^p$, $s = p - 1$, respectively, and using (11), we obtain the inequalities

$$|v_j|^q \leq |u_j|^q + |u_2|^q, \qquad t \in [t^*, t_2],$$

and

$$|v_1|^q \leq |u_1|^q + |u_2|^q, \qquad t \in [t_1, t^*],$$

analogous to (13) and (14), which are valid for $p \geq 2$ (or, equivalently, for $1 \leq q \leq 2$). The rest of the argument is now the same as before except that, at the end, the roles of p and q have to be reversed. This establishes (10) and thus concludes the proof of Lemma I.

We now turn to the proof of Theorem I. We may assume that two of the endpoints of C coincide with points at which two of the components vanish, since, if necessary, we can replace C by a subarc C_1 for which this is true; evidently, (3) is true for C if it holds for C_1. Applying now Lemma I, we find that (3) will be a consequence of the inequality

(17)
$$\int_C \frac{\|dv\|}{\|v\|} > v$$

where $v(t) = [v_1(t), v_2(t)]$ is continuous and piecewise differentiable on an interval $[t_1, t_2]$ and such that $v_1(t_1) = v_2(t_2) = 0$. We note that v_1 and v_2 cannot vanish at the same point t since, as is apparent from the proof of Lemma I, this would imply that C passes through the origin. We also may assume that v_1 and v_2 do not vanish on (t_1, t_2), since otherwise we might again replace C by a subarc. Accordingly, the continuous function $R(t) = |v_1(t)|\,[|v_2(t)|]^{-1}$ will take the value 0 at t_1 and tend to ∞ as $t \to t_2$. At a point at which both v_1' and v_2' exist, we have

$$|R'| = \left|\frac{|v_2|\,|v_1|' - |v_1|\,|v_2|'}{v_2^2}\right| \leq \frac{|v_2|\,|v_1'| + |v_1|\,|v_2'|}{v_2^2}$$

$$\leq \frac{\|v'\|_p \|v\|_q}{v_2^2} = \frac{\|v'\|_p}{\|v\|_p} \cdot \frac{\|v\|_p \|v\|_q}{v_2^2} .$$

Since

$$\frac{\|v\|_p \|v\|_q}{|v_2|^2} = (1+R^p)^{\frac{1}{p}}(1+R^q)^{\frac{1}{q}},$$

this implies

$$\frac{|R'|}{(1+R^p)^{\frac{1}{p}}(1+R^q)^{\frac{1}{q}}} \leq \frac{\|v'\|_p}{\|v\|_p} .$$

If t varies from t_1 to t_2, R varies from 0 to ∞. Thus, an integration leads to (17), where c_p is the constant (4). This proves the inequality (3).

To show that the constant c_p in (3) is the best possible, we consider a two-dimensional vector $u = [u_1(t), u_2(t)]$ whose components are obtained from the system of differential equations

$$u_1' = \rho u_2^{\,q-1}$$

(18)

$$u_2' = -\rho u_1^{\,q-1}$$

with the boundary conditions $u_1(t_1) = u_2(t_2) = 0$, where ρ is an arbitrary positive continuous function. From (18), we have

$$u_1^{q-1}u_1' + u_2^{q-1}u_2' = 0$$

and thus

$$u_1^q + u_2^q = A^q, \tag{19}$$

where A is a positive constant (it is clear from (18) and the boundary conditions that both v_1 and v_2 are positive in (t_1,t_2)). Again from (18),

$$(|u_1'|^p+|u_2'|^p)^{\frac{1}{p}} = \rho(u_1^q+u_2^q)^{\frac{1}{p}}$$

and thus by (19),

$$\|u'\|_p = \rho A^{\frac{q}{p}}. \tag{20}$$

On the other hand, setting $R = u_1(u_2)^{-1}$ we have

$$R' = \frac{u_2u_1'-u_1u_2'}{u_2^2} = \frac{\rho(u_1^q+u_2^q)}{u_2^2} = \frac{\rho A^q}{u_2^2}.$$

Hence, by (20),

$$\frac{\|u'\|_p}{\|u\|_p} = \frac{R'u_2^2A^{\frac{q}{p}}}{A^q\|u\|_p} = \frac{R'u_2^2}{A\|u\|_p} = \frac{R'u_2^2}{\|u\|_p\|u\|_q} = \frac{R'}{(1+R^p)^{\frac{1}{p}}(1+R^q)^{\frac{1}{q}}}.$$

Since $R(t_1) = 0$, $R(t_2) = \infty$, this leads to

$$\int_C \frac{\|du\|_p}{\|u\|_p} = c_p,$$

where c_p is the constant (4), and confirms that inequality (3) is sharp. This completes the proof of Theorem I.

Turning now to Theorem II, we note that, because of Lemma I, it is again sufficient to consider the two-dimensional case (it may be observed that, in accordance with (9), a vector u such that $\|u\|_p = 1$ is transformed into a two-dimensional vector v such that $\|v\|_p = 1$). Our problem thus reduces to finding the minimum of

$$\int_{t_1}^{t_2} [|v_1'|^p + |v_2'|^p]^{\frac{1}{p}} |dt| \tag{21}$$

under the side conditions $v_1^p + v_2^p = 1$ (v_1, v_2 may obviously be taken nonnegative without loss of generality) and $v_1(t_1) = v_2(t_2) = 0$. If we eliminate v_2 and set $v_1^p = s$, the integral (21) transforms into

$$\frac{1}{p}\int_0^1 [s^{1-p} + (1-s)^{1-p}]^{\frac{1}{p}} |ds|.$$

Evidently, this expression attains its minimum if s grows monotonically from 0 to 1, and the value of the minimum is the constant (6). This proves Theorem II. It is clear from our argument that, e.g., there will be equality in (5) for the vector $(\sin^{\frac{2}{p}} t, \cos^{\frac{2}{p}} t, 0, \ldots, 0)$ and the interval $[0, \frac{\pi}{2}]$.

Turning now to Theorem III, we note that, by Lemma I, it is sufficient to consider the arc C'' described in the statement of the Lemma. As in the proof of Theorem I, we introduce the function $R(t) = |v_1(t)|[|v_2(t)|]^{-1}$ (where $v = v(t) = [v_1(t), v_2(t)]$

is the parametric representation of C"). Since, by (10), the assumption $\|u\|_q = 1$ implies $\|v\|_q = 1$, we have

$$|R'(t)| \leq \frac{|v_2||v_1'|+|v_1||v_2'|}{v_2^2} \leq \frac{\|v'\|_p\|v\|_q}{v_2^2} = \frac{\|v'\|_p(\|v\|_q)^2}{v_2^2},$$

and thus, because of

$$\frac{\|v\|_q}{|v_2|} = (1+R^q)^{\frac{2}{q}}$$

$$\frac{|R'|}{(1+R^q)^{\frac{2}{q}}} \leq \|v'\|_p.$$

Integrating, and noting that $R(t_1) = 0$, $R(t_2) = \infty$, we obtain (7), with the value (8) of the constant δ_p.

The fact that this constant is the best possible can again be shown by means of the vector function $u = [u_1(t), u_2(t)]$ whose components are determined from the differential equations (18) with the boundary conditions $u_1(t_1) = u_2(t_2) = 0$. Elementary considerations show that, for $q \neq 2$, multiplication of the function ρ by a suitable positive constant will give to the constant A in (19) the value 1, and we will thus have $\|u\|_q = 1$, as required. Using (20) and the ensuing computation, we obtain

$$\|u'\|_p = R'u_2^2 = \frac{R'u_2^2}{(\|u\|_q)^2} = \frac{R'}{(1+R^q)^{\frac{2}{q}}},$$

where $R = u_1(u_2)^{-1}$. Since $R(t_1) = 0$, $R(t_2) = \infty$, we thus have

$$\int_C \|du\|_p = \delta_p$$

where δ_p is the constant (8). This completes the proof of Theorem III.

We conclude with an example illustrating the application of geometric results of the type considered in this paper to problems in the theory of differential equations.

The differential equation

$$x' = Ax, \tag{22}$$

where x is an n-vector and A a continuous $n\times n$ matrix, is said to be nonoscillatory on an interval I if every nontrivial solution vector x has at least one component which does not vanish on I. In order to obtain a condition which guarantees nonoscillation on I, we suppose that, on the contrary, there exists a nontrivial solution vector x such that each of its components vanishes at some point of I. In this case, the curve $C = \{x = x(t),\ t \in I\}$ will satisfy the hypotheses of Theorem I, and we thus have

$$\int_I \frac{\|x'(t)\|_p}{\|x(t)\|_p}\, dt \geq c_p,$$

where c_p is the constant (4). Since, by (22),

$$\|x'\|_p \leq \|A\|_p \|x\|_p,$$

where $\|A\|_p$ is the matrix norm induced by the vector norm $\|x\|_p$, it follows that the existence of an oscillatory solution implies the inequality

$$\int_I \|A\|_p dt \geq c_p.$$

This leads to the following nonoscillation criterion (cf. [5] and, for the case p = oo , [7]).

If

$$J \wedge Npdt < c_p, \qquad p \wedge 1,$$

where c_p ±s the constant (4), then the equation (22) is non-oscillatory on I.

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