A DERIVATION OF THE POINCARÉ GROUP

by

Andrew Vogt*

Report 72-17

August 1972

* This work was partially supported by NSF Science Development Grant GU-2056.

A DERIVATION OF THE POINCARÉ GROUP

by

Andrew Vogt

Abstract

Let $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ be a bijection such that whenever p and q lie on a common light ray, fp and fq lie on a common light ray. Then f is in the group generated by the Poincaré group and dilatations.

The proof of this fact is based on Zeeman's theorem that causality implies the Lorentz group.

A DERIVATION OF THE POINCARE GROUP

by

Andrew Vogt

Department of Mathematics
Carnegie-Mellon University

§1 Introduction: A point p in $\mathbb{R}^4$ is represented here in the form $p = (t_p, x_p)$ where t_p is in $\mathbb{R}$ and x_p is in E^3. For elements x and y in $\mathbb{R}^3$, let $x \cdot y$ denote their Euclidean scalar product and let $\|x\| = \sqrt{x \cdot x}$. We define the Minkowski metric $\hat{G} : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}$ by $\hat{G}(p,q) = t_p t_q - (x_p \cdot x_q)$, and we define $G : \mathbb{R}^4 \to \mathbb{R}$ by $G(p) = \hat{G}(p,p) = t_p^2 - \|x_p\|^2$. A bijection is a one-to-one, onto map.

THEOREM 1: Let $f : \mathbb{R}^4 \to \mathbb{R}^4$ be a bijection which satisfies for each p and q in $\mathbb{R}^4$:

$$G(p-q) = 0 \implies G(fp-fq) = 0.$$

Then f is in the group generated by the Poincaré group and dilatations.

This theorem appears to be new or at least widely unknown. An ambiguous footnote in Einstein's original paper [3], p.46, states that "the equations of the Lorentz transformation may be more simply deduced from the condition that in virtue of these equations the relation $x^2 + y^2 + z^2 = c^2t^2$ shall have as its consequence the second relation $\xi^2 + \eta^2 + \zeta^2 = c^2\tau^2$."

This work was partially supported by NSF Science Development Grant GU-2056.

A theorem with the same conclusions as Theorem 1 but slightly stronger hypotheses -- namely, that $G(p-q) = 0 \Leftrightarrow G(fp-fq) = 0$ -- has been proved by Barucchi [1] and apparently much earlier by Aleksandrov. For references to Aleksandrov's work see Pimenov's book [5], p.21. The author discovered this theorem as well as Theorem 1 independently.

Zeeman's theorem that causality implies the Lorentz group plays an important role in the proof of Theorem 1. We reprove this theorem here in a manner suggested by [6]. For other proofs see Rothaus [6] and Zeeman [7]. Lemma 3 of Zeeman's paper has been clarified by Barucchi and Teppati [2].

In the following three sections we develop parts of the proof of Theorem 1 in the form of Theorems 2, 3, and 4. In §5 we assemble these parts and complete the proof. All of our results generalize from four dimensions to n dimensions $(n \geq 3)$, and the notation of the paper should allow one to follow the argument with the generalization in mind. If $n = 2$, Theorems 2 and 3 are false but Theorem 4 remains valid.

§2: Our definitions are taken from Zeeman [8]. For p in $\mathbb{R}^4$, we define:

the space cone through $p = C_p^S = \{q : q = p \text{ or } G(q-p) < 0\}$;

the time cone through $p = C_p^T = \{q : q = p \text{ or } G(q-p) > 0\}$;

the light cone through $p = C_p^L = \{q : G(q-p) = 0\}$.

A line through p is called a space line, a time axis, or a light ray accordingly as it lies in the space cone, the time cone, or the light cone through p. A plane through p is called a

<u>space-time plane</u> if and only if it contains a time axis through p. Each of these characterizations is independent of the choice of p in the line or plane. A space-time plane can be characterized alternatively as the plane defined by two distinct, intersecting light rays.

If p and q are distinct points and $G(q-p) = 0$, then $C_p^L \cap C_q^L$ is the light ray through p and q, and we denote this light ray by $L(p,q)$.

We introduce two relations from [7], which we define as follows:

$$p < q \quad \text{if and only if} \quad G(q-p) > 0 \quad \text{and} \quad t_p < t_q;$$
$$p \lessdot q \quad \text{if and only if} \quad G(q-p) = 0 \quad \text{and} \quad t_p < t_q.$$

$<$ is a partial ordering, but $\lessdot$ is not since it is not transitive. Both relations are preserved by translations and positive dilatations (Zeeman's theorem discovers <u>all</u> transformations which preserve these relations), and both relations are reversed by negative dilatations.

Let $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$. We consider the following conditions which might be imposed on f:

(2.1) for each p and q in $\mathbb{R}^4$ $G(p-q) = 0 \Rightarrow G(fp-fq) = 0$;

(2.2) for each p and q in $\mathbb{R}^4$ $G(p-q) = 0 \Leftrightarrow G(fp-fq) = 0$;

(2.3) for each p and q in $\mathbb{R}^4$ $p \lessdot q \Leftrightarrow fp \lessdot fq$;

(2.4) for each p and q in $\mathbb{R}^4$ $p < q \Leftrightarrow fp < fq$.

LEMMA 2.1; Let $f : \mathbb{R}^4 \to \mathbb{R}^4$ be a bijection. Then (2.4) ⟹ (2.3) ⟹ (2.2).

Proof: Since $G(p-q) = 0$ if and only if $p = q$, $p < q$, or $q < p$ and since f is injective (one-to-one), (2.3) ⟹ (2.2).

To prove (2.4) ⟹ (2.3), we note that f is surjective (onto). Then (2.4) ⟹ (2.3) is a consequence of the fact that $p < q$ if and only if $p < q$ is false and for each r $q < r$ implies $p < r$. Likewise, (2.3) ⟹ (2.4) is a consequence of the fact that $p < q$ if and only if $p < q$ is false and there exists an r such that $p < r$ and $r < q$.

LEMMA 2.2; Let $f : \mathbb{R}^4 \to \mathbb{R}^4$ be a bijection satisfying (2.2). Then $f(C_p^L) = C_{fp}^L$, $f(C_p^L \cap C_q^L) = C_{fp}^L \cap C_{fq}^L$ and $f(L(p,q)) = L(f_p, f_q)$.

Proof: $q \in C_p^L \Rightarrow G(q-p) = 0 \Rightarrow G(fq-fp) = 0 \Rightarrow fq \in C_{fp}^L$. Thus $f(C_p^L) = C_{fp}^L \cap \text{range } f = C_{fp}^L$ since f is surjective. $f(C_p^L \cap C_q^L) = f(C_p^L) \cap f(C_q^L) = C_{fp}^L \cap C_{fq}^L$ since f is bijective. Finally, if $G(q-p) = 0$, then $f(L(p,q)) = f(C_p^L \cap C_q^L) = C_{fp}^L \cap C_{fq}^L = L(f_p, f_q)$ since $G(f_q - f_p) = 0$.

LEMMA 2.3:- Let $u > 0$ and $v > 0$. Then $\hat{G}(u,v) \geq 0$ and $\hat{G}(u,v) = 0$ if and only if u and v are dependent.

Proof; Since $u > 0$, $t_u^2 = \|x_u\|^2$ and $t_u > 0$. Thus $t_u = \|x_u\| \neq 0$. Likewise, since $v > 0$, $t_v = \|x_v\| \neq 0$. Then $G(u,v) = t_u t_v - (x_u \cdot x_v) \geq t_u t_v - \|x_u\| \, \|x_v\| = 0$. Equality holds if and only if $x_u \cdot x_v = \|x_u\| \, \|x_v\|$. In this case $x_u = \lambda x_v$ for some $\lambda > 0$, $t_u = \|x_u\| = \lambda\|x_v\| = \lambda t_v$, and $u = \lambda v$.

<u>THEOREM 2</u> (<u>Zeeman's Theorem</u>): Let $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ be a bijection satisfying (2.4). Then f is affine-linear.

<u>Proof</u>: By lemmas 2.1 and 2.2 f satisfies (2.2) and takes light rays onto light rays. If f is a bijection of $\mathbb{R}^n$ onto $\mathbb{R}^n$ $(n \geq 2)$ which takes lines onto lines, then f must be affine-linear (see [4], p.110). Hence, it suffices to show that f takes time axes and space lines onto lines. Since each time axis or space line is the intersection of two space-time planes ($n \geq 3$!) and f is a bijection preserving intersection, we may further reduce the problem to showing that f takes space-time planes onto planes.

Let $\mathcal{P}$ be a space-time plane. Without loss of generality we suppose that $\mathbb{O}$ is in $\mathcal{P}$ and $\mathcal{P} = \text{span}\,\{u,v\}$ for independent vectors $u \underset{\cdot}{>} \mathbb{O}$ and $v \underset{\cdot}{>} \mathbb{O}$. In addition, we assume $f(\mathbb{O}) = \mathbb{O}$. Let $\mathcal{P}' = \text{span}\,\{fu, fv\}$. By lemma 2.1 f satisfies (2.3). Hence, $fu \underset{\cdot}{>} f(\mathbb{O}) = \mathbb{O}$ and $fv \underset{\cdot}{>} f(\mathbb{O}) = \mathbb{O}$. Moreover, fu and fv are independent since f maps distinct light rays onto distinct light rays. Thus $\mathcal{P}'$ is itself a space-time plane.

If w is any element of $\mathcal{P}$, w lies on a light ray parallel to v which intersects the two parallel light rays $L(\mathbb{O},u)$ and $L(v,u+v)$. Hence, fw lies on a light ray which intersects the two disjoint light rays $L(\mathbb{O},fu)$ and $L(fv,f(u+v))$. If these light rays are in $\mathcal{P}'$, so is fw. But $L(\mathbb{O},fu)$ is in $\mathcal{P}'$ and fv is in $\mathcal{P}'$. Hence, $f(\mathcal{P}) \subseteq \mathcal{P}'$ provided $f(u+v)$ is in $\mathcal{P}'$.

For λ real $f(\lambda u)$ is in $L(\mathbb{O},fu)$. Moreover, if $\lambda_1 < \lambda_2$, $f(\lambda_1 u) \underset{\cdot}{<} f(\lambda_2 u)$. Thus, there exists an increasing bijection

p_1 : IR—*yiR* such that f(Au) = p_1(A)f(u). Likewise, there exists an increasing bijection p_2 : 3R—*IR such that f(Au+v) - f v = p_2(A) (f (u+v)-fv) . So, 6(f (u+v) -fv,f (Au+v) -fv) = p_2(A) G(f (u+v) -fv) = 0 since G(u+v-v) = G(u) = 0. Applying lemma 2.3 and the inequalities f (u+v) - fv ;> <D, f(Au+v) - f(Au) > ©, we obtain:

0 ^ £(f (u+v)-fv, f(Au+v) - f(Au))

= 6(f(u+v)-fv, f(Au+v) - fv + fv - p_x(A)fu)

= Ĝ(f (u+v)-fv,fv) - p_1(A)6(f (u+v)-fv,fu) .

As A tends to +oo , p_1 (A) tends to +oo also. For the inequality to hold, we must have &(f(u+v)-fv,fu) <^ 0. However, f (u+v) - f v > <D and fu > (D. By lemma 2.3 Ĝ(f (u+v) - fv, fu) = 0 and so (f(u+v)-fv) and fu are dependent vectors. It follows that f(u+v) is in span {fu,fv} = P'.

Tiius f takes space-time planes into space-time planes. Since f^{-1} fulfills the hypotheses of Theorem 2, f^{-1} does the same and we conclude that f maps space-time planes onto (space-time) planes, completing the proof.

§2:

LEMMA 3^: Let G(u) > 0 and G(v) =0 with v ^ <D. Then Ĝ(u,v) ^ 0. In fact, if u > <D and v ^> d), Ĝ(u,v) > 0.

Proof; G(u) > 0 implies $t_u^2 > \|x_u\|^2$, and G (v) =0 with v ^ <D implies $|t_v| = \|x_v\| \neq 0$. If 0 = 6(u,v) = $t_u t_v - (x_u \cdot x_v)$, it follows that $|t_u t_v| = |x_u \cdot x_v| \leq \|x_u\| \|x_v\| < |t_u| |t_v|$ for a contradiction. If $u > \Phi$ and $v \geq \Phi$, then $t_u > \|x_u\|$ and $t_v = \|x_v\| \neq 0$. Hence, $\hat{G}(u,v) = t_u t_v - (x_u \cdot x_v) \geq t_u t_v - \|x_u\| \|x_v\| = (t_u - \|x_u\|) t_v > 0$.

<u>LEMMA 3.2</u>: Let $G(u) < 0$. Then there exists w such that $w \neq \Phi$ and $0 = G(w) = \hat{G}(u,w)$.

<u>Proof</u>: Since $G(u) < 0$, $t_u^2 < \|x_u\|^2$. Choose x_w in $\mathbb{R}^3$ such that $\|x_w\| = 1$ and $x_u \cdot x_w = t_u$. This is possible since x_w is in a space of more than one dimension and $-\|x_u\| < t_u < \|x_u\|$. Let $w = (1, x_w)$.

<u>LEMMA 3.3</u>: Let $u > \Phi$ and $v > \Phi$. Then there exists w such that $w > \Phi$, $G(w-u) < 0$, and $G(w-v) < 0$.

<u>Proof</u>: Let $\mathfrak{G} = \{q : t_q > 0, G(q-u) < 0, G(q-v) < 0\}$. $\mathfrak{G}$ is an open subset of $\mathbb{R}^4$ in the Euclidean topology. Choose $w_1 \geq \Phi$ then $G(\lambda w_1 - u) = -2\lambda \hat{G}(u,w_1) + G(u)$ and $G(\lambda w_1 - v) = -2\lambda \hat{G}(v,w_1) + G(v)$. By lemma 3.1 $\hat{G}(u,w_1)$ and $\hat{G}(v,w_1)$ are both positive. Hence for λ sufficiently positive, $G(\lambda w_1 - u) < 0$ and $G(\lambda w_1 - v) < 0$, and λw_1 is in $\mathfrak{G}$. But λw_1 is in C_Φ^L = the boundary of C_Φ^T. Hence, since $\mathfrak{G}$ is a neighborhood of λw_1, there exists w in $\mathfrak{G} \cap C_\Phi^T$. w has all the properties required by the lemma.

For p and q such that $G(p-q) \neq 0$, we define a set

$$C_{p,q} = \{u : u \in C_p^L, \text{ and } u = p \text{ or } L(p,u) \cap C_q^L \neq \emptyset\}.$$

$C_{p,q}$ is another "cone" through p. If u is in $C_{p,q}$ and $u \neq p$, then $L(p,u) \cap C_q^L \subseteq C_p^L \cap C_q^L$. The set $C_p^L \cap C_q^L$ is a section of C_p^L forming an <u>ellipsoid</u> or a <u>hyperboloid</u> with two branches accordingly as q is in C_p^T or C_p^S. This observation makes the following two lemmas geometrically obvious.

<u>LEMMA 3.4</u>: If q is in C_p^T, then $C_p^L = C_{p,q}$.

<u>Proof</u>: If u is in C_p^L and $u \neq p$, consider for λ real the expression $G(p+\lambda(u-p)-q) = G(p-q) + 2\lambda \hat{G}(p-q,u-p)$. Since $G(p-q) > 0 = G(u-p)$, lemma 3.1 guarantees that $G(p-q,u-p) \neq 0$. Thus λ can be found so that $0 = G(p+\lambda(u-p)-q)$. Set $u^* = p+\lambda(u-p)$. Then u^* is in $L(p,u) \cap C_q^L$, and u is in $C_{p,q}$.

<u>LEMMA 3.5</u>: If q is in C_p^S, then $C_p^L \supseteq C_{p,q}$ but $C_p^L \neq C_{p,q}$.

<u>Proof</u>: By lemma 3.2 we can select $w \neq \Phi$ such that $0 = G(w) = G(p-q,w)$. Then $(p+w)$ is in C_p^L, but $G(p+\lambda w-q) = G(p-q) < 0$ for all λ. Thus $L(p,p+w) \cap C_q^L = \emptyset$ and $(p+w)$ is not in $C_{p,q}$.

<u>LEMMA 3.6</u>: Let $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ be a bijection satisfying (2.2). Then if $p < q$, fq is in C_{fp}^T.

<u>Proof</u>: Suppose there exist p and q with $p < q$ and fq not in C_{fp}^T. By lemma 2.2 we may rule out fq being in C_{fp}^L. Thus fq is in C_{fp}^S.

Lemma 3.5 now tells us that we may choose w in C_{fp}^L such that w is not in $C_{fp,fq}$. Since f is bijective, $w = fu$ for some u in C_p^L. Hence by lemma 3.4 u is in $C_{p,q}$. Since $u \neq p$, there exists u^* in $L(p,u) \cap C_q^L$. Thus fu^* is in $f(L(p,u) \cap C_q^L) = f(L(p,u)) \cap f(C_q^L) = L(fp,fu) \cap C_{fq}^L = L(fp,w) \cap C_{fq}^L \neq \emptyset$. Thus w is in $C_{fp,fq}$ contrary to hypothesis. We have no choice but to affirm the lemma.

<u>LEMMA 3.7</u>: Let $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ be a bijection satisfying (2.2). If $p < q_1$, $p < q_2$, and $fp < fq_1$, then $f_p < fq_2$.

<u>Proof</u>: By lemma 3.3 choose w such that $w > \Phi$, $G(p+w-q_1) < 0$, and $G(p+w-q_2) < 0$. Let $q_3 = p + w$. Then $p < q_3$. So by lemma 3.6 fq_3 is in C^T_{fp}. If $fq_3 < fp_1$, by transitivity $fq_3 < fq_1$. Since f^{-1} fulfills the hypotheses of lemma 3.6, q_1 is in $C^T_{q_3}$. But then $G(q_3-q_1) = G(p+w-q_1) > 0$, which is false. Since fq_3 is in C^T_{fp} and $fq_3 \neq fp_1$, we must conclude that $f_p < fq_3$.

Now fq_2 is in C^T_{fp}. If $fq_2 < fp$, then $fq_2 < fq_3$ and q_3 is in $C^T_{q_2}$. This contradicts $G(q_3-q_2) = G(p+w-q_2) < 0$, and so we finally conclude that $fp < fq_2$.

<u>THEOREM 3</u>: Let $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ be a bijection satisfying (2.2). Then f or $-f$ satisfies (2.4).

<u>Proof</u>: Suppose that there exists no pair p_1 and q_1 such that $p_1 < q_1$ and $fp_1 < fq_1$. Then $p < q \Longrightarrow fp > fq$ by lemma 3.6. Applying the same lemma to f^{-1}, we find that $fp < fq \Longrightarrow q$ is in C^T_p and because of our supposition $q < p$. Thus $p < q \Longleftrightarrow -fp < -fq$, and $-f$ satisfies (2.4).

Alternatively suppose there exists a pair p_1 and q_1 such that $p_1 < q_1$ and $fp_1 < fq_1$. Let $p_2 < q_2$. Choose p_o such that $p_1 < p_o$ and $p_2 < p_o$. By lemma 3.7 $fp_1 < fp_o$. Now consider the map $g = -1 \circ f \circ -1$ where $-1(p) = -p$ for each p in $\mathbb{R}^4$. g is a bijection satisfying (2.2). Thus we can apply lemma 3.7 to the inequalities $-p_o < -p_1$, $-p_o < -p_2$, and

$g(-p_o) < g(-p_1)$ to conclude $g(-p_o) < g(-p_2)$. In other words, $fp_2 < fp_o$. Then $p_2 < p_o$, $p_2 < q_2$, and $fp_2 < fp_o$ imply $fp_2 < fq_2$.

We have succeeded in proving that $p < q \Rightarrow fp < fq$. Since we have a pair satisfying $p_1 < q_1$ and $fp_1 < fq_1$, we can apply the same argument to f^{-1}, obtaining $fp < fq \Rightarrow p < q$. Thus f satisfies (2.4).

§4:

LEMMA 4.1: Let $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ be injective and satisfy (2.1). Then $f(C_p^L) \subseteq C_{fp}^L$, $f(C_p^L \cap C_q^L) \subseteq C_{fp}^L \cap C_{fq}^L$, and $f(L(p,q)) \subseteq L(fp,fq)$.

Proof: Compare with lemma 2.2.

LEMMA 4.2: If r is in C_p^T and q is not in C_p^L, then $C_r^L \cap C_{p,q}$ contains at least two points.

Proof: If q is in C_p^T, then by lemma 3.4 $C_{p,q} = C_p^L$. Thus $C_r^L \cap C_{p,q} = C_r^L \cap C_p^L$ = an ellipsoid, which of course contains two points.

If q is in C_p^S, choose u in $C_r^L \cap C_p^L$ but not in $C_{p,q}$. As in the proofs of lemmas 3.4 and 3.5, we must have $\hat{G}(p-q,u-p) = 0$, and so u is in the set $p + [\text{span}\{p-q\}]^{\perp}$. The latter set is a hyperplane passing through p. It meets the ellipsoid $C_r^L \cap C_p^L$ in a closed subset of $C_r^L \cap C_p^L$. The complement in $C_r^L \cap C_p^L$ of this closed subset is the open subset $C_r^L \cap C_{p,q}$. If $C_r^L \cap C_{p,q}$ is non-empty, it contains two points.

Assume C^L_T 0 $C_{p),q}$ is empty. Then the ellipsoid C^L_r n C^L_p is a subset of the hyperplane. But since p is on' the hyperplane, all light rays joining p to the ellipsoid are in the hyperplane. Such rays form C^p_L, and the only space containing C^L_P is $]R^4$ itself. We thus have a contradiction.

<u>THEOREM 4</u>; Let f : IR^4 —> $]R^4$ be injective and suppose that $f(HR^4)$ is not a subset of a light ray. Then if f satisfies (2.1) , f satisfies (2. 2) .

<u>Proof</u>; Assume (2.2) fails. Then there exist p and q such that G(p-q) ^ 0 = G(fp-fq). Hence, $f(C^L_p n C^L_q)$ c $C^\wedge_p$ 0 $C^\wedge_q$ = L(fp,fq) = L where we adopt the abbreviation L for the light ray L(fp, fq) .

Let u be in C_{PJSL}. If u / p, choose u* in L(p,u) fl C^L_q.

Then fu is in f(L(p,u)) = f(L(p,u*)) CL(fp,fu*). But fu* is in $f(C^p O C^q)$ c L. Hence, L(fp,fu*) = L and fu is in L. Thus $f(C_{P*q})$ c L.

Let r be in C^T_p• By lemma 4.2 choose v and w distinct in C^ 0 $C_{p,g}$. Then ffv,fw} c $f(C^\wedge n C_{p,g})$ c f(C^) n $f(C_{p,q})$ c $C^\wedge_r$ ∩ L. But C^L_{fr} n L is a singleton unless the light ray L contains fr. So fr is in L, and $f(C^T_p)$ c L.

Finally, let u be in IR^4. choose a light ray *I* through u which meets C^T_p in two points v and w. Then fu is in *f(l)* = f(L(v,w)) c L(fv,fw) = L since fv and fw are in $f(C^T_p)$ c L. Thus $f(IR^4)$ c L contrary to hypothesis, and we must accept the validity of (2.2).

§5:

LEMMA 5.1: Let $G(u) > 0$ and v be arbitrary. Then $(\hat{G}(u,v))^2 \geq G(u)G(v)$.

Proof: If $G(v) \leq 0$, the proof is trivial. So assume $G(v) > 0$. Then $(\hat{G}(u,v))^2 - G(u)G(v) = (t_u t_v - x_u \cdot x_v)^2 - (t_u^2 - \|x_u\|^2)(t_v^2 - \|x_v\|^2)$
$= -2t_u t_v (x_u . x_v) + (x_u \cdot x_v)^2 + t_v^2\|x_u\|^2 + t_u^2\|x_v\|^2 - \|x_u\|^2\|x_v\|^2$
$\geq -2|t_u||t_v||x_u \cdot x_v| + |x_u \cdot x_v|^2 + |t_v|^2\|x_u\|^2 + |t_u|^2\|x_v\|^2 - \|x_u\|^2\|x_v\|^2$
$= (|t_v|\|x_u\| - |t_u|\|x_v\|)^2 + (2|t_u||t_v| - \|x_u\|\|x_v\| - |x_u \cdot x_v|)(\|x_u\|\|x_v\|$
$- |x_u \cdot x_v|) \geq 0$ since $|t_u||t_v| > \|x_u\|\|x_v\| \geq |x_u \cdot x_v|$.

We are now prepared to prove Theorem 1 of the first section. We note first that the Poincaré group is the group generated by the (homogeneous) Lorentz transformations and translations. A (homogeneous) Lorentz transformation is a linear map $L : \mathbb{R}^4 \to \mathbb{R}^4$ which satisfies for each p in $\mathbb{R}^4$ $G(p) = G(Lp)$. One may easily verify from this definition that such a map L satisfies $\hat{G}(p,q) = \hat{G}(Lp,Lq)$ for all p and q in $\mathbb{R}^4$, that L is a bijection, and that L^{-1} is also a Lorentz transformation.

Proof of Theorem 1: Let $f : \mathbb{R}^4 \to \mathbb{R}^4$ be a bijection satisfying (2.1). By Theorem 4 f satisfies (2.2). By Theorem 3 $\pm f$ satisfies (2.4), and by Theorem 2 $\pm f$ is affine-linear. Thus, f itself is affine-linear.

To establish that f is in the group generated by the Poincaré group and dilatations, it suffices to show that f differs from a Lorentz transformation by no more than a translation and/or a dilatation.

Since f is affine-linear, $f - f(0) = F$ is a linear map. Moreover, since f has the property (2.2), F does also -- i.e., $G(p) = 0 \iff G(Fp) = 0$.

Choose u_1 in $\mathbb{R}^4$ with $G(u_1) > 0$. By Theorem 3 $G(Fu_1) > 0$. Let $\lambda = \sqrt{G(Fu_1)/G(u_1)}$, and let $L = \frac{1}{\lambda}F$. Then $L : \mathbb{R}^4 \to \mathbb{R}^4$ is a linear map such that $G(p) = 0 \iff G(Lp) = 0$ and $G(u_1) = G(Lu_1)$.

Let v be any element of $\mathbb{R}^4$ such that $G(v) = 0$ and $v \neq 0$. Then $G(u_1 - \lambda v) = G(u_1) - 2\lambda\hat{G}(u_1,v)$. Since $\hat{G}(u_1,v) \neq 0$ by lemma 3.1, λ may be chosen so that $G(u_1 - \lambda v) = 0$. In this case let $w = u_1 - \lambda v$. Then λ is non-zero and $\hat{G}(u_1,v) = \hat{G}(\lambda v+w,v) = \hat{G}(w,v) = \frac{1}{2\lambda} G(\lambda v+w) = \frac{1}{2\lambda} G(u_1) = \frac{1}{2\lambda} G(Lu_1) = \frac{1}{2\lambda} G(\lambda Lv+Lw) = \hat{G}(Lw,Lv) = \hat{G}(\lambda Lv+Lw,Lv) = \hat{G}(Lu_1,Lv)$. Thus $\hat{G}(u_1,v) = \hat{G}(Lu_1,Lv)$ for any v such that $G(v) = 0$.

Now let p be an arbitrary element of $\mathbb{R}^4$. Consider $G(p-\lambda u_1) = G(p) - 2\lambda\hat{G}(u_1,p) + \lambda^2 G(u_1)$. As a polynomial in λ this expression has discriminant equal to $4(\hat{G}(u_1,p))^2 - 4G(u_1)G(p)$. By lemma 5.1 this is non-negative. Hence, $G(p-\lambda u_1) = 0$ for a suitable choice of λ. Let $v = p - \lambda u_1$ for such a λ. Then $G(p) = G(\lambda u_1+v) = 2\lambda\hat{G}(u_1,v) = \lambda^2 G(u_1) + 2\lambda\hat{G}(u_1,v) = \lambda^2 G(Lu_1) + 2\lambda\hat{G}(Lu_1,Lv) = G(\lambda Lu_1+Lv) = G(Lp)$. Thus L is a Lorentz transformation.

Hence, $f = f(0) + \lambda L$ is in the group generated by translations ($f(0)$), dilatations (λ), and Lorentz transformations (L).

References

[1] Barucchi, G., "The group of timelike permutations", Nuovo Cimento 55A(1968), 385-395.

[2] Barucchi, G., and G. Teppati, "On lemma 3 of Zeeman", Nuovo Cimento 51A(1967), 529-532.

[3] Einstein, A., "On the electrodynamics of moving bodies", translation in The Principle of Relativity, A. Sommerfeld, ed., Dover, New York, 1923.

[4] Gans, D., Transformations and Geometries, Appleton-Century-Crofts, New York, 1969.

[5] Pimenov, R. I., Kinematic Spaces, (translation from Russian), Consultants Bureau, New York, 1970.

[6] Rothaus, O. S., "Order isomorphisms of cones", Proceedings AMS 17 (part 2) (1966), 1284-1288.

[7] Zeeman, E. C., "Causality implies the Lorentz group", J. Math. Phys. 5(1964), 490-493.

[8] ____________, "The topology of Minkowski space", Topology 6 (1967), 161-170.