A NOTE ON SMOOTH BANACH SPACES

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Abstract

The purpose of the present note is to obtain a complete characterization of inner product spaces in terms of smoothness and to show that the theorem in [5] is the best possible in the sense that the isomorphism cannot be replaced by an isometric isomorphism. This also solves the following problem posed by Phelps in the affirmative: Are there Banach spaces B not isometric with inner product spaces such that the norms of B and B^* are twice Fréchet differentiable away from the origin.

A Note on Smooth Banach Spaces

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Let B be a real Banach space and B^* its dual. Let the norms on B and B^* be denoted by $\|\cdot\|$ and $\|\cdot\|^*$ respectively. If B is a smooth Banach space then the map $\sigma : B \longrightarrow B^*$ defined by $\sigma(x) = \|x\| G_{\frac{x}{\|x\|}}$ if $x \neq 0$, and $\sigma(0) = 0$, G_x denoting the Gâteaux gradient of the norm at x, is known as the <u>extended spherical image map</u>. It is known that the extended spherical image map is a homeomorphism if and only if $\|\cdot\|$ and $\|\cdot\|^*$ are Fréchet differentiable away from the origin, Cudia [2]. It is also known that if the norms of B and B^* are twice Fréchet differentiable away from the origin (in particular if the extended spherical image map is a C^1- diffeomorphism) then B is isomorphic to a Hilbert space, Sundaresan [5; theorem 5].

The purpose of the present note is to obtain a complete characterization of inner product spaces in terms of smoothness and to show that the theorem in [5] cited above is the best possible in the sense that the isomorphism cannot be replaced by an isometric isomorphism. This also solves the following problem posed by Phelps in the affirmative: Are there Banach spaces B not isometric with inner product spaces such that the norms of B and B^* are twice Fréchet differentiable away from the origin. Before proceeding to the main results of the paper, a few definitions and known facts are stated.

B denotes a real Banach space and B , B are the first and second duals of B. B(B) £(B,B) denote, respectively, the Banach space of bounded bilinear functionals on B and the Banach space of bounded linear operators on B into B with the usual supremum norms. The norms of B and B^* are denoted respectively by ||«|| and $||.||^*$ while the norms of ti*(B) and «£(B,B^) will be denoted by ||.|| itself as there will be no occasion for confusion. Since the mapping m : £(B,B*) —» &(B) defined by m(T) (x,y) = T(x) (y) for x,y e B is a linear isometry we identify these spaces. U(U), S(S) denote respectively the unit ball and unit sphere of B(B).

Definitions; Let (B,||.||) be a Banach space. The norm is said to be differentiable at x ^ 0 if there exists a functional f € B such that

$$\lim_{\|h\| \to 0} \frac{1 \ \|x+h\| - \|x\| - f(h)|}{\|h\|} = 0$$

The functional f is the first derivative of the norm at x and will be denoted by G_x. If the norm is once differentiable in a neighborhood of x ^ 0 and if there exists a linear operator T_x e£(B,B*) such that

$$\lim \frac{G_{x+h} - G_x - T_x(h)}{\|h\|} = 0$$

then the norm is said to be twice differentiable at x and T_x is the second derivative of the norm at x. it is well known that T_x is a symmetric bilinear form on B, e.g. Cartan [1].

Let B_1, B_2 be two Banach spaces and U be an open subset of B_1. Let $f : U \to B_2$ be a differentiable (twice differentiable) function with Df ($D^2 f$) as the first (second) derivative of f. Then f is said to be of class C^1 (C^2) if Df (D^2f) is continuous on $U \to \mathcal{L}(B_1,B_2)$ ($U \to \mathcal{L}(B_1,\mathcal{L}(B_1,B_2))$). If $f: U \to B_2$ is a homeomorphism such that f, f^{-1} are of class C^1 then f is said to be a C^1-diffeomorphism. For more on diffeomorphisms we refer to Cartan [1].

If the functional $\|\cdot\|$ on B is directionally differentiable at all $x \in B\setminus\{0\}$ then B is said to be smooth. If it is twice directionally differentiable on $B\setminus\{0\}$ it is said to be 2-smooth, [5].

The first two directional derivatives of the norm $\|\cdot\|$ ($\|\cdot\|_*$) if they exist at $x(f)$ will be denoted respectively by G_x, T_x (G_f, T_f).

Some properties of the first two directional derivatives of the norm of B are stated in lemma 1 below and are known [5].

Lemma 1; Let the first two directional derivatives of the norm of B exist at some $x \neq 0$. Then

(1) $G_{\lambda x}$ exists for all $\lambda \neq 0$, $G_{\lambda x} = \operatorname{sgn}\lambda \, G_x$ and $\|G_{\lambda x}\| = \|G_x\| = 1$. Indeed G_x is the support functional of the unit ball at $\frac{x}{\|x\|}$.

(2) $T_{\lambda x}$ exists for all $\lambda \neq 0$, $T_{\lambda x} = \frac{1}{|\lambda|} T_x$ and T_x is a symmetric bilinear form, $T_x(y,y) \geq 0$ for all $y \in B$ and range $T_x \subseteq \{x\}^{\perp}$, the submanifold of B annihilating x.

Let g (g^*) be the real valued function on B (B^*) defined by $g(x) = \frac{1}{2}\|x\|^2$ ($g^*(f) = \frac{1}{2}\|f\|_*^2$). We denote the second order directional derivatives at $x(f)$ by T_x^1 (T_f^2) respectively.

Further if $x \in B$ then $\sigma(x)$ denotes the functional $\|x\| G_{\frac{x}{\|x\|}}$ in B^* whenever $G_{\frac{x}{\|x\|}}$ exists. It is easily verified that if the norm is twice directionally differentiable at a point $x \neq 0$ then $T^1_{\lambda x} = T^1_x$ if $\lambda \neq 0$.

The main theorem provides a characterization of Hilbert spaces up to isometric isomorphisms in terms of differentiability properties of the norm in B and B^*.

Theorem 1. The following properties of a Banach space are equivalent.

(1) B is isometrically isomorphic to a Hilbert space.

(2) The norms in B and B^* are once differentiable away from zero and there exists an $x \neq 0$ such that $\|\cdot\|$ and $\|\cdot\|^*$ are twice differentiable at x and $\sigma(x)$ respectively, with $\|T^1_x\| \leq 1$ and $\|T^2_{\sigma(x)}\| \leq 1$.

(3) The extended spherical image map σ on $B \sim \{0\}$ into $B^* \sim \{0\}$ is a C^1-diffeomorphism and $\|T^1_x\| \leq 1$ for all $x \in S$.

(4) The norm in B is of class C^2 away from 0 and $\|T^1_x\| \leq 1$ for all $x \in S$.

Proof: (1)$\Longleftrightarrow$(2). It is easily verified that the norm in a Hilbert space is of class C^2. Further for all $x \neq 0$ $T^1_x(h,h) = (h,h)$. Hence $\|T^1_x\| = 1$. Thus (1)$\Rightarrow$(2). We proceed to show that (2)$\Rightarrow$(1). Since the norm in B^* is Fréchet differentiable, it follows from the results on pages 113-114 in

Day [3] that B is reflexive. Let $x \in S$ be such that (2) holds. Let σ^* be the extended spherical image map on B^*. Since σ^* is differentiable at $\sigma(x)$ it follows that

$$\begin{aligned}\sigma^*(\sigma(x+ty)) &= \sigma^*(\sigma(x)) + T^2_{\sigma(x)}[\sigma(x+ty) - \sigma(x)] + \theta_x(t) \\ &= \sigma^*(\sigma(x)) + T^2_{\sigma(x)}[tT^1_x(y) + \varphi_x(t)] + \theta_x(t)\end{aligned}$$

where $\dfrac{\varphi_x(t)}{t} \to 0$, $\dfrac{\theta_x(t)}{t} \to 0$ as $t \to 0$.

So $$\frac{\sigma^*(\sigma(x+ty)) - \sigma^*(\sigma(x))}{t} \to T^2_{\sigma(x)}(T^1_x(y))$$

as $t \to 0$. On the other hand it is easily verified that $\sigma^*(\sigma(y)) = Q(y)$ for all $y \in B$, where Q is the canonical embedding of B onto B^{**}. Thus

$$T^2_{\sigma(x)}(T^1_x(y)) = Q(y)$$

Hence if $y \in S$

$$1 = \|T^2_{\sigma(x)}T^1_x(y))\| \leq \|T^2_{\sigma(x)}\| \; \|T^1_x(y)\|^* \leq \|T^2_{\sigma(x)}\| \; \|T^1_x\| \leq 1$$

Since $\|T^2_{\sigma(x)}\| \leq 1$, $\|T^1_x\| \leq 1$. It follows that $\|T^1_x(y)\|^* = 1$. Further, since B is reflexive there exists $z \in S$ such that

$$1 = \|T^1_x(y)\| = T^1_x(y,z) \leq (T^1_x(y,y))^{1/2}(T^1_x(z,z))^{1/2} \leq 1.$$

Hence $T^1_x(y,y) = 1$ for all $y \in S$, and therefore $T^1_x(y,y) = \|y\|^2$ for all $y \in B$. Hence the norm on B is determined by an inner product. Thus (2) $\Rightarrow$ (1).

If B is a Hilbert space, then since $\sigma(x)(h) = (x,h)$, σ is a C^1-diffeomorphism on $B \sim \{0\}$.

Further, $\|T_x^1\| = 1$ for all $x \neq 0$. Hence (2) $\Rightarrow$ (3). Next, if B is a Banach space such that σ is a C^1-diffeomorphism, then it is easily verified that $\|\cdot\|$ on B is of class C^2 away from 0. Hence (3) $\Rightarrow$ (4).

We now verify that (4) $\Rightarrow$ (1). Let $x \neq 0$. By the mean value theorem, for $h \in B$, there exist $\theta = \theta(x,h)$, $\theta_1 = \theta_1(x,h)$, $0 \leq \theta, \theta_1 \leq 1$, such that

$$\|x + h\|^2 = \|x\|^2 + 2\|x\|G_x(h) + T^1_{x+\theta h}(h,h)$$

and $$\|x - h\|^2 = \|x\|^2 - 2\|x\|G_x(h) + T^1_{x-\theta_1 h}(h,h)$$

Hence

$$\|x + h\|^2 + \|x - h\|^2 = 2\|x\|^2 + [T^1_{x+\theta h}(h,h) + T^1_{x-\theta_1 h}(h,h)] \leq$$

$$\leq 2\|x\|^2 + 2\|h\|^2$$

Since $\|T^1_y\| = \|T^1_{\lambda y}\| \leq 1$ if $y \neq 0$ and $\lambda \neq 0$. Thus

$$\|x + h\|^2 + \|x - h\|^2 \leq 4 \quad \text{if} \quad \|x\| = \|h\| = 1.$$

Hence by a known theorem, Schoenberg [4], $\|\cdot\|$ is determined by an inner product.

q.e.d.

Remark: In connection with the above it might be mentioned that if B is a Banach space such that $\|\cdot\|$ is of class C^2 away from zero and there exists an $L > 0$ such that $T^1_x(y,y) \geq L\|y\|^2$ for all $y \in B$ and for all $x \in S$, then B is uniformly convex. To see this, if $x \in B \sim \{0\}$ and $h \in B$, by the mean value theorem there exists $\theta = \theta(x,h)$, $\theta_1 = \theta_1(x,h)$

$0 \leq \theta,\ \theta_1 \leq 1$, such that

$$\|x + h\|^2 = \|x\|^2 + 2\|x\|G_x(h) + T^1_{x+\theta h}(h,h),$$

$$\|x - h\|^2 = \|x\|^2 - 2\|x\|G_x(h) + T^1_{x-\theta_1 h}(h,h).$$

Hence $\|x + h\|^2 + \|x - h\|^2 = 2\|x\|^2 + [T^1_{x+\theta h}(h,h) + T^1_{x-\theta_1 h}(h,h)]$.

Therefore,

$\|x + h\|^2 + \|x - h\|^2 \geq 2\|x\|^2 + 2L\|h\|^2$, for all $x \in B\sim\{0\}$, $h \in B$,

and if $\|p\| = \|q\| = 1$ and $\|p - q\| \geq \epsilon$ where $\epsilon > 0$ then

$$2 = \|p\|^2 + \|q\|^2 \geq 2\|\frac{p+q}{2}\|^2 + 2L\|\frac{p-q}{2}\|^2 \geq 2\|\frac{p+q}{2}\|^2 + \frac{L}{2}\epsilon^2$$

and
$$\|\frac{p+q}{2}\|^2 \leq 1 - \frac{L}{4}\epsilon^2 .$$

Therefore B is uniformly convex with modulus of convexity $\delta(\epsilon) = 1 - \{1 - \frac{L}{4}\epsilon^2\}^{\frac{1}{2}}$.

Example: Now we provide an example to show that theorem 1 is the best possible.

Consider the two dimensional space $(\mathbb{R}^2, \|\cdot\|)$ where for $x = (x_1,x_2)$ we let

$$\|x\|^2 = x_1^2 + \alpha^2 x_2^2(1 + \exp(-\frac{x_1^2}{x_2^2})) \qquad \text{if } x_2 \neq 0$$

$$\|x\|^2 = x_1^2 \qquad \text{if } x_2 = 0$$

with $0 < \alpha < 1$. Then $(\mathbb{R}^2, \|\cdot\|)$ is a Banach space and the extended spherical image map is given by

$$\sigma(x) = \left(x_1\left(1 - \alpha^2 \exp\left(-\frac{x_1^2}{x_2^2}\right)\right),\ \alpha^2 x_2 + \alpha^2 \frac{(x_1^2 + x_2^2)}{x_2^2} \exp\left(-\frac{x_1^2}{x_2^2}\right)\right) \qquad x_2 \neq 0$$

$$\sigma(x) = (x_1, 0) \qquad x_2 = 0$$

The Jacobian of σ is positive at all $x \neq 0$. Hence the spherical image map is a C^1-diffeomorphism. However, the $\|\cdot\|$ is not determined by an inner product.

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