ASYMPTOTIC STABILITY FOR A CLASS OF FUNCTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT

The equations studied have the form

$$\dot{u} = -mg(u(t)) - \int_0^t a(t-\tau)g(u(\tau))d\tau + £(t,u(t)), \qquad m \; 2 \; 0$$

on a Hilbert space W. g can be nonlinear and unbounded. The case $m = 0$ was studied earlier by the author (Rep. 70-10 and 71-24). a is to be a strongly positive kernel in the terminology of the earlier reports. It is shown that if g is a nonlinear elliptic operator of a special type and $f = f_0 + f_1 + p(t)$, $p(t)$ -* 0 as t-*oo, the equation has a generalized solution and that this solution has a finite limit as t-x©. It is shown that the provisional asymptotic stability results of the earlier reports can be extended to a larger class of perturbations f when $m > 0$ and the results are compared to the differential equation

$$\dot{u} = -mg(u(t)) + f(t,u(t)) .$$

Implications in the theory of approach to steady state are discussed.

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1. Introduction.

This paper concerns functional differential equations of the form,

$$(1.1) \qquad \dot{u}(t) = -T_m(g(u))(t) + f(t,u(t)), \qquad t > 0, \; m \geq 0,$$

on a Hilbert space $\mathcal{H}$. Here $g: S_g \to W$ may be unbounded and nonlinear, f is a map from $[0,\infty) \times \mathcal{S}_g$ into H and T_m denotes the linear Volterra operator,

$$(1.2) \qquad (T_m\zeta)(t) = m\zeta(t) + \int_0^t a(t-\tau)\zeta(\tau)d\tau,$$

on R^1. This study is intended as a contribution to the investigation of systems which possess a "memory" and hence depend on their past history.

The special case, $W = R^1$, $m = 0$, of this equation has been studied in great detail by Levin and Nohel [6], Hannsgen [5] and many others. The object has been to determine conditions on a, g and f which guarantee asymptotic stability of solutions. The physical interpetation of this is that one is looking for systems which possess an internal damping mechanism.

The author and James Wong [8] have obtained some rather natural extensions of the results of [6], that is $m = 0$, to Hilbert spaces W, with unbounded g's. The goal was to be able to treat g's which are partial differential operators thus allowing one to consider, for example, the mechanics of continuous media. Unfortunately the results of [8] are of a conditional nature. They state that <u>if</u> $g(u(t))$ is weakly bounded and weakly uniformly continuous <u>then</u> $g(u(t))$ tends weakly to zero as t approaches infinity. We call this <u>provisional asymptotic</u> stability. For partial differential operators it proved possible to verify the boundedness and continuity hypotheses only for linear equations.

In [7] the author pursued the asymptotic stability question, for $m = 0$, when f has the special form

$$(1.3) \qquad f(t) = f_1 t + f_0 + p(t), \qquad p \to 0 \text{ as } t \to \infty .$$

The object was the study of approach to steady state of solutions of the second order equations,

$$(1.4) \qquad \ddot{u}(t) = \frac{d}{dt} T_0(g(u))(t) + f_1 + \mu(t) \qquad \mu \to 0 \text{ as } t \to \infty .$$

The result, under essentially the same conditions as in [8], was that if $a(t) \to a(\infty) > 0$ as $t \to \infty$, then (provisionally) $g(u(t)) \to a(\infty)^{-1} f_1$ weakly.

The preceding result indicates that the integral term does indeed provide a damping mechanism to systems governed by equations of the form (1.4) <u>provided</u> that a has a non-zero limit at infinity. The implications in the study of elastic materials with memory are discussed in [7]. Interest in the result is enhanced by the fact, [7], that the steady state limit, when $a(\infty) > 0$, is the same as that obtained by quasi-static theory in which one merely drops the "acceleration" $\ddot{u}$ in (1.4). This is a common approximation in engineering since it results in simpler problems.

When $m = 0$ and g is unbounded, equation (1.1) possesses a second defect in addition to the provisional nature of the stability results. This is that existence of solutions is very hard to establish. See [4] for a linear case.

Equation (1.1) with $m > 0$ represents an attempt to create a model for systems with memory which retains the damping properties described above but for which the two defects are partially remedied. In section 4 we consider (1.1), with $m > 0$, when g is a non-linear elliptic operator, on a region Ω, of the same type as that used in the "monotone operator theory" [2] of the parabolic differential equation,

$$\dot{u} = -mg(u(t)) + f. \tag{1.5}$$

(The conditions on g are very natural ones in the context of elasticity or heat flow). We show that (1.1) then possesses a (generalized) solution, under essentially the same conditions on a as in [8], provided that f has the form (1.3). We show, moreover, that this solution is unique, provided that m is sufficiently large. In section 5 we show that if m is large enough for uniqueness then solutions satisfy,

$$\lim_{t \to \infty} u(\cdot,t) = g^{-1}(a(\infty)^{-1}f_1) \qquad \text{in } L_2(\Omega). \tag{1.6}$$

Condition (1.6) is weaker than the steady state result when $m = 0$ but is, of course, consistent with it. It is, moreover, again the quasi-static limit obtained by dropping $\ddot{u}$ and $\dot{u}$ in the differentiated version of (1.1). We emphasize that the result here is no longer provisional.

The question of whether the requirement of sufficiently large m is really necessary is most intriguing and remains open.

The monotone operator methods of [2] can be applied here, with only minor modifications, provided one first uses the technical device of inverting the equation (1.1) to solve for $g(u(t))$. The interesting feature is that the necessary requirements on the resolvent kernel in this inversion follow automatically from the conditions perviously imposed on a. These considerations are discussed in section 3.

It is not at all surprising that if we make $m > 0$ in (1.1) we obtain considerable improvement in the provisional asymptotic stability results over the case $m = 0$ of [8]. This means that the class of allowable perturbations f, for which one can obtain provisional asymptotic stability, is larger when $m > 0$ than when $m = 0$. When $m > 0$ this class is essentially the same as that for (1.5), again not a surprising result. We discuss these questions in section 3. We show there that making $m > 0$ enlarges the class f in two ways, by allowing autonomous perturbations ($f = f(u)$), which are not permitted when $m = 0$, and by allowing perturbations which depend on derivatives of u when g is a partial differential operator. Both of these improvements are shared by (1.5). On the other hand the special properties of approach to steady state which are produced by the assumption $a(\infty) > 0$ hold for all $m \geq 0$ and are not shared by (1.6) for any m.

The equation of nonlinear heat flow is a special case of (1.6). Similarly equations of the form (1.1) can be viewed as a highly specialized case of the equation of heat flow in a rigid heat conductor exhibiting memory [1]. We have indicated that one case of (1.4) yields a model in elastic materials with memory. The differentiated version of (1.1) yields another model for the same subject. Although the general dynamic characteristics of the two elasticity models are very different the results here show that any experiment which is based on approach to steady state will not be able to distinguish between them.

2. Positive Kernels and Inversion.

We want to discuss here properties of the kernels a in the operator T_m,

$$T_m(\zeta)(t) = m\zeta(t) + \int_0^t a(t-\tau)\zeta(\tau)d\tau. \tag{2.1}$$

For $T > 0$ and $\zeta \in C[0,T]$ we set,

$$Q_a[\zeta;T] = \int_0^T \zeta(t)\int_0^t a(t-\tau)\zeta(\tau)d\tau. \tag{2.2}$$

Definition 2.1. a is a positive kernel if $Q_a[\zeta;T] \geq 0$ for all T.

Definition 2.2. a is a strongly positive kernel if there exist constants $\epsilon > 0$, $\alpha > 0$ such that $a(t) - \epsilon e^{-\alpha t}$ is a positive kernel.

The concept of strong positivity was introduced in [8]. Both definitions can be extended to Hilbert spaces, with a replaced by a family $A(t)$ of bounded linear operators and $e^{-\alpha t}$ replaced by a contractive semi-group. This general theory of [8] could be carried through here also but, for brevity, we consider only the case where $A(t) = a(t)I$, I the identity. Then for a positive we have, on any Hilbert space $\mathcal{H}$,

$$(2.3) \quad Q_a[\zeta;T] = \int_0^T (\zeta(t), \int_0^t a(t-\tau)\zeta(\tau)d\tau)dt \geq 0 \qquad \text{for all } \zeta \in C[0,\tau;\mathcal{H}],$$

with a similar result for strong positivity.

We introduce another concept from [8].

Definition 2.3. Let $\mathcal{H}$ be a Hilbert space. Then $u \in C([0,\infty);\mathcal{H})$ is called weakly stable if for every $\eta \in \mathcal{H}$ the function $(u(t),\eta)$ is uniformly bounded and uniformly continuous on $[0,\infty)$.

The basic result from [8] which connects these concepts with asymptotic stability is the following.

Theorem 2.1. *Let $\mathcal{H}$ be a Hilbert space and a be strongly positive. Suppose $u(t)$ is weakly stable. If $Q_a[u;T]$ is bounded independently of T then $u(t)$ must tend weakly to zero as t tends to infinity.*

It is shown in [8] that positivity is closely connected to the Laplace transform of a. We impose on a the following conditions:

(2.4) $$a(t) = a(\infty) + b(t), \quad a(\infty) \geq 0, \quad b \in L_1(0,\infty),$$

(2.5) $$a \in C^{(2)}[0,\infty), \quad a^{(k)} \in L_1(0,\infty) \qquad k = 1,2.$$

It follows from these that a has a Laplace transform $\hat{a}$ satisfying,

(2.6) $$\hat{a}(s) = \frac{a(\infty)}{s} + \hat{b}(s), \quad \hat{b} \text{ continuous in } \operatorname{Re} s \geq 0 \text{ and analytic in } \operatorname{Re} s > 0,$$

(2.7) $$\hat{a}(s) = a(0)s^{-1} + \dot{a}(0)s^{-2} + o(s^{-2}) \qquad \text{as } s \quad \infty \quad \text{in } \operatorname{Re} s \geq 0.$$

The following results are proved in [8]*.

Lemma 2.1. (i) a *is positive if* $\operatorname{Re} \hat{a}(s) > 0$ *in* $\operatorname{Re} s > 0$. (ii) a *is strongly positive if* $\dot{a}(0) < 0$ *and* $\operatorname{Re} \hat{a}(s) > 0$ *in* $\operatorname{Re} s \geq 0$. (iii) a *is strongly positive if* $\dot{a} \not\equiv 0$ *and* $(-1)^k a^k(t) \geq 0 \qquad k = 0,1,2.$

Our main concern in this section is with the inverse T_m^{-1} of the operator T_m when $m > 0$. It is a standard result that T_m^{-1} exists if $m > 0$ and has the form:

(2.8) $$T_m^{-1}(h)(t) = \frac{1}{m}h(t) + \int_0^t k(t-\tau)h(\tau)\,d\tau.$$

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*The condition in (iii) implies those in (i) and (ii) but the converse does not hold, [8].

We strengthen (2.5) to,

(2.5)' $aeC^3[O,c3D)$ $a^{(k)} el\wedge(0, OD)$ $k = 1,2,3,$

and we require,

(2.9) $(-l)^k a^{(k)}(0) > 0$ $k = 0,1,2.$

We set $d(s)$ equal to the transform of $a^{\bullet}$ that is $s\hat{a}(s) - a(0)$.

Theorem 2.2. Suppose (2.4), (2.5)^T and (2.9) hold and that,

(2.10) $\text{Re}\ \hat{a}(s) > 0,$ $\text{Re}\ d(s) < 0$ in $\text{Re}\ s \geq 0.$

Then $\dot{k}$ jjs a strongly positive kernel and

(2.11) $k(0) = -a(0)/m^2.$

Remark 2.1. It follows from Lemma 2.1, and its footnote, that the hypotheses of Theorem 2.2 will be satisfied if $(-1)^{k} a^{(k)}(t) \geq 0$, $k = 0,1,2,3.$ It is interesting to note that this was the hypothesis used in [6] . It was only later that it was shown in [5] that these could be weakened to $k = 0,1,2.$

Proof; We solve $T(£) = h$ by Laplace transforms. The result is,

(2.12) $\hat{£} = (m+\hat{a})"^1\hat{!}! = m"^1\hat{h} + \hat{k}\hat{h},$ $\hat{k} = (m+\hat{a}) - m"^1.$

It is readily checked that $\hat{k}$ exists in $\text{Re } s \geq 0$ and is analytic in $\text{Re } s > 0$. (It has a removable singularity at $s = 0$ if $a(\infty) \neq 0$. (2.6) yields,

$$\hat{k}(s) = -\frac{a(0)}{m^2}\frac{1}{s} + O\left(\frac{1}{s^2}\right) \quad \text{as} \quad s \to \infty. \tag{2.13}$$

k may be recovered by the complex inversion formula and (2.11) follows.

Condition (2.5)' yields, in analogy to (2.7),

$$\hat{a}(s) = a(0)s^{-1} + \dot{a}(0)s^{-2} + \ddot{a}(0)s^{-3} + o(s^{-3}) \quad \text{as} \quad s \to \infty. \tag{2.14}$$

From this we obtain,

$$\hat{\alpha}(s) = s\hat{k}(s) + \frac{a(0)}{m^2} = -\left(\frac{\dot{a}(0)}{m^2} + \frac{a(0)^2}{m^3}\right)\frac{1}{s} - \frac{\Gamma}{s^2} + o\left(\frac{1}{s^2}\right) \tag{2.15}$$

$$\text{as} \quad s \to \infty,$$

where $\Gamma = m^{-3}(\ddot{a}(0) + a(0)^3 - 2a(0)\dot{a}(0))$. Condition (2.9) implies Γ is positive and therefore there exists $\alpha > 0$ and $\epsilon > 0$ such that

$$\text{Re}\left(\hat{\alpha}(s) - \frac{\epsilon}{s+\alpha}\right) > 0, \quad \text{on} \quad s = i\eta \quad \text{for} \quad |\eta| \quad \text{sufficiently large.} \tag{2.16}$$

If we can show that $\text{Re } \hat{\alpha}(i\eta) > 0$ for η in a compact set $|\eta| \leq M$ then it will follow from (2.16) that we can choose α and ϵ so that (2.16) holds for all η. Then the maximum principle implies that (2.16) holds for $\text{Re } s \geq 0$. This shows that $\dot{k} - \epsilon e^{-\alpha t}$ is positive.

We have, by (2.12),

$$s\hat{k}(s) - k(0) = s\hat{k}(s) + \frac{a(0)}{m} = -\frac{s\hat{a}(s)}{m(m+\hat{a})} + \frac{a(0)}{m^2}$$

$$= -\frac{s\hat{a}(s)\,(m+\bar{\hat{a}}(s))}{m(m^2+|\hat{a}|^2) + 2m\,\mathrm{Re}\,\hat{a}(s)} + \frac{a(0)}{m^2}.$$

Hence,

$$\text{(2.17)} \qquad \mathrm{Re}(s\hat{k}(s) - k(0)) = \frac{a(0)}{m^2} - \frac{\mathrm{Re}\,s\hat{a}(s)}{m^2+|\hat{a}|^2 + 2m\,\mathrm{Re}\,\hat{a}(s)} .$$

Now $\mathrm{Re}\,\hat{a}(s) < 0$ implies $\mathrm{Re}\,s\hat{a}(s) < a(0)$, hence, with $\mathrm{Re}\,\hat{a}(s) > 0$, we have,

$$\frac{\mathrm{Re}\,s\hat{a}(s)}{m^2+|\hat{a}|^2 + 2m\,\mathrm{Re}\,\hat{a}(s)} \le \frac{\mathrm{Re}\,s\hat{a}(s)}{m^2} < \frac{a(0)}{m^2}$$

and the right side of (2.17) is positive. This completes the proof of Theorem 2.2.

When we study approach to steady state we will need another result.

Theorem 2.3. <u>Let</u> a <u>satisfy</u> (2.4) <u>and</u> (2.5), <u>with</u> $a(\infty) > 0$, <u>and also the condition</u>,

$$\text{(2.18)} \qquad \int_t^\infty b(\tau)\,d\tau \in L_2(0,\infty) .$$

<u>Let</u> b <u>be positive</u>. <u>Suppose</u> h has the form $h = f_1 + f_2 + p$

where $\rho \in L_2(0,\infty)$. Then, for $m > 0$, $T_m(h)(t) = a(\infty)^{-1} f_1 + r$ where $r \in L_2(0,\infty)$.

Proof; We let $\eta = \xi - a(\infty)^{-1} f_1$ and write the equation $T_m(\xi) = h$ in the form,

$$m\eta(t) + \int_0^t a(t-\tau)\eta(\tau)\,d\tau = c + \gamma(t), \tag{2.19}$$

where,

$$c = f - \frac{m f_1}{a(\infty)} - \frac{f_1}{a(\infty)} \int_0^\infty b(\tau)\,d\tau; \tag{2.20}$$

$$\gamma(t) = \rho(t) + \frac{f_1}{a(\infty)} \int_t^\infty b(\tau)\,d\tau.$$

The hypotheses show that $\gamma \in L_2(0,\infty)$. Multiply (2.19) by $\eta(t)$ and integrate from 0 to T. Since b is positive we obtain then,

$$m\int_0^T \eta^2(t)\,dt + \frac{a(\infty)}{2}\left(\int_0^T \eta(t)\,dt\right)^2 \le c\int_0^T \eta(t)\,dt + \int_0^T \gamma(t)\eta(t)\,dt \tag{2.21}$$

$$\le \frac{c}{2a(\infty)} + \frac{m}{2}\int_0^T \gamma^2(t)\,dt + \frac{2}{m}\int_0^T \eta^2(t)\,dt,$$

and we deduce that,

$$\int_0^T \eta^2(t)\,dt \le M \qquad \text{for all } T.$$

This proves the theorem.

Corollary 2.1. *If the hypotheses of Theorem* 2.3 *hold the function* k *in* (2.8) *belongs to* $L_2(0,\infty)$.

Proof: The hypotheses (2.4) and (2.5) imply that $b \in L_2(0,\infty)$. It is known that k itself satisfies the equation $T_m\zeta = a(t)$, hence the result follows immediately from Theorem 2.3.

This theorem indicates the central role played throughout the study of steady state by the condition $a(\infty) > 0$. If $a(\infty) = 0$ it is easy to see that the result is not true. Notice that the limit of $T_m(h)$ is independent of m in $m > 0$.

3. Provisional Asymptotic Stability.

We consider functional differential equations of the form,

$$(E_m) \quad \dot{u}(t) = -mg(u(t)) - \int_0^t a(t-\tau)g(u(\tau))d\tau + f(t,u(t)), \quad t > 0, m \geq 0,$$

on a Hilbert space $\mathcal{H}$. Here g is a transformation (possibly nonlinear) with domain $\mathcal{D}_g \subset \mathcal{H}$ and f is a mapping from $[0,\infty) \times \mathcal{D}g$ into $\mathcal{H}$. We want to consider simultaneously the differential equation,

$$(\mathcal{E}_m) \qquad \dot{u}(t) = -mg(u(t)) + f(t,u(t)), \qquad t > 0, \ m > 0.$$

Definition 3.1. We say that (E_m) or $(\mathcal{E}_m)$ are *provisionally asymptotically stable* if any solution which is weakly stable

(Definition 2.3) satisfies,

$$\text{weak} \lim_{t \to \infty} g(u(t)) = 0. \tag{3.1}$$

Our object here is to compare the provisional asymptotic stability of (E_m) for $m > 0$ and $m = 0$ and both of these with $(\mathcal{E}_m)$. The results were discussed qualitatively in the introduction. We list various conditions on f which guarantee provisional asymptotic stability in the various cases. We will state the theorems first and give proofs at the end of the section. These proofs are essentially the same as those in [7] and [8] hence we devote most of the section to an analysis of the meaning of the conditions on f. These conditions are stated in a rather tedious manner. The reason for this is that our major interest is the application to partial differential operators; in this context the conditions on f become inequalities between differential operators and these are extremely delicate.

We first list the technical hypotheses on a and g which will be used. It is assumed throughout that a is strongly positive. In addition to this four other hypotheses will sometimes be used; all assume that (2.4) holds:

(A_1) $a(\infty) > 0$

(A_2) $a(\infty) = 0$

(A_3) b is strongly positive,

(A_4) $\beta(t) = \int_t^\infty b(\tau)d\tau \in L_1(0,\infty)$.

Concerning g we assume throughout that there exists a functional $G(u)$, defined on $\mathcal{D}_g$, such that for all $u \in C^{(1)}(0,\infty;\mathcal{H})$ with $u(t) \in \mathcal{D}_g$ for $t > 0$,

$$\frac{d}{dt} G(u(t)) = -(g(u(t)),\dot{u}(t)). \tag{3.2}$$

We will impose one of the following two conditions on G:

(G_1)
$$\inf_{v \in \mathcal{D}_g} G(v) > -\infty$$

(G_1') $\quad G(u) \geq \varphi(\|u\|)\|u\| \quad$ where $\varphi(\xi) \to \infty$ as $\xi \to \infty$.

The perturbations f we consider have the form,

$$f(t,u) = f_1 t + f_0 + h(t,u). \tag{3.3}$$

Here f_1 and f_0 are fixed elements of $\mathcal{H}$ and h will always be subject to a condition of the form:

$(H_{m'})$ $\quad (h(t,u),g(u)) \leq p(t)(1+G(u)) + m'\|g(u)\|^2,$

$$p \in L_1(0,\infty), \quad 0 \leq m' \leq m.$$

f_0 and f_1 will sometimes be required to satisfy the conditions:

(F_0) $\quad (f_0,g) \leq k(1+G(u))$ for some k, $|(f_0,h)| \leq q(t)(1+G(u)),$
$$q \in L_1(0,\infty),$$

(F_1) $\quad (f_1,g) \leq k(1+G(u))$ for some k, $|(f_1,h)| \leq q(t)(1+G(u)),$
$$q \in L_1(0,\infty).$$

We are now ready to state the results. Throughout this statement we use the notation $g(u(t)) \rightharpoonup \gamma$ to mean that *if* $g(u(t))$ is weakly stable *then* $g(u(t))$ approaches γ weakly.

Theorem 3.1. *For equation* $(\mathcal{E}_m)$ *we have:*

(i) $f_1 = f_0 = 0$, (G_1), $(H_{m'})$ for $m' < m$ *imply* $g(u(t)) \rightharpoonup 0$

(ii) $f_1 = 0$, (G_1'), (H_0), (F_0), *imply* $g(u(t)) \rightharpoonup m^{-1} f_1$.

Theorem 3.2. *For equation* (E_m) *we have:*

(i) $f_1 = f_0 = 0$, (G_1), (H_m) *imply* $g(u(t)) \rightharpoonup 0$.

(ii) $f_1 = 0, (G_1)$, (A_2), (A_3), (H_m) *imply* $g(u(t)) \rightharpoonup 0$ (for any f_0)

(iii) $f_1 = 0$, (G_1'), (A_1), (A_4), (H_0), (F_0), *imply*

$$g(u(t)) \rightharpoonup (m + \int_0^\infty b(\tau)d\tau)^{-1} f_0$$

(iv) (G_1'), (A_2), (A_4), (H_0), (F_1), *imply* $g(u(t)) \rightharpoonup a(\infty)^{-1} f_1$

Remark 3.1. It is easy to see, by looking at linear cases on R^1, that if $f_1 \neq 0$ in $(\mathcal{E}_m)$, or in (E_m) when $a(\infty) \neq 0$, then solutions will, in general, grow linearly.

Remark 3.2. Case (iii) of Theorem (3.2) holds for $m = 0$ also provided we assume $\int_0^\infty b(\tau)d\tau = \hat{b}(0) > 0$. Notice that this would follow from (ii) of Lemma 3.1.

Remark 3.3. The theorems show that when $a(\infty) = 0$ equations (E_m) and $(\mathcal{E}_m)$ have essentially the same class of allowable perturbations. Notice, however, that we can take $m = 0$ in (E_m) and still have a class of allowable perturbations while $(\mathcal{E}_m)$ degenerates so as to have none. On the other hand if $a(\infty) > 0$ and either f_0 or f_1 is non-zero then the provisional asymptotic stability properties of (E_m) are clearly much stronger than those of $(\mathcal{E}_m)$. Notice that in case (i) for $(\mathcal{E}_m)$ and cases (i) and (ii) for (E_m) the class of allowable perturbations increases with m. In the remaining cases, however, the result is, at least qualitatively, independent of m.

There are two difficulties in the application of the preceding results. The first is that (3.2) represents a severe restriction on the admissable g's. For scalar equations we can simply take $G(u) = \int_0^u g(\xi)\,d\xi$ but in more than one dimension (3.2) is a real limitation. (See Remark (5.1)). The second difficulty arises from the restrictions $(H_{m'})$, (F_0), (F_1). Let us illustrate with two simple examples.

Example 3.1. $\mathcal{H} = R^1$, $g(u) = u$.

We take $G(u) = \frac{u^2}{2}$ which satisfies (G_1'). Then (H_m) becomes,

$$h(t,u)u \leq \alpha(t)\left(1 + \frac{u^2}{2}\right) + mu^2 . \tag{3.3}$$

If $h(t,u) = \mu u$ then (3.3) will be satisfied if $\mu < m$ provided $m > 0$ but can never be satisfied if $m = 0$.

Example 3.2. $M = L_2(0,1)$, $\mathcal{D}_g = \{w: w_{xx} \in L_2(0,1),\ w(0) = w(1) = 0\}$, $g(u) = -u_{xx}$. Then, for $u \in \mathcal{D}_g$,

$$-\int_0^1 u_t u_{xx} dx = \int_0^1 u_{xt} u_x dx = \frac{1}{2}\frac{d}{dt}\int_0^1 u_x^2 dx = \frac{d}{dt} G(u).$$

Let $h(t,u) = \mu(\varphi(t)u + \psi(t)u_{xx})$. Then (H_m) becomes,

$$(3.4) \qquad (f,g(u)) = -\mu\varphi(t)\int_0^1 u_x^2 dx + \mu\psi(t)\int_0^1 u_{xx}^2 dx$$

$$\leq a(t)\left(1 + \frac{1}{2}\int_0^1 u_x^2 dx\right) + m\int_0^1 u_x^2 dx, \qquad a \in L_1(0,\infty)$$

For $w \in \mathcal{D}_g$ we have the inequality,

$$(3.5) \qquad \int_0^1 u_x^2 dx \leq c \int_0^1 u_{xx}^2 dx \qquad \text{for some } c > 0.$$

If $m > 0$ it follows that (3.4) is satisfied if, for instance $\varphi \in L_1$, and $|\mu| \|\psi\|_\infty < m$. For $m = 0$ however (3.4) can be satisfied only if $\psi = 0$ and $\varphi \in L_1$.

These two examples illustrate that conditions (H_m) gives qualitatively stronger results when $m > 0$ than when $m = 0$. There are two major improvements when $m > 0$:

(1) Autonomous perturbations $h(t,u) = H(u)$ are allowed.

(2) For partial differential operators f can depend on higher order derivatives.

An important special case of (H_m) for $m > 0$ is,

(H_m^*) $\|h(t,u)\| \leq m'\|g(u)\| + \beta(t)$, $m' < m$, $\beta \in L_2(0,\infty)$

(H_m^*) clearly illustrates improvement (1) above. We consider next an example which extends Example 3.2 and which illustrates improvement (2).

Example 2.3. (Linear partial differential functional equations).

Let Ω be a bounded domain in R^n and $\mathcal{H} = L_2(\Omega)$. We introduce the spaces $H_k(\Omega)$ consisting of functions with strong L_2 derivatives up to order k in Ω and set,

$$(3.6) \qquad \|u\|_k^2 = \sum_{|\alpha| \leq k} \int_\Omega (D^\alpha u)^2 dx.$$

(Here $\alpha = (\alpha_1, \ldots, \alpha_n)$, α_k non-negative integers and

$$D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}, \qquad |\alpha| = \sum_{j=1}^{k} \alpha_j\ ).$$

$\overset{o}{H}_k(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in H_k.

We let g be a linear elliptic operator of order $2M$ of the form,

$$(3.7) \qquad g(u) = \sum_{\substack{|\alpha| \leq M \\ |\beta| \leq M}} D^\alpha (a_{\alpha\beta} D^\beta u) \qquad a_{\alpha\beta} = a_{\beta\alpha},$$

with smooth coefficients a_{Q0}. For $\&_g$ we take H_{2M} D H_M^O. This means we are imposing Dirichlet boundary conditions of the form,,

(3.8) $\quad D^a u = 0 \quad$ on an $\quad |a| \leq M-l.$

We assume that the coefficient of $|l^u|\!\!\!\!\urcorner_0$ in Garding's inequality is zero, that is, if

(3.8) $$B(u,v) = \sum_{\substack{|a| \leq M \\ \leq M}} \int_Q D^Q u \, a_{\alpha\beta} D^{\beta} v \, dx ,$$

then

(3.9) $\quad B(u,u) \geq c||u||\text{^}, \quad c > 0$ for all $u \in H^{\circ}_{M}$.

(Condition (3.9) guarantees that the problem $g(u) = f$ has a unique solution $u \in \&_g$ for all $f \in L_2(Q)$, see [3]).

This situation was discussed in [8] where it was pointed out that a suitable $G(u)$ is,

(3.10) $$G(u) = \frac{1}{2} B(u,u) .$$

(3,9) shows that (G_1) is satisfied. We consider perturbations of the form,

(3.11) $\quad f(t,u) = f_x(x)t + f_Q(x) + h(t,x,D\text{^}u), \quad |_y| \leq 2M.$

Suppose that h is sublinear in the variables D^u that is satisfies,

(3.12) $$|h(t,x,\pounds^y)| \; 1 \; ju \sum_{j=0}^{2M} \sum_{|y|=j} \backslash i^7\backslash + J(t)$$

where $J(t) \in L_2(0,\infty; L_2(\Omega))$. Then the following result from the theory of partial differential equations [3] shows that (H_m^*) will be satisfied if μ is sufficiently small.

<u>Lemma</u> 3.1. <u>If</u> g <u>satisfies</u> (3.9) <u>there exists a constant</u> $c' > 0$ <u>such that</u>,

$$\|g(u)\|_0^2 \geq c'\|u\|_{2M}^2. \tag{3.13}$$

When $m = 0$ we cannot use this idea and must work directly with (H_0). This requires that

$$(h,g) = \sum_{\substack{|\alpha|\leq M \\ |\beta|\leq M}} \int D^{\alpha}h(t,x,D^{\alpha}u)D^{\alpha}u\,dx \tag{3.14}$$

be estimated by $B(u,u)$. However (3.9) shows that $B(u,u)$ is $O(\|u_m\|)$ hence (3.14) shows that we cannot allow h to depend on $D^{\gamma}u$ for $\gamma \neq 0$. It is clear that (H_0) will be satisfied if, for instance, $h = h(t,u)$ with

$$|h_u(t,\xi)| \leq \varphi(t), \qquad \varphi \in L_1(0,\infty) \quad \text{for all } \xi. \tag{3.15}$$

These considerations show that the result of Example 3.2 is quite general.

Let us consider the meaning of conditions (F_0) and (F_1) in the present example. We have, (3.16) $$(f_o,g) = \sum_{\substack{|\alpha|\leq M\\|\beta|\leq M}} \int_\Omega D^\alpha f_O a_{\alpha\beta} D^\beta u \, dx,$$

hence by 3.9, the first of conditions (F_O) is satisfied if $f_Q \in H_m(Q)$; similarly for the first of $(F_\pm)$. Note that (F_Q) and $(p_\perp)$ are to be applied only in the case when h satisfies (H_O). Hence we assume as above that h = h(t,x,u). Then the second of conditions (F_O) is satisfied if,

(3.17) $$|h(t,x,u)| \leq \psi(t)\,(1+|u|) \qquad \psi\in L_1(0,\infty),$$

and $f_O \in L_2(0)$. Similarly for $(F_\perp)$. Thus the results of Theorems 3.1 (ii) and 3.2 (iii) and (iv) are applicable if h satisfies (3.15) and (3.17) and f_O (or $f_1) \in H_m(fi)$.

<u>Remark</u> 3.4. It is shown in [8] that, in this case, weak convergence of g(u(t)) to *y* in $L_2(Q)$ implies strong convergence of g(u(t)) to *y* in $\overset{O}{H}_m(Q)$.

<u>Example</u> 3.4. (A nonlinear partial differential functional equation).

Let $Jt = L_\infty(0,1)$ and take $g(v) = -\frac{\partial}{\partial x} a(v_x)$, with $v(0) = v(l) = 0$. It is shown in [8] that an appropriate G in this case is,

(3.18) $$G(u) = \int_0^1 \int_0^{u_x} \sigma(\xi)\, d\xi.$$

We impose the condition a' (£) $\geq$, € > 0. Then (3.18) yields,

$$(3.19)\qquad G(u) \geq L_Z \int_0^1 u^2_x dx \geq c \int_0^1 u^2 dx, \qquad \text{for} \quad u(0)=u(l)=0.$$

Hence (G_1) is satisfied. The condition $o^1 \geq e$ also yields,

$$(3.20)\qquad ||g(u)||^2 = \int_0^1 a' (u_x)^2 u^2_{xx}\, dx \geq e \int_0^1 u^2_{xx}\, dx \geq e'N|^2.$$

We consider perturbations of the form,

$$f(t,u) = f_x(x)t + f_Q(x) + h(t,x,u,u_x,u_{xx})^2.$$

Suppose in analogy to (3.12),

(3.2D $h(t,K,i_o,i_r i_2)$ i $M(k_0 l^4 |\xi_1| + |\xi_2|) + J(t,x)$

$$\int_0^1 J^p dx \; G \; L^0_2 (\mathcal{O},\infty).$$

A few calculations based on (3.20),will show one that estimates of the form (H^*_m) or (H_O) are not possible unless *a* is actually sublinear in u_x (a^1(£) must be bounded). Hence if one wants to apply the results of Theorems 3.1 and (3.2) for a genuinely non-linear *a* one must be content with the case h = 0. The question is whether (F_Q) and (F_1) can ever be satisfied with non-linear g's. We have,

$$(3.22)\qquad (f_o,g) = \int_0^1 \frac{\partial f_O}{\partial x} \sigma(u_x)\, dx$$

Suppose σ satisfies a condition of the form $|\sigma(\xi)| \leq K'(1+|\xi|^r)$ $1 < r < 2$. Then by Hölder's inequality,

$$(3.23) \qquad (f_0, g) \leq K'\left(\|f_0\|_1 + \left(\int_0^1 \left|\frac{\partial f_0}{\partial x}\right|^q\right)^{1/q} \|u\|_1\right), \qquad q = \frac{2}{2-r} > 2.$$

Hence (F_0) is satisfied if $\int_0^1 \left|\frac{\partial f_0}{\partial x}\right|^q dx$ is bounded.

Remark 3.5. This example, and Theorem (3.2), show that for this special g the equation (E_m) for $m > 0$ provides no change in the results for $m = 0$. The latter was discussed in [7] together with an application to elasticity. The equation for $m > 0$ yields a new model for the elasticity problem but this new model yields the same results on approach to steady state. The importance of the case $m > 0$ is that in the next two sections we will demonstrate that the conditional results of this example can then be made rigorous.

Remark 3.6. Equations (3.13) and (3.20) illustrate an essential limitation to our methods. They represent a priori bounds for differential operators. The best one can ever expect in this direction, for an operator of order $2m$ is $\|g(u)\| \geq c\|u\|_{2m}$. This fact will always limit the theory to linear perturbations h.

Proofs of Theorems 3.1 and 3.2.

(3.1)(i) Multiply the equation by $g(u(t))$ and integrate from 0 to T. (3.2) and $(H_{m'})$ yield,

$$(3.23)\quad G(u(T)) + (m-m')\int_0^T \|g(u(t))\|^{-2}dt \le \int_0^T p(t)\,(1+G(u(t))dt + G(u(0)).$$

It follows that $G(u(T))$ is bounded for all T and $\|g(u(t)\| \in L_2(0,\infty)$. Thus for any $v \in H$ $(g(u(t),v) \in L_2(0,\infty)$ and, if this quantity is uniformly continuous, it must tend to zero. (Note that, here strong uniform continuity of $g(u(t))$ implies strong convergence to zero.

(3.2)(i),(ii). The proofs of these are essentially the same as in [8]. Again one multiplies by $g(u(t))$ and integrates from 0 to T. In case (i)(3.2) and H_m imply that $G(u(T))$ and $Q_a[g(u);T]$ are uniformly bounded the result then follows from Theorem 2.1. In case (ii) one obtains, by (3.2) and (H_m),

$$G(u(T)) + Q_b[g(n);T] \le \frac{a(\infty)}{2}\|f_0\|^2 + \int_0^T b(t)\,(1+G(u(t))dt + G(u(0)).$$

Hence $G(u(T))$ and $Q_b[g(u);T]$ are bounded and the result follows from Theorem 2.1.

The remaining cases can be reduced to ones already treated. It is here that condition (G_1') enters. We have the obvious result:

(3.24) If (G_1') is satisfied and $\tilde{G}(u) = G(u) - (\beta,u)$ then $(1+G(u)) \le K(1+\tilde{G}(u))$ for some constant K.

(3.1)(ii) Write the equation as,

$$(3.25)\qquad \dot{u} = -m\,\tilde{g}(u(t)) + h(t,u), \qquad \tilde{g}(u) = g(u) - \frac{f_0}{m}.$$

Then we have $\tilde{G}(u) = g(u) - (\frac{f_0}{m}, u)$. By (G_1'), $\tilde{G}$ satisfies (G_1). From (H_0), (3.24) and (F_0) we have,

$$(3.26) \quad (h,\tilde{g}) = (h,g) - \frac{1}{m}(h,f_0) \leq p(t)(1+G(u)) + \frac{1}{m} q(t)(1+G(u))$$
$$\leq K(p(t) + q(t))(1+\tilde{G}(u)).$$

Hence the result follows from 3.1(i).

(3.2)(iii). Let $\Gamma = (m + \int_0^\infty b(\tau)d\tau)^{-1} f_0$ and write the equation as,

$$(3.27) \quad \dot{u} = -m\tilde{g}(u(t)) - \int_0^t a(t-\tau)\tilde{g}(u(\tau))d\tau + h + \Gamma\beta(t),$$
$$\tilde{g}(u) = g(u) - \Gamma$$

We have, by (H_0), (F_0), (A_4), (G_1') and (3.24),

$$(3.28) \quad (h + \Gamma\beta(t), \tilde{g}) = (h,g) + (\beta(t)\Gamma, g) - (h,\Gamma) - \beta(t)\|\Gamma\|^2$$
$$\leq p(t)(1+G(u)) + \beta(t)(m + \int_0^\infty b(\tau)d\tau)^{-1}|(f_0,g)|$$
$$+ (m + \int_0^\infty b(\tau)d\tau)^{-1}|(h,f_0)| + \beta(t)\|\Gamma\|^2$$
$$\leq p(t)(1+G(u)) + \beta(t)K'(1+G(u)) + K''q(t)(1+G(u)) + p(t)\|\Gamma\|^2$$
$$\leq r(t)(1 + \tilde{G}(u))$$

Then the result follows from (3.2)(i).

(3.2.(iv) Write the equation as,

$$(3.29)\quad \dot{u}(t) = -m\tilde{g}(u(t)) - \int_0^t a(t-\tau)\tilde{g}(u(\tau))d\tau + \tilde{f}_0 + \tilde{h},$$

where $\tilde{g}(u) = g(u) - \frac{f_1}{a(\infty)}$, $\tilde{f}_0 = f_0 - \frac{f_1}{a(\infty)}\int_0^\infty b(\tau)d\tau$,

$\tilde{h} = h + \frac{f_1}{a(\infty)}\beta(t)$.

Then one checks, by calculations like those above, that the hypothesis of case 3.2(iii) are satisfied. Hence the result follows from that case.

4. <u>Existence Theory</u>.

We consider equations of the form,

$$(4.1)\qquad u_t(x,t) = T_m(g(u)(x,\cdot))(t) + f(x,t)$$

with the initial condition,

$$(4.2)\qquad u(x,0) = 0.$$

Here x lies in a bounded domain of R^n and g is a non-linear partial differential operator of the form,

$$(4.3)\qquad g(u)(x,t) = \sum_{|\alpha|\le m}(-1)^{|\alpha|}D^\alpha A_\alpha(x,D^\gamma u(x,t)),\qquad |\gamma| \le m.$$

u is to satisfy the Dirichlet boundary conditions*,

(4.4) $D^{a}u = 0$ on ad, $|a| \leq m-1$.

We assume that the hypotheses of Theorem (2.1) are in effect. Then we may invert equation (4.1) and if we integrate by parts in the resulting equation we obtain,

(4.5) $\hat{m}\, u_{,t}(x,t) + L(x,u)(x,t) + \int_0 \dot{k}(t-T)u(x,T)dT = cp(x,t),$

where,

(4.6) $Lu = g(u) - M;$ $cp(x,t) = T^{-1}(f(x,\cdot))(t),$

with $3 = \frac{k(0)}{m} > 0$ and k strongly positive.

Our main object is to establish the existence of a generalized solution of (4.6) under certain conditions on g. We will see, (Remark 4.4) that this solution can also be considered giving a generalized solution of (4.1). The solutions are to reflect the boundary conditions (4.4) while allowing for algebraic non-linearities in g. Appropriate for this are the Sobolev spaces $\bar{W}_p^m(Q)$ consisting of functions w with generalized derivatives $D^{\alpha}w$ up to order m belonging to L_p. We set

---

*Inhomogeneities in either (4.2) or (4.4) can also be treated in a standard way.

$$\|w\|_{p,m}^{p} = \sum_{|\alpha|\leq m} \int_{\Omega} |D^{\alpha}w|^{p}dx. \tag{4.7}$$

$\overset{o}{W}{}_p^m$ consists of the completion of $C_0^{\infty}(\Omega)$ in the norms (4.7). $\overset{o}{W}{}_p^{-m}$ consists of the dual of $\overset{o}{W}{}_p^m$. We will use also the space $L_p[0,T; \overset{o}{W}{}_p^m]$ $T > 0$, of functions $u: [0,T] \to \overset{o}{W}{}_p^m$ which are measurable with $\|u\|_{p,m}(t) \in L_p[0,T]$, and also $L_q[0,T; \overset{o}{W}{}_p^{-m}]$.

In order to avoid certain technical complications we will assume throughout that $p \geq 2$. We impose on the A_{α} the growth conditions,

(I) $\qquad |A_{\alpha}(\xi_{\gamma})| \leq K_1 (\sum_{|\gamma|\leq m} |\xi_{\gamma}|^{p-1} + 1)$ for some constant k_1.

Observe that $p \geq 2$ implies that the A_{α}'s are of at least linear growth. We set,

$$\begin{aligned} \mathcal{L}(u,v) &= \sum_{|\alpha|\leq m} \int_{\Omega} A^{\alpha}(D^{\gamma}u)D^{\alpha}v \, dx \\ L(u,v) &= L(u,v) - \beta \int_{\Omega} u\,v\,dx \end{aligned} \tag{4.8}$$

It follows from (I) and Hölder's inequality that $\mathcal{L}$ satisfies,

$$|L(u,v)| \leq C_1\|u\|_{m,p}^{p/q} \|v\|_{m,p}, \qquad \frac{1}{p} + \frac{1}{q} = 1. \tag{4.9}$$

This means that $\mathcal{L}$ takes $u \in L_p[0,T; \overset{o}{W}{}_p^m]$ into an element of $L_q[0,T; \overset{o}{W}{}_p^{-m}]$ (as a functional of v). We write, (4.10) $\qquad \mathcal{L}[u,v] = \int_0^T \mathcal{L}(u,v)\,dt, \qquad u,v \in L_p[0,T;\ \overset{o}{W}{}_p^m].$

Now we consider the expression $I(u,v)$ defined by,

(4.11) $$I(u,v) = \int_0^t \dot{k}(t-\tau) \int_\Omega v(x,t)u(x,\tau)\,dx\,d\tau.$$

We have, by Hölder's inequality,

(4.12) $$|I(u,v)| \le C_2\|v\|_{p,m}(t)\left(\int_0^t \|u\|_{p,m}^p(\tau)\right)^{1/pq}$$

Thus I acts as a mapping from the function space $L_p[0,T;\ \overset{o}{W}{}_p^m]$ into $L_q[0,T;\ \overset{o}{W}{}_p^{-m}]$. (Notice that I is <u>not</u> a map from $\overset{o}{W}{}_p^m$ into $\overset{o}{W}{}_p^{-m}$ as is $\mathcal{L}$. We write,

(4.13) $\qquad I[u,v] = \int_0^T I(u,v)\,dt; \qquad u,v \in L_p[0,T;\ \overset{o}{W}{}_p^m].$

We are now ready to define our generalized solutions.

<u>Definition 4.1</u>. Let $\varphi \in L_q[0,T;\ \overset{o}{W}{}_p^{-m}]^*$. Then a <u>generalized solution</u> of (4.5), (4.4), (4.2) is a function $u \in L_p[0,T;\ \overset{o}{W}{}_p^m]$ such that,

(i) u has a derivative $u_t \in L_q(0,T;\ \overset{o}{W}{}_p^{-m}]$

(ii) $[u_t,v] + L[u,v] + I[u,v] = [\varphi,v]$ for all $v \in L_p[0,T;\ \overset{o}{W}{}_p^m]$,

(iii) $\lim_{t\to 0} \int_\Omega u^2(x,t)\,dx = 0.$

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*This will be true, in the present case, if f in (3.1) $\in L_2((0,T);\ L_2(\Omega))$, for this property is preserved under T_m^{-1}. Hence $\varphi \in L_2((0,T);\ L_2(\Omega))$ but then $\int \varphi\, v\,dx \in W_q[0,T;\ \overset{o}{W}{}_p^{-m})$ since $q \le 2$.

Remarks 4.1. (a) For $\phi \in L_p[0,T; \overset{\circ}{W}{}^m_p]$ and $X \in L_q[0,T; \overset{\circ}{W}{}^{-m}_q]$, $[\chi;\phi]$ denotes $\int_0^T \langle \chi, \phi \rangle dt$. We indicate in the outline of the proof the sense in which (i) and (ii) are to be interpreted, (b) (iii) may, of course, require redefining u on a set of measure zero.

We obtain an existence theorem if we impose conditions on g which make L monotone and positive. These conditions are:

(II) $\quad \mathcal{L}(u,u) \geq a_0 ||u||^p_{m,p}$ for all $u \in S_1$, $a_0 > 0$;

(III) $\quad \mathcal{L}(u,u-v) - \mathcal{L}(v,u-v) \geq a_1 ||u-v||^p_{m,p}$, $a_1 > 0$, for all $u, v \in \overset{\circ}{W}{}^m_p$.

Remark 4.2. In usual terminology $\mathcal{L}$ would be called monotone if the right side of III were replaced by zero. This is not enough for us because of the presence of the term $\beta \int_G uv\,dx$ in L (see Lemma 4.1 below). We need the stronger condition III. Notice that if $A_\alpha(0) = 0$ then (III) implies (II). It is shown in [2] (lemma 1) that in any case (III) implies

(II*) $\quad \mathcal{L}(u,u) \geq a_0 ||u||^p_{m,p} - k$ for some k

It will be clear from the calculations following that (II*) suffices for our purposes hence it is enough to require (III).

Remark 4.3. Conditions (II) and (III) are very restrictive. We discuss equations for which they hold at the end of this section. They are the conditions given in [2] under the name ellipticity and strong ellipticity. They can, presumably, be weakened in the same way as in [2].

Theorem 4.1. *Suppose* (II) *and* (III) *hold*. *Then* (4.5) *has a generalized solution*. *This solution is unique if* m *is sufficiently large*.

Outline of Proof: The proof involves only minor variations of the proof of Theorem 9 (or Theorem 13) of [2] hence we present only an outline. We begin with a lemma which establishes certain positivity and monotonicity requirements.

Lemma 4.1. *Let* L *satisfy* (II) *and* (III). *Set* $W(u,v) = L(u,v)+I(u,v)$ *and* $W[u,v] = L[u,v] + I[u,v]$. *Then we have*,

(II') $$W[u,u] \geq a_0' \int_0^T \|u\|_{m,p}^p dt - K_1 ,$$

(III') $$W[u,u-v] - W[v,u-v] \geq a_1' \int_0^T \|u\|_{m,p}^p dt - K_2 ,$$

for constants $a_0' > 0$, $a_1' > 0$, K_1 *and* K_2, *for all* $u,v \in L_p[0,T;\overset{o}{W}{}_p^m]$. *If* m *is sufficiently large we can take* K_1 *and* $K_2 = 0$.

Proof: Consider first the expression $I[u,u]$. We have,

(4.14) $$I[u,u] = \int_\Omega \int_0^T u(x,t) \int_0^t \dot{k}(t-\tau)u(x,\tau)d\tau \, dt \, dx \geq 0$$

since $\dot{k}$ is positive and $W[u,u-v] - W[v,u-v] = W[u-v,u-v] \geq 0$. Hence it suffices to consider $L[u,v]$. We have,

$$L[u,u] = \mathcal{L}[u,u] - \beta \int_0^T \|u\|_{0,2}^2 dt. \tag{4.15}$$

The problem is that β is positive. If m is so large that $\beta < \min(a_0, a_1)$ we see that (II') and (III') hold from (II) and (III). Otherwise we have, by Young's inequality,

$$\|u\|_{0,2} \leq \epsilon \|u\|_{0,s} + K(\epsilon), \tag{4.16}$$

for any ϵ and any $s > 2$, where K is a constant depending on s and on the domain Ω. It follows from the embedding theorems for Sobolev spaces, [3], that,

$$\|u\|_{0,s} \leq C\|u\|_{m,p} \qquad \text{if} \quad s \leq \frac{1}{\frac{1}{p} - \frac{m}{n}} \tag{4.17}$$

if $\frac{1}{p} - \frac{m}{n} \geq 0$ and for all s if $\frac{1}{p} < \frac{m}{n} \leq 0$. Since $p \geq 2$ and $\frac{1}{\frac{1}{p} - \frac{m}{n}} > p$ we see that we can always choose an s, $2 < s$, such that $\|u\|_{0,s} \leq C\|u\|_{m,p}$. It follows that,

$$\beta\|u\|_{0,2}^2 \leq \frac{a_0}{2} \|u\|_{m,p}^p + K_3 \tag{4.18}$$

for some K_3. Then (II') follows from (II) with $K_1 = K_3 T$. (III') is obtained from (III) in an analogous manner.

We proceed by Galerkin's method. Let $v_1, v_2, \ldots$ be a complete system in $\mathcal{V}_p$ which, for convenience, we assume to be orthonormal in $L_2(\Omega)$. We define approximate solutions u^k,

$$u^k(x,t) = \sum_{j=1}^{k} c_j^k(t) v_j(x),$$

where the c_j are determined by the equations,

$$(4.19) \qquad \int_\Omega \dot{u}^k(x,t) v_j(x)\,dx + W(u^k(\cdot,t), v_j) = \langle h; v_j \rangle$$

for $j = 1,\ldots,k$. Equations (4.19) are of the form,

$$(4.20) \qquad \dot{c}(t) = -\int_0^t k(t-\tau)c(\tau)\,d\tau + f(t,c(t)),$$

on the finite dimensional space R^k with $\dot{k}(t)$ a (strongly) positive kernel on R^k. We have, by (II') and (4.14),

$$(4.21) \qquad (c(t), f(t, c(t)))_{R^k} = -L\Big(\sum_{j=1}^{k} c_j^k(t) v_j, \sum_{j=1}^{k} c_j^k(t) v_j\Big)$$

$$\leq -a_0 \|\textstyle\sum_j c_j^k\|^2 + K_1$$

It follows from the positivity of a then, that the c's are bounded independently of $t \in [0,T]$ and k. The existence of a solution of (4.19) then follows just as in [2] (where 4.19 was an ordinary differential equation.)

If one multiplies (4.19) by $c_j^k(t)$ sums over j, and integrates from 0 to T (the calculation just indicated) one finds as in [2] that,

$$\int_\Omega (u^k(x,t))^2 dx \le M_1 , \tag{4.22}$$

$$\int_0^T \|u^k\|_{m,p}^p dt \le M_2 , \tag{4.23}$$

for all k. It follows that one can extract a subsequence converging weakly in $L_p[0,T; \overset{o}{W}{}_p^m]$ to $u \in L_p[0,T; \overset{o}{W}{}_p^m]$. From (I) and (4.23) one deduces that a further subsequence can be extracted so that the $A_\alpha(D^\gamma u^k)$'s all converge weakly in $L_q[(0,T) \times \Omega]$ to some a_α's. The main problem is to show that $a_\alpha = A_\alpha(D^\gamma u)$ on which we comment later. The validity of (ii) of Definition (4.1) then follows from (4.19) and a passage to the limit through the subsequences (see [2]).

Equations (4.19) yield,

$$\int_0^T \int_\Omega u_t^k \zeta \, dxdt = -W[u^k,\zeta] + \langle h,\zeta\rangle \tag{4.24}$$

for all ζ's which are of the form $\zeta = \sum_1^k c_j(t)v_j(x)$. The fact that the $A_\alpha(D^\gamma u^k)$ converge weakly in $L_q[(0,T) \times \Omega]$ to a_α while the u^k's converge weakly in $\overset{o}{W}{}_p^m$ (hence in $L_2[(0,T)\times\Omega]$) shows that

$$(4.25)\qquad \lim_{k\to\infty}\int_0^T\int_\Omega u^k_t\,\zeta\,dx\,dt = -\int_0^T\int_\Omega a_0(x,t)\,\zeta(x,t)\,dx\,dt$$

$$+\ \beta\int_0^T\int_\Omega u(x,t)\zeta(x,t)\,dx\,dt - \int_0^T\int_\Omega \zeta(x,t)\int_0^t k(t-\tau)u(x,\tau)\,d\tau\,dx\,dt$$

$$+\ \langle h,\zeta\rangle,$$

where the limit is through the subsequence and ζ is any linear combination of terms $c_j v_j$. The right side of (4.25) defines a bounded linear functional of ζ for all $\zeta \in L_p[0,T;\overset{\circ}{W}{}^m_p]$ and this functional is in $L_q[0,T;W^{-m}_q]$. We define u_t* by the right side of (4.25), that is,

$$(4.26)\qquad \int_0^T \langle u_t,\zeta\rangle dt = \text{right side of (4.25) for all } \zeta\in L_p[0,T;\overset{\circ}{W}{}^m_p].$$

We return to (4.24). In this formula let $\zeta(x,t) = u^k(x,t)\,\chi[0,\tau)$, $0 < \tau < T$. We obtain, since $u^k(x,0) = 0$,

$$(4.27)\qquad \int_\Omega (u^k(x,\tau))^2dx = -W[u^k,\zeta] + \langle h,\zeta\rangle$$

---

*Note that for any $\zeta = \sum c_j(t)v_j(x)$ with smooth c_j's such that $c_j(T) = 0$, (4.24) yields, $-\int_0^T\int \zeta_t u^k\,dx\,dt = -W[u^k,\zeta] + \langle h,\zeta\rangle$. Here we can let $k\to\infty$ (through the subsequence) in the term on the left. We compare with (4.26) and obtain, $\int_0^T\langle u_t,\zeta\rangle dt = -\int_0^T\int_\Omega u\,\zeta_t\,dx\,dt$. This justifies the term derivative for u_t.

Now from the boundedness of the $A_\alpha(D^\gamma u^k)$ in $L_q[(0,T)\times\Omega]$ and (4.22) one deduces from (4.27) that,

$$(4.28) \qquad \int_\Omega u^k(x,\tau)^2 dx \leq N_1 \left(\int_0^\tau \int_\Omega |u^k|^p dx\, dt\right)^{1/p} + N_2(\tau)$$

where $N_2(\tau) \to 0$ as $\tau \to 0$. Now we observe that the embedding of $L_p[0,T; \overset{o}{W}{}^p_m]$ into $L_p[0,T; \overset{o}{W}{}^p_0]$ is compact. Hence by (4.23) there is a subsequence which converges strongly to $u \in L_p[0,T; \overset{oo}{W}{}_p]$ hence a fortiori in $L_2[(0,T)\times\Omega]$. Still a further subsequence must converge strongly to u in $L_2(\Omega)$ pointwise almost everywhere in t. If we let k tend to infinity through all these subsequences we obtain, then from 4.28,

$$(4.29) \qquad \int_\Omega u^2(x,\tau)\, dx \leq N_1 \left(\int_0^\tau \int_\Omega |u|^p dx\, dt\right)^{1/p} + N_2(\tau)$$

Result (iii) of Definition 4.1 follows from this formula.

The verification that $a_\alpha = A_\alpha(D^\gamma u)$ proceeds exactly as in [2] and is a standard argument. It is based on the monotonicity result (III') and the fact that $W[u,v]$ is hemicontinuous. This means that $W[u-\xi\zeta,\zeta] \to 0$ as $\xi \to 0$.

The uniqueness is again standard, as in [2]. One subtracts two solutions u and u^* and uses (III') to obtain,

$$(4.30) \qquad \int_0^T \|u-u^*\|^p_{m,p} dt \leq 0,$$

from which the result follows. This completes the proof of Theorem 4.1.

<u>Remark 4.4</u>. We can write equations (4.19) in the form,

$$(4.31) \qquad \int_\Omega u_t^k v_j \, dx = T_m^{-1} \left\{ \mathcal{L}(u^k, v_j) + \langle h, v_j \rangle \right\}$$

or, on applying T_m,

$$(4.32) \qquad \int_\Omega u_t^k v_j \, dx = -m \mathcal{L}(u^k, v_j) - \int_0^t a(t-\tau) \mathcal{L}(u^k(\cdot,\tau) v_j) d\tau$$

$$+ \langle f, v_j \rangle \qquad\qquad j = 1, \ldots, k.$$

We multiply by smooth functions of t and integrate from 0 to T. This yields

$$(4.33) \qquad \int_\Omega u_t^k v \, dx = -m \mathcal{L}[u^k, v] - \int_0^T \int_0^t a(t-\tau) \mathcal{L}(u^k(\cdot,\tau), v(\cdot,t) d\tau \, dt$$

$$+ \int_0^T \langle f, v \rangle dt,$$

for all v's which are linear combinations of $c_j(t) v_j(x)$. Now however we can pass to the limit through the subsequence. Since the $L_q[(0,T) \times \Omega]$ limits a_α of the $A_\alpha(D^\gamma u^k)$ have already been identified as $A_\alpha(D^\gamma u)$ the result will be,

$$(4.34)\quad \int_\Omega u_t v\,dx = -m\,\mathcal{L}[u,v] - \int_0^T\int_0^t a(t-\tau)\mathcal{L}(u(\cdot,\tau),v(\cdot,t)d\tau\,dt + \int_0^T \langle f,v\rangle\,dt.$$

Equation (4.34) holds first for the span of $c_j v_j$ and then by closure for all of $L_p[0,T;\overset{o}{W}{}_p^m]$. Thus u is a generalized solution of (4.1) in the same sense as it is a generalized solution of (4.5). Our first efforts on this problem were devoted to establishing directly that the u^k's in (4.34) converged to a solution. This failed because in this generalized formulation there seems to be no analog of multiplying the equation by $g(u)$ and integrating, the device we used in section 3 to establish a priori bounds. This is the reason we passed to equation (4.5) thus having u appear only linearly in the integral term.

<u>Remark 4.5</u>. It is shown in [2] (Theorem 5) that a sufficient condition for (III), and hence, by Remark 4.2 for our result, is that the A_α satisfy conditions of the form,

$$(\text{III}'')\qquad \sum_{\substack{|\alpha|\le m\\ |\beta|\le m}} \frac{\partial A_\alpha}{\partial \xi_\beta}(x,\xi_\gamma)\eta_\alpha\eta_\beta \ge a_3 \sum_{|\alpha|=m} |\xi_\alpha|^{p-2}\eta_\alpha^2$$

for all ξ and η. Conditions (I) and (III') are very restrictive. They state, essentially that $A_\alpha(D^\gamma u)$ depends only on $D^\gamma u$ for $|\gamma| = m$ and that these occur to the power (p-1). Let us illustrate with the case discussed in Example 3.4. Let $n = 1$, $\Omega = (0,1)$, $m = 1$. Then,

$$g(u) = -\frac{\partial}{\partial x}\,\sigma(x,u,u_x). \tag{4.35}$$

Here $\alpha = 0$ or 1, $\xi_\gamma = (\xi_0,\xi_1)$, $A_0 = 0$ $A_1 = \sigma(x,\xi_0,\xi_1)$. (I) yields

$$\sigma(x,\xi_1,\xi_2) \le K_1(|\xi_1|^{p-1} + |\xi_2|^{p-1} + 1), \tag{4.36}$$

while (III") becomes,

$$\frac{\partial\sigma(x,\xi_0,\xi_1)}{\partial\xi_0}\,\eta_1\eta_0 + \frac{\partial\sigma(x,\xi_0,\xi_1)}{\partial\xi_1}\,\eta_1^2 \ge a_3|\xi_1|^{p-2}\eta_1^2. \tag{4.37}$$

Thus we need $\frac{\partial\sigma}{\partial\xi_0} = 0$ and $\frac{\partial\sigma(x,\xi_1)}{\partial\xi_1} \ge a_3|\xi_1|^{p-2}$. An acceptable σ would be

$$\sigma(x,u,u_x) = -\frac{\partial}{\partial x}\,(|u_x|^{p-1}\,\text{sign}\,u_x), \qquad p > 2.$$

5. Validity of Asymptotic Stability Results.

In this section we demonstrate the validity of the asymptotic stability results of section 3 in a special case. We take g as in section 4. We assume that m is so large that (II') and (III') hold with K_1 and K_2 equal to zero, see Lemma 4.1. We consider the equation, (5.1) $u_t(x,t) = T_m(g(u)(x,\bullet))(t) + f_x(x)t + f_Q(x) + p(x,t)$

$u(x,0) = 0$*

Here f, and f_ are in $L_o(0)$ and $p \in L_9(0, OD, L\sim(Q))$ and the hypotheses of Theorem 2.3 hold. The results of section 3 then yield the provisional result,

(5.2) $$\underset{t - C3D}{\text{weak lira}}\; g(u(t)) = -\frac{1}{7a} t \; f_{-}(x),$$ (aD)

provided that g is weakly stable.

Remark 5.1. The provisional result (5.1) requires the existence of a G satisfying (3.2). In general, there will be no such G for the operator (4.3). Thus it is remarkable that the rigorous result of this section is obtained without this assumption.

The results of [2] show that under the conditions (II) and (III) which we have placed on g the problem*,

(5.3) $$g(u(x)) = \hat{r}£) \; {}^{f}i<^{x})$$

(5.4) $$D^a u = 0 \quad \text{on} \quad dft \qquad |a) \leq m - 1$$

*Our result holds with minor modifications for any initial condition.

has a unique generalized solution $u_0(x)$. This is a function $u_0 \in \overset{\circ}{W}{}^m_p$ such that,

$$\mathcal{L}(u,v) = \int_\Omega \frac{1}{a(\infty)} f_1(x)v(x)\,dx \qquad \text{for all} \quad v \in \overset{\circ}{W}{}^m_p . \tag{5.5}$$

<u>Theorem 5.1</u>. <u>Suppose</u> a <u>satisfies the conditions of Theorem 2.2 with</u> m <u>large enough for uniqueness</u>. <u>Let</u> u <u>be the (unique) generalized solution of the inverted equation associated with</u> 5.1. <u>Then</u> u <u>satisfies,</u>

$$u(\cdot,t) \to u_0(\cdot) \qquad \text{in} \quad L_2(\Omega). \tag{5.6}$$

Before we prove this theorem we indicate its implications in the study of approach to steady state. Consider the second order equation,

$$u_{tt} = \frac{\partial}{\partial t} T_m(g(u)) + f_1(x), \qquad u(x,0) = u_t(x,0) = 0. \tag{5.7}$$

Suppose u is a classical solution. Then u will also be a classical solution of (5.1) with $f_0 \equiv 0$. It is then a classical solution of the inverted equation and must be the same as the generalized solution. It follows that u satisfies (5.6).

Notice that if $g(0) = 0$ then $u_0 = 0$ if $f_1 = 0$, by uniqueness. Hence if $g(0) = 0$ then solutions of (5.1) for $f_1 = 0$ must tend to zero in $L_2(\Omega)$.

<u>Proof of Theorem 5.1</u>. Theorem 2.3 shows that the inversion of (5.1) yields an equation of the form,

$$(5.8)\quad \frac{1}{m}u_t + \mathcal{L}u + \int_0^t \dot{k}(t-\tau)u(x,\tau)d\tau = a(\infty)^{-1}f_1(x) + r(x,t)$$

where $r \in L_2(0,\infty; L_2(\Omega)]$, and, by Corollary 3.1, $k \in L_2(0,\infty)$. The associated generalized solution then satisfies,

$$(5.9)\quad \frac{1}{m}\int_0^T \langle u_t, v\rangle + L[u,v] + I[u,v]$$

$$= \frac{1}{a(\infty)}\int_0^T\int_\Omega f_1 v \; dx\,dt + \int_0^T\int_\Omega r(x)v \; dx\,dt.$$

From (5.5) we have,

$$(5.10)\quad \mathcal{L}[u_0,v] = \frac{1}{a(\infty)}\int_0^T\int_\Omega f_1 v \; dx\,dt\,.$$

We set,

$$(5.11)\quad w = u - u_0$$

Then if we subtract (5.10) from (5.9) we can write the result as,

$$\frac{1}{m}\int_0^T \langle w_t, v\rangle + \tilde{L}[w,v] + I[w,v]$$

$$(5.12)\quad = \int_0^T\int_\Omega r(x,t)v(x,t)dx\,dt - \int_0^T k(t)\int_\Omega u_0(x)v(x,t)dx\,dt\,,$$

where,

$$(5.13)\qquad L[w,v] = \mathcal{L}[w+u_0,v] - \mathcal{L}[u_0,v] - \lambda\int_0^T\int_\Omega wv\,dx\,dt .$$

Conditions (II) and (III) for $\mathcal{L}$ imply that (II) and (III) are also satisfied for $\tilde{\mathcal{L}}[w,v] = \mathcal{L}[w+u_0,v] - \mathcal{L}[u_0,v]$ ((II) for $\tilde{\mathcal{L}}$ follows from (III) for $\mathcal{L}$). Since r and k are in $L_2(0,\infty)$, hence in $L_2(0,T)$, the right side of (5.12) has the form $[\varphi,v]$ where $\varphi \in L_q[0,T; \overset{\circ}{W}{}^m_p]$ ($p \geq 2$ means $q \leq 2$). Thus we conclude that we can find a generalized solution. We assume that m is large enough so that the constants K_1 and K_2 in Lemma (4.1) are zero. Then we have uniqueness. Moreover we obtain, in this case, by choosing v equal w in (5.12) and using (II') of Lemma (4.1)*,

$$(5.14)\qquad \|w(\cdot,T)\|^2_{L_2(\Omega)} + a\int_0^T \|w(\cdot,t)\|^p_{m,p}\,dt \leq \int_0^T a(t)\|w(\cdot,t)\|^2_{L_2(\Omega)}\,dt$$

where $a \in L_2(0,\infty)$.

From (5.14) we deduce that $\|w(\cdot,t)\|_{L_2(\Omega)}$ is bounded for all t and that $\|w\|_{m,p} \in L_p(0,\infty)$. The second statement yields the following provisional result:

(5.15) If $\|D^\alpha w(\cdot,t)\|_{L_r(\Omega)}$ is uniformly continuous on $[0,\infty)$ for any α, $|\alpha| \leq m$, and any $r \leq p$, then $\|D^\alpha w\|_{L_r(\Omega)} \to 0$.

---

* The calculation giving the first term is justified as in section 4 by passage to the limit through the subsequence.

We prove Theorem (5.1) by using the equation to show that $\|w(\cdot,t)\|_{L_2(\Omega)}$ is uniformly continuous. The calculation is like that for (4.29). Given $t_1 < t_2$ we choose w in (5.12) to be $w(x,t)\chi_{[t_1,t_2]}$ and obtain,

$$\frac{1}{2}\|w(\cdot,t_2)\|_{L_2(\Omega)} - \frac{1}{2}\|u(\cdot,t_1)\|_{L_2(\Omega)} \tag{5.16}$$

$$= -\Sigma\int_{t_1}^{t_2}\int_\Omega A_\alpha(D^\gamma u)D^\alpha u\,dx\,dt + \beta\int_{t_1}^{t_2}\int_\Omega u^2\,dx\,dt$$

$$-\int_{t_1}^{t_2}\int_0^t \dot{k}(t-\tau)\int_\Omega u(x,t)u(x,\tau)\,dx\,d\tau\,dt$$

$$+\int_{t_1}^{t_2}\int_\Omega (r(x,t) - k(t)u_0(x))v(x,t)\,dx\,dt$$

The boundedness of $\|u(\cdot,t)\|_{L_2(\Omega)}$ and the fact that r and k are in L_2 on $(0,\infty)$ show that the last three terms on the right side of (5.16) are uniformly small with $t_2 - t_1$. For the first term we have by Hölder's inequality

$$\left|\int_{t_1}^{t_2}\int_\Omega A_\alpha(D^\gamma u)D^\alpha u\,dx\,dt\right| \tag{5.17}$$

$$\le \left(\int_{t_1}^{t_2}\|A^\alpha(D^\gamma u)\|^q_{L^q(\Omega)}\,dt\right)^{1/q}\left(\int_{t_1}^{t_2}\|D^\alpha u\|^p_{L_p(\Omega)}\,dt\right)^{1/p}$$

We have indicated before that $u \in L_p[0,T; \overset{o}{W}{}_p^m]$ and condition I imply that $A^{\alpha}(D^{\gamma}u) \in L_q[(0,T) \times \Omega]$. But we have shown that $u \in L_p[0,\infty; \overset{o}{W}{}_p^m]$ hence $A^{\alpha}(D^{\gamma}u) \in L_q[(0,\infty) \times \Omega]$. Thus both terms on the right of (5.17) are uniformly small with $t_2 - t_1$. This completes the proof that $\|u(\cdot,t)\|_{L_2(\Omega)}$ is uniformly continuous and hence the proof of Theorem 5.1.

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