ON ASYMPTOTICALLY BALANCED INFINITE GAMES

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ABSTRACT

In this paper an infinite player game is treated by an infinite linear program and its dual. When the value of one of the programs is finite, then the result is a class of games called asymptotically balanced games. An equivalent characterization of these games is proved. Some necessary and sufficient conditions for non-emptiness of the core are proved along with some results on the relationship between the set of undominated payoffs and the core.

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1. Introduction.

Finite minimal balanced collections have played an important role in combinatorial analysis and in n-person game theory, see Shapley [15]. They are intimately related to various solution concepts such as the core and the nucleolus, as developed in Schmeidler [14], Charnes-Kortanek [5], Kohlberg [11], and Billera [4].

In the finite context, the set of integers $N = \{1,2,...,n\}$ is given and one considers positive weightings of the characteristic functions of subsets of N which sum to the characteristic function of the set N itself. Thus, when a payoff function $v : 2^N \to R$ is imposed, one may then relate such possible weighting schemes, called "balanced collections" to solution concepts in n-person game theory.

In this paper, we explore the analogous situation when the number of players N becomes the set of positive integers $1,2,...$. Thus, as in the finite case, let the characteristic function e^S, for any $S \subseteq N$, be given by

$$(e_S)_k = \begin{cases} 1 & \text{if } k \in S \\ 0 & \text{if } k \notin S, \end{cases} \qquad (1)$$

where the notation $(\ )_k$ means the k^{th} component in the vector within the parentheses. We shall also denote the payoff value on $S \subseteq N$ by v^S, which we shall take as given.

In the case of finite N, attention is focused on a constraining set of equations and inequalities which define balanced and minimal balanced collections:

$$\sum_{S \subseteq N} \eta_S e_S = e_N \quad \text{and} \quad \eta_S \geq 0 \quad \text{for all} \quad S \subseteq N \tag{2}$$

Now when N is infinite, (2) can be investigated in terms of various locally convex topologies which are related to perhaps game theoretic interpretations. In this paper we view the possibilities of coalition formation by <u>any</u> finite subset of players in N. Thus, within a total finite communication time members of any finite set of players may communicate to each other and thus decide to form or not to form. However, there are infinitely many players from which coalitions may form.

2. <u>The Structure of the Infinite Programs for the Infinite Player Set</u>.

In accordance with the above interpretation, we assume,

$$v_S = 0 \text{ if } S \neq N \text{ and } S \text{ contains infinitely many players, and } v_S \geq 0 \text{ for all } S \subseteq N. \tag{3}$$

This leads to the following underlying topological spaces for relations (2). Let

$$E = \sum_{\substack{S \subset N \\ |S| < \infty}} \oplus R_S, \tag{4}$$

the direct sum of copies of the real field endowed with the convex core topology.

Let

$$F = \prod_{i \in N} R_i . \tag{5}$$

the direct product of copies of the real field. In this setting the game $v = (v(S) : S \subseteq N, |S| < \infty)$ is actually a member of the conjugate dual of E denoted by $E^{\#}$,[1]

$$E^{\#} = \prod_{\substack{S \subseteq N \\ |S| < \infty}} R_S \tag{6}$$

endowed with the topology of pointwise convergence. Analogously the conjugate dual of F is given by

$$F^{\#} = \sum_{i \in N} \oplus R_i . \tag{7}$$

Thus, for example, the pairing between E and $E^{\#}$ is defined by

$$(\eta, \gamma) = \sum_{\substack{S \subseteq N \\ |S| < \infty}} \eta_S \gamma_S \tag{8}$$

for any $\eta \in E$ and $\gamma \in E^{\#}$. See Ben-Israel-Charnes-Kortanek [1].

Thus, the spaces E and F are finite sequence spaces and with them we obtain the special dual infinite linear programs studied recently in Duffin [9], Needed also are the designations of polar cones and a linear transformation which we now develop.

Introduce closed convex cones as follows:

$$C = \{\eta \in E : \eta_S \geq 0\}, \tag{9}$$

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[1] Since E has the convex core topology all linear functionals on E are continuous and $E^{\#}$ coincides with the algebraic dual.

so that $C^{\#} = \{y \in E^{\#} : y_S \geq 0$ for all $S \subseteq N, |S| < \infty\}$. Define a continuous linear transformation A on E by:

$$A(\eta) = A(\sum_S \eta_S) = \sum_S \eta_S e_S \in F. \tag{10}$$

It then follows that $A^T : F^{\#} \longrightarrow E^{\#}$ is given by:

$$A^T x = \{(x, e_S) : S \subseteq N, |S| < \infty\} \in E^{\#}. \tag{11}$$

Therefore, the formal infinite linear programs:

(I)

$$\sup_{\eta \in E}(v, \eta)$$

subject to $A(\eta) = e_N$, $\eta \geq 0$

(II)

$$\inf_{x \in F^{\#}}(x, e_N) \tag{12}$$

subject to $A^T x - v \in C^{\#}$

become

PROGRAM I

$$\sup_{\eta \in E} \sum_S \eta_S v_S$$

$$\text{subject to} \quad \sum_S \eta_S e_S = e_N \tag{13}$$

$$\eta_S \geq 0$$

and

PROGRAM II

$$\inf_{x \in F^{\#}}(x, e_N) \tag{14}$$

subject to $(x, e_S) \geq v_S$, for all $S \subseteq N$, $|S| < \infty$.

Thus, the equations and inequalities of (2) become part of a pair of infinite linear programs in duality.

In Duffin [8], it was shown that a "perfect duality" can be insured between infinite linear programs such as Programs I and II, if the constraints of I are relaxed to accomodate inconsistent but asymptotically consistent solutions. The relevant topology for Program I is the pointwise convergence topology. This means that a sequence,

$$\{\eta(\nu)\}^{\infty} \quad \text{in} \quad E \tag{15}$$

is consistent for Program I if $\lim_{k\to\infty} \sum_S \eta_S(k) e_S = e_N$ pointwise, i.e., for each $i \in N$,

$$\lim_{k\to\infty} \sum_S \eta_S(k)\,(e_S)_i = (e_N)_i . \tag{16}$$

The <u>value</u> associated to a consistent solution (15) is

$$\lim \sum \eta_Q(k) v_Q , \tag{17}$$

which may be finite or infinite. With these definitions, then, Duffin's Theorem 1 [9] states that one of the programs (I) or (II) is consistent and has finite value if and only if the other is consistent and has finite value, in which case the values are the same. Note that for problem II, we are using the <u>ordinary</u> notion of consistency, i.e., there exists $x \in F^{\#}$ such that $(x, e_S) \geq v_S$ for all $S \subseteq N$, $|S| < \infty$. Duffin [8] has called the sequences in (15) consistent solutions to a <u>subordinary</u> form of Program I.

We shall also denote the value of Program I by $M(e_N)$, which is defined to be the supremum of all the values stemming from consistent solutions, (15). We simply denote the value of

Program II by i(II). The value of Programs I or II are not necessarily finite.

3. Asymptotically Balanced Collections.

When N is finite, consistent solutions to (2) are called balanced collections and extreme points of the closed convex set determined by (2) are called minimal balanced collections.

When N is infinite however, the relations (2), or as they appear in Program I in (13), do not admit consistent solutions because (a) $|S| < \infty$ and (b) only finitely many $\eta_S > 0$. Our objective is to use the perfect duality of infinite linear programming to introduce asymptotically balanced collections.

DEFINITION. The game $\{v_S : S \subseteq N\}$ is said to be asymptotically balanced, AB, if a subordinary form of Program I is consistent and $M(e_N) < \infty$. This simply means that Program I is subconsistent, Duffin [8],and its subvalue is finite. The possibility of $M(e_N) = -\infty$ is excluded because of assumption (1), i.e. $v_S \geq 0$ for all $S \subseteq N$.

Henceforth, the word "consistent" when applied to Program I shall be taken to mean "subconsistent", as in (15), since from programming theory we shall be dealing only with the perfect duality between Programs I and II .

DEFINITION. A finite coalition $T \subseteq N$ is said to be a carrier of the game v if:

$$S \subseteq N,\ v(S) > 0 \text{ implies } S \cap T \neq \emptyset. \tag{17}$$

We now show that any consistent solution, x, to Program II determines a carrier

$$T_x = \{i \in N : x_i > 0\} \dots \quad . \tag{18}$$

In fact, carriers yield the following characterization of AB.

<u>THEOREM 1</u>. v is AB if and only if there exist carrier coalitions, and for each carrier T, $\sup\{v_S : i \in S\} < \infty$ for all $i \in T$.

<u>Proof</u>. Assume v is AB. By perfect duality this means that Program II has a consistent solution, say x. Set $T = T_x$. Let $S \subseteq N$ with $v_S > 0$. If to the contrary $S \cap T = \emptyset$, then $0 = (x, e_S) \geq v_S > 0$, which is a contradiction. Hence $S \cap T \neq \emptyset$. Furthermore $(x, e_T) \geq (x, e_{T \cap S}) = (x, e_S) \geq v_S$, since $x_i \geq 0$ for all i. Since (x, e_T) is independent of S, this shows that $\sup\{v_S : v_S > 0,\ S \subseteq N\} < \infty$ and hence $\sup\{v_S : i \in S\} < \infty$.

Assume the conditions of the theorem hold. Let T be any carrier and define

$$x_i = \begin{cases} \sup\{v_S : i \in S\}, & i \in T \\ 0 & i \notin T \end{cases} \quad . \tag{19}$$

It follows that x is consistent for Program II. For if $v_S > 0$, then there exists $i \in T \cap S$ since T is a carrier. Hence $x_i \geq v_S$ and again

$$(x, e_S) = (x, e_{T \cap S}) \geq v_S. \tag{20}$$

If $v_S = 0$, then $(x,e_S) \geq 0 = v_S$, and x is consistent. Since $x \geq 0$ for any consistent solution and $(e_N)_i = 1$ for all $i \in N$, it follows that Program II has finite value. Hence by perfect duality Program I also has finite value and therefore by definition, v is AB. Q.E.D.

4. <u>The</u> M (<u>majorant</u>) <u>Operator</u>.

For $|S| \geq 2$ consider the following dual programs.

<u>PROGRAM</u> I_S

$$\sup \sum_Q \eta_Q v_Q$$

$$\text{subject to} \quad \sum_Q \eta_Q e_Q = e_S \tag{21}$$

$$\eta_Q \geq 0, \text{ where } Q \subseteq S,$$

and $|Q| < \infty$, and η is in the associated finite sequence space.

<u>PROGRAM</u> II_S

$$\inf(x,e_S)$$

$$\text{subject to} \quad (x,e_Q) \geq v_Q, \text{ for all } Q \subseteq S, |Q| < \infty. \tag{22}$$

Let $M(S)$ denote the value of Program I_S, finite or not. $M(\cdot)$ is an extension of the "M-operator" introduced in Charnes-Kortanek [5] and also studied in a different infinite setting by Charnes-Eisner-Kortanek [7].

<u>LEMMA 1</u>. Let $S \subseteq N$, $|S| \geq 2$. Then Program I_S has (asymptotic) consistent solutions.

Proof, Since all $Q \subseteq S$, $|Q| < \infty$ are present in the summation of (21), $e_{\{i\}}$, $i \in S$, appears in the summation. If S is a finite coalition, then simply set $e_{\{i\}} = 1$ for $i \in S$. If S is infinite, then proceed through the singletons of $i \in S$, i.e., $(1,0,\ldots,0\ldots) = e^{(1)}$, $(1,1,0\ldots) = e^{(2)}$, ..., $(1,1,\ldots,1,0,\ldots) = e^{(k)}$ so that

$$\lim_{K} (e^{(K)})_i = (e_S)_i \text{ for all } i \in S. \tag{23}$$

Remark. Lemma 1 shows that Program I is always (asymptotic) consistent. This fact is analogous to the status of relations (2) when N is finite, namely they have consistent solutions also.

LEMMA 2. If v is AB, then for $S \subseteq N$, $|S| \geq 2$, it follows that $M(S) < \infty$.

Proof. By Lemma 1, Program I_S is consistent. Assume to the contrary that $M(S) = +\infty$. Therefore there exists a sequence $\{\eta_Q(k)\}_{k=1}^{\infty}$, $Q \subseteq S$, $|Q| < \infty$, such that

$$\lim_{k\to\infty} \sum_{Q} \eta_Q(k) v_Q = +\infty\ldots \quad . \tag{24}$$

Now form another sequence if necessary, $\{\eta_{Q'}(k)\}_{k=1}^{\infty}$ converging pointwise to e_{N-S} i.e.

$$\lim_{k} \left(\sum_{Q'} \eta_{Q'}(k) e_{Q'}\right)_i = (e_{N-S})_i \quad \text{for all } i \in N-S, \tag{25}$$

where $Q' \subseteq N-S$, $|Q'| < \infty$. Therefore since S and $N-S$ are disjoint, the sequences in (24) and (25) may be combined to form a consistent solution to Program I_S in (21) which is also unbounded. This contradicts the fact that v is AB. Hence $M(S) < \infty$ for all $S \subseteq N$, $|S| < \infty$, $|S| \geq 2$. Q.E.D.

Analogous to the case of finite N, the M-operator can be used to characterize redundant inequalities in Program II, and also to make contact with the cover of the game v, i.e. the smallest propergame covering v following Lloyd Shapley's idea around 1968 or so.

5. Domination and the Core of a Game.

The game value v_S is zero for all infinite player coalitions except possibly v_N, and in fact, generally $v_N > 0$. Using the definition of $F^{\#}$ in (7), let

$$X = \{x \in F^{\#} : (x,e_N) = v_N, \ x \geq 0\} \tag{26}$$

called the set of payoff vectors. A member $y \in X$ dominates $x \in X$ through a coalition S, written $y >_S x$ if

$$(y,e_S) > (x,e_S) \tag{27a}$$

and

$$(y,e_S) \leq v_S. \tag{27b}$$

Let

$$U = \{x \in X : x \text{ cannot be dominated by some } y \in X \text{ through some } S \subseteq N\}. \tag{28}$$

U is called the set of undominated payoff vectors in X.

The <u>core</u> of the game v is defined as

$$C(v) \equiv C = \{x \in X : (x, e_S) \geq v_S \text{ for all } S \subseteq N\}. \tag{29}$$

When N is finite, it is known that the core $C(v)$ is empty if and only if

$$M(e_N) > v_N. \tag{30}$$

See for example, Charnes-Kortanek [1], Proposition 4. When N is infinite, (30) no longer characterizes core emptiness because in this case $C(v)$ may be empty even when $M(e_N) = v_N$. We present an example later to illustrate.

We do, however, obtain the following elementary result.

<u>THEOREM 2</u>. Assume the game v is AB. Then

(a) $M(e_N) < v_N$ implies $C(v)$ not empty and

(b) $C(v)$ not empty implies $M(e_N) \leq v_N$.

<u>Proof</u>. (a) Assume $M(e_N) < v_N$. By the perfect duality of Programs I and II and a property of an infimum, there exists x consistent for Program II such that

$$I(II) = M(e_N) \leq (x, e_N). \tag{31}$$

Construct a member $x' \in C$ as follows:

$$x'_i = \begin{cases} x_i + (v_N - (x, e_N))/|T_x|, & \text{if } i \in T_x \\ 0, & \text{otherwise,} \end{cases} \tag{32}$$

where T_x is the carrier (18) of x.

Hence

(i) $(x', e_S) \geq (x, e_S)$ for all S and

(ii) $(x', e_N) = (x, e_N) + v_N - (x, e_N) = v_N$, proving $x \in C$ and hence part (a).

Proof of (b). First, since v is AB, it follows by perfect duality that $i(II) = M(e_N)$. Let $x \in C(v)$. This means x is consistent for Program II and hence

$$v_N = (x, e_N) \geq i(II) = M(e_N) \quad (33)$$

proving (b) of the theorem.

We are interested in conditions under which $C(v) = U$. For N finite, necessary and sufficient conditions exist which do not extend to infinite N. These conditions however lead to the following approach.

Define a new game v' by

$$v'_S = \min\{v_S, v_N\} \text{ for all } S \subseteq N. \quad (34)$$

Then if v is AB, so is v' and also for the grand coalition N, we have $v'_N = v_N$. Furthermore it follows that

$$M(e_N) \geq M'(e_N) \text{ since } v'_S \leq v_S \text{ for all } S, \quad (35)$$

where $M'(e_N)$ is the value of Program I for game v'.

THEOREM 3. If $U = C$, then either

(a) $v_S \leq v_N$ for all $S \subseteq N$ or

(b) $M'(e_N) \geq v_N$.

Proof. Assume $U = C$. We consider various cases.

1. $U = C \neq \emptyset$. Assume there exists S such that $v_S > v_N$. Let $x \in C$. Then $v_N = (x, e_N) \geq (x, e_S) \geq v_S > v_N$, a contradiction. Hence in this case property (a) of the theorem holds.

2. $U = C = \emptyset$. By Theorem 1, it follows that $M(e_N) \geq v_N$. We consider two situations.

1.2.1. If $v' = v$, i.e., $v_S \leq v_N$ for all S and $M(e_N) = M'(e_N)$, it follows that $M'(e_N) \geq v_N$ which is (b).

1.2.2. There exists S such that $v_S > v_N$. Assume to the contrary of $M'(e_N) \geq v_N$, that $M'(e_N) < v_N = v'_N$. Thus by (31) and (32), the game v' has a non-empty core. Observe here that $v'_S = v_N$. Let $x \in C(v')$, the core of v'. To show $x \in U$.

Case 1. There exist subsets T satisfying $v_T = v'_T$ and $y \in X$ satisfying $(y, e_T) \leq v_T$. Then $(x, e_T) \geq v'_T = v_T \geq (y, e_T)$ so that y cannot dominate x through such a T.

Case 2. There exist T such that $v_T > v_N$, and $y \in X$ such that $(y, e_T) \leq v_T$ and $(y, e_T) > (x, e_T)$. In this situation we have $(x, e_T) \geq v'_T = v_N$. But $y \in X$ and therefore $(y, e_N) = v_N$. Hence

$$v_T > v_N = (y, e_N) \geq (y, e_T) > (x, e_T) \geq v'_T = v_N$$

which is a contradiction. Hence there are no T's in this case. Therefore cases 1 and 2 show that $x \in U$.

But throughout 1.2, $U = \emptyset$, and $x \in U$ is a contradiction to this. Hence also in 1.2.2 we conclude $M'(e_N) \geq v_N$, which is (b) of the theorem. This completes the proof of Theorem 3.

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A sufficient condition for $U = C$ is given in the next theorem.

THEOREM ±. Assume $v_S \le v_N$ for all $S \subseteq N$. Then $U = C(v)$.

Proof, Case 1. $C(v) \neq \emptyset$. Let $x \in C$ and assume to the contrary that $x \notin U$, i.e. $y >_S x$. Hence $v_S \geq (y, e_S) > (x, e_S) \geq v_S$, a contradiction. Hence $C(v) \subseteq U$.

Let $x \in U$ and assume to the contrary that $x \notin C(v)$. Then there exists a coalition, S_0, of smallest finite cardinality such that

$$(x, e_{S_0}) < v_{S_0} \le v_N \tag{36}$$

Since $(x, e_{T_x}) = (x, e_N) = v_N$, it follows that $S_0 \cap N - T_x \neq \emptyset$. Construct $y \in X$ as follows.

$$y_i = \begin{cases} x_i + (v_{S_0} - (x, e_{S_0}))/|S_0 \cap T_x| & \text{if } i \in S_0 \cap T_x \\ (v_N - v_{S_0})/|S_0 \cap (N - T_x)| & \text{if } i \in S_0 \cap (N - T_x) \\ 0 & \text{if } i \notin S_0 . \end{cases}$$

Then

$$(y, e_{S_0}) = (x, e_{S_0}) + v_{S_0} - (x, e_{S_0}) = v_{S_0} \tag{37}$$

and

$$(y, e_N) = (y, e_{T_x}) + (y, e_{N-T_x}) = v_{S_0} + v_N - v_{S_0} = v_N$$

implying $y \in X$. Finally $(y, e_{S_0}) = v_{S_0} > (x, e_{S_0})$ which shows $y >_{S_0} x$, contradicting $x \in U$. Hence $x \in C(v)$ and $U \subseteq C(v)$. Thus in this case $C(v) = U$.

When N is finite we have shown elsewhere (C. Bird, for example) that conditions (a) and (b) as stated in Theorem 3 form necessary and sufficient conditions for $U = C(v)$. The following example shows that this is not true for N infinite.

We now present an example of a game with empty core, but which satisfies $v_N = M(e_N)$ and $v_S < v_N$ for all $S \subseteq N$. By Theorem 4, it follows that the set of undominated payoffs U is also empty.

Example. $C(v) = U = \emptyset$.

Let $v_1 = v_2 = 1/2$ and for all integers n, K satisfying $n \geq K \geq 3$ define

$$v_{(1,K,K+1,\ldots,n)} = \frac{1}{2} + \sum_{j=K}^{n} (\frac{1}{2})^{j-1} \tag{38}$$

and $v_{(2,K,K+1,\ldots,n)} = v_{(1,K,K+1,\ldots,n)}$. For all other $S \neq N$, set $v_S = 0$. Since Program II has consistent solutions ($v_1 = v_2 = 1$ for example), it follows from perfect duality that v is AB. We shall define v_N a little later.

For each integer $k > 3$ we construct a II consistent solution, $x(k)$ as follows:

$$x_1 = x_2 = (\frac{1}{2})^{k-2} + \frac{1}{2} \tag{39}$$

$$x_3 = (\frac{1}{2})^2 - (\frac{1}{2})^{k-2} \text{ and for } j \leq k-1,$$

$$x_j = (\frac{1}{2})^{j-1}, \; x_k = (\frac{1}{2})^{k-2}.$$

LEMMA 4. $x(k)$ defined in (34) is consistent for Program II for for the game in (36).

Proof. Let $S = \{1, K, K+1, \ldots, n\}$ where $n \geq K \geq 3$, where we have $\{x_1, \ldots, x_k\}$ on hand, $k > 3$.

1. Clearly $x_1 \geq v_1$, $x_2 \geq v_2$.

2. $K = 3$.

2.1. $n < k$. In this case,

$x_1 + x_K + \ldots + x_n = \frac{1}{2} + (\frac{1}{2})^2 + \ldots + (\frac{1}{2})^{n-1} = v_S$ for $n \geq 3$.

2.2. $n \geq k$. Here

$x_1 + x_K + \ldots + x_n \geq \frac{1}{2} + (\frac{1}{2})^2 + \ldots + (\frac{1}{2})^{n-1} = v_S$.

2. $K > 3$.

2-1. $K < k$.

3.1.1. $n < k$. In this case,

1

3.1.2. $n \geq k$. Here

$x_1 + x_K + \ldots + x_n = (\frac{1}{2})^{k-2} + \frac{1}{2} + (\frac{1}{2})^{K-1} + \ldots + (\frac{1}{2})^{k-2} + (\frac{1}{2})^{k-2} >$

$\frac{1}{2} + (\frac{1}{2})^{K-1} + \ldots + (\frac{1}{2})^{k-2} + [(\frac{1}{2})^{k-1} + \ldots + (\frac{1}{2})^{n}]$ since

$(\frac{1}{2})^{k-2} > (\frac{1}{2})^{k-1} + \ldots + (\frac{1}{2})^{n}$ for all $n \geq k-1$.

4. $K = k$, $n \geq k$. Here

1

since $(\frac{1}{2})^{K-2} > (\frac{1}{2})^{K-1} + \ldots + (\frac{1}{2})^{n-1}$. Q.E.D.

We now show that there is no consistent solution $x \in E$ satisfying $(x, e_N) = 1\frac{1}{2}$. It suffices to show that if x is consistent for Program II, then $x_2 \geq \frac{1}{2}$ and

$$\sum_{\substack{i\in T_x \\ i\neq 2}} x_i \geq 1, \tag{40}$$

since then $(x,e_N) > 1\frac{1}{2}$.

Now if to the contrary, there were a solution x satisfying $x_2 \leq \frac{1}{2}$. Set $K = \max\{i : i \in T_x\}$. Then $v_{\{2,K+1\}} = \frac{1}{2} + (\frac{1}{2})^{K}$. But $(x, e_{\{2,K+1\}}) = x_2 \leq \frac{1}{2}$ which is a contradiction. Hence any consistent solution x satisfies $x_2 > \frac{1}{2}$. Assume to the contrary of (38), that $\sum_{\substack{i\in T_x \\ i\neq 2}} x_i < 1$ for some consistent x. But there is a coalition S containing 1 but not 2 such that $\sum_{\substack{i\in T_x \\ i\neq 2}} x_i < v_S < 1$, since the v_S values grow (but are less than) 1. Therefore $v_S > (x,e_S) \geq v_S$ which is a contradiction. Therefore (38) is proven.

On the other hand $(x(k),e_N) = (\frac{1}{2})^{k+2} + \frac{1}{2} + 1 = 1\frac{1}{2} + (\frac{1}{2})^{k+2}$ and hence $\lim_{k\to\infty} (x(k),e_N) = 1\frac{1}{2}$, i.e., $\bar{x}(\Pi) = 1\frac{1}{2}$. By perfect duality, it follows that $M(e_N) = 1\frac{1}{2}$ also. Hence setting $v_N = 1\frac{1}{2}$, the game has no core and satisfies $M(e_N) = v_N$ and

6. Stronger Conditions for $C(v) \neq \emptyset$.

In Theorem 2 we proved that $v_N > M(e_N)$ implies a non-empty core and that if $C(v) \neq \emptyset$ then $M(e_N) \leq v_N$. This still leaves unresolved the case of $v_N = M(e_N)$. Example 1 shows that there are games where $M(e_N) = v_N$ that do not have cores. Obviously, there are many games when $M(e_N) = v_N$ which have cores. In this section we will prove some stronger theorems.

<u>THEOREM</u> <u>5</u>. Let T be a carrier of v which has a finite number of elements. Let $Q_i = \sup_{s \in F_i} v_S$ where $F_i = \{S \mid i \in S\}$. If for each $i \in T$ there is a sequence of coalitions $\{S_i^K\}_{K=1}^{\infty}$ such that $\lim_{K \to \infty} v_{S_i^K} = Q_i$ and $S_i^K \cap S_j^K = \emptyset$ for all $i,j \in T$, then v has a non-empty core.

<u>Proof</u>. First we will show that $\sum_{i \in T} Q_i \leq M(e_N)$. Since $S_i^K \cap S_j^K = \emptyset$ then $\sum_{i \in T} v_{S_i^K} \leq M(e_N)$ for $K = 1,2,\ldots$. However for each $i \in T$ $\lim_{K \to \infty} v_{S^K} = Q$, thus $\sum_{i \in T} Q_i \leq M(e_N)$. Now let

$$x_i = \begin{cases} Q_i & i \in T \\ 0 & i \notin T. \end{cases}$$

If $v_N = \sum_{i \in T} Q_i$ then $x \in C(v)$. For any S if $S \cap T = \emptyset$ then $v_S = 0$ and trivially $(x, e_S) \geq v_S$ holds. If $S \cap T \neq \emptyset$ then for any $i \in T \cap S$

$$(x, e_S) \geq Q_i \geq v_S$$

since $S \in F_i$. Thus $x \in C(v)$. If $v_N > \sum_{i \in T} Q_i$ then trivially there is an element in its core also. Note that this proof also shows that $\sum Q_i = M(e_N)$.

Definition. The family of coalitions carried only by $S \subset T$ is

$$G_S = \{R \mid S \subset R \quad R \cap (T-S) = \emptyset\}.$$

Note that any non-zero-valued coalition belongs to one such G_S. Let $P_S = \sup_{R \in G_S} v_R$.

Definition. The set of players in a family of coalitions, H, not carried by coalition C at a level greater than b is

$$K_{H,b,C} = \{n \mid \exists S \in H,\ v_S > b,\ n \in S,\ C \cap S = \emptyset\}.$$

We know that setting $x_i = Q_i$ for $i \in T$ will yield a consistent solution to Program II so that if $v_N \geq \sum_{i \in T} Q_i$ then $C(v) \neq \emptyset$. We can prove a stronger result along the same lines. Let Program II' be

$$\min \sum_{i \in T} x_i \quad \text{such that for each} \quad S \subset T \quad (x, e_S) \geq P_S.$$

Thus we have the following result.

THEOREM 6. Any consistent solution of II' satisfies the constraints of Program II and the minimum of $II' \leq \sum_{i \in T} Q_i$. As a result if $v_N \geq \min II'$ then $C(v) \neq \emptyset$.

Proof. Let x be consistent for II'. Then if R is any coalition such that $v_R > 0$ there is only one $S \subset T'$ with R in G_S. Now, $(x, e_S) \geq P_S$ so that $(x, e_R) \geq (x, e_S) \geq P_S \geq v_R$ since $R \in G_S$. Therefore x is consistent for Program II. Because II' is a finite linear program the minimum is attained. To see that min II' $\leq \sum_{i \in T} Q_i$ all that is necessary is to show that $x_i = Q_i$ for $i \in T$ is a consistent solution to II'.

Let $S \subset T$, for each i in S $Q_i \geq P_S$ since $G_S \subset F_i$. Therefore $\sum_{i \in S} Q_i = (x, e_S) \geq P_S$. This proves the theorem.

In Theorem 5 we proved that $v_N \geq M(e_N)$ is equivalent to $C(v) \neq \emptyset$ when the supremum for each family of coalitions whose intersection with similar coalition in other families was empty. In the next theorem the same equivalence is shown under an extremely different condition.

<u>THEOREM 7</u>. If for each $S \subset T$, for every $\varepsilon > 0$ and each finite coalition C with $C \cap T = \emptyset$, $K_{G_S, P_S - \varepsilon, C} \neq \emptyset$ then $v_N \geq M(e_N)$ is equivalent to $C(v) \neq \emptyset$.

<u>Proof</u>. We will show that x is a consistent solution of II' iff x is a consistent solution of II. By Theorem 6 one direction has been shown. Let x be a consistent solution to II. Then for any $S \subset T$ and for each R in G_S, $(x, e_R) \geq v_R$. Let $C = T_x - T$. For any $\varepsilon > 0$ $K_{G_S, P_S - \varepsilon, C} \neq \emptyset$; therefore there is an R in G_S such that $v_R > P_S - \varepsilon$ with $C \cap R = \emptyset$. This implies that $(x, e_S) = (x, e_R) \geq v_R \geq P_S - \varepsilon$. Since this holds for arbitrary $\varepsilon > 0$, $(x, e_S) \geq P_S$. Thus Programs II and II' are equivalent. Because II' is a finite linear program there is an x' with $(x', e_N) = M(e_N)$ which satisfies the constraints of Program II. Therefore the theorem is proved.

The intuitive meaning of the previous theorem is that there are an infinite number of players who can join S and have the value of their coalitions approach P_S. If, instead, there were only a finite number of players who had to belong to a coalition which contained S in order for P_S to be approached then one could formulate a new linear program using $T \cup C$ instead of T. As example 1 shows it is possible to have a series of finite linear programs, each of which has a smaller minimum and includes more players than the previous one. Thus the infimum need not be attained.

7. Some Interpretations and Justification of the Model.

One of the characteristics of our model is that it makes relatively few assumptions, and as a result, it should be rather general. Nevertheless, some of the consequences tend to reduce the generality of the model. In this section we will present some justifications for making the few restrictions which were made, support some of the conclusions which we reached and present some situations which are modeled rather well by this game.

The restriction that $v_S = 0$ when S contains an infinite number of players seems rather reasonable, since some people have questioned the idea of an infinite number of players uniting to block an outcome. We made (I) AB since to do otherwise would have meant that the value of the game was infinite, which we do not see as a very useful situation. The restriction of making v_S non-negative was for mathematical purposes; however, we do not see that any real advantage is gained by dropping this restriction.

The two consequences which may need justification are the result of the characterization via Theorem 1. The result that only a finite set of players get non-zero payoffs even if they play by themselves seems reasonable as there is only a finite quantity of material being payed out, so that not everyone will get some if he goes it alone. In fact, this model seems to depict quite accurately the situation where a small group of players relative to the entire set of players hold the power and the rest of the players must choose up sides.

For example, consider a dictatorship where the dictator, the head of the army and the head of the secret police are T' and all other individuals can ally themselves with one of the leaders. Or consider a union leader, the head of management and a government negotiator as T. Other individuals such as union members and non-union members would play a role in the negotiation, depending on the level at which they were willing to settle. For example, the more disaffection there is from a union, the less power the union has at the bargaining table. In general one might consider the individuals in T' as leaders with the amount of power coming from the kind and number of followers that they have.

The second result which may need some defending is that $S \cap T' = \emptyset$ implies that $v_S = 0$. In other words, a leaderless group has no power. This seems to be a philosophical question rather than mathematical, for many people will agree or disagree depending on their outlook on life. Whether or not there are leaderless groups with power, there certainly are enough of these groups without power so that our model has wide application.

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