# NONLINEAR SEMIGROUPS AND A HYPERBOLIC CONSERVATION LAW

by

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## Abstract

This paper is concerned with the hyperbolic conservation law

(1.1) $$\frac{\partial u}{\partial t} + \frac{\partial b(u)}{\partial x} = 0$$

(1.2) $$u(x,0) = f(x).$$

By means of the Crandall-Liggett theory of nonlinear semigroups ([1]), it will be shown that the operator $A : u \mapsto \frac{d}{dx}b(u)$ generates a semigroup $S = \{S_t;\ t \geq 0\}$ of contraction operators on $L^1(\mathbb{R})$. The function $t \mapsto S_t f$ may then be thought of as a generalized solution of the initial value problem (1.1-.2), and it will be seen that for $f \in L^1 \cap L^\infty$, this function is a weak solution in the usual sense. Finally, semigroup methods will be employed to derive the "ordering principle" for the solutions of (1.1-.2), in a form due to Kružkov [3].

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§1. This paper is concerned with the hyperbolic conservation law

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$$u(x,0) = f(x). \tag{1.2}$$

By means of the Crandall-Liggett theory of nonlinear semigroups ([1]), it will be shown that the operator $A : u \mapsto \frac{d}{dx} b(u)$ generates a semigroup $S = \{S_t; t \geq 0\}$ of contraction operators on $L^1(\mathbb{R})$. The function $t \mapsto S_t f$ may then be thought of as a generalized solution of the initial value problem (1.1-.2), and it will be seen that for $f \in L^1 \cap L^\infty$, this function is a weak solution in the usual sense. Finally, semigroup methods will be employed to derive the "ordering principle" for the solutions of (1.1-.2), in a form due to Kružkov [3].

The statement of the Crandall-Liggett theorem, and an overview of the technical aspects of the paper, will be given in §2. The remainder of this introductory section is devoted to a general comparison of the present approach and the existing theory of equation (1.1).

The conservation law (1.1) has been studied quite thoroughly, both because of its mathematical interest, and because it

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is the prototype of some important hyperbolic systems describing physical phonomena. First of all, simple examples show that solutions of (1.1) can develop discontinuities ("shocks") even for smooth data (1.2); consequently, a solution can only be expected to satisfy the equation in a weak sense:

(1.3) Definition. A bounded, measurable function $u$ of $(x,t)$ is said to be a weak solution of (1.1), (1.2), if

i) $u(t,x) \longrightarrow f(x)$ for a.e. $x$, as $t \longrightarrow 0$;

ii) $\int_{t>0} \int_{x \in \mathbb{R}} \{u\varphi_t + b(u)\varphi_x\} dxdt = 0$ for all twice continuously differentiable functions $\varphi$ with compact support in the half-plane $t > 0$.

The fundamental result in the theory, and the source of most of the complications, is this: a weak solution exists for any $f \in L^{\infty}$, but it need not be unique. One now introduces an additional constraint, which has the effect of choosing a unique solution from all the possible weak solutions of (1.1-.2). This restriction, called the "entropy condition", imposes a certain behavior on the solution near a curve of discontinuity.* The semigroup property (or "principle of causality") in the class of weak solutions satisfying the entropy condition follows

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* There are several formulations of the entropy condition for functions whose discontinuities have a less regular structure; see [3], [4], [6], [9].

immediately from uniqueness. Finally, arguments based, again, on the entropy condition, can be used to demonstrate the $L^1$-stability of piecewise smooth solutions [7]:

$$(1.4) \qquad |u(\cdot,t) - v(\cdot,t)|_{L^1} \le |u(\cdot,0) - v(\cdot,0)|_{L^1},$$

as well as the ordering principle [3],

$$(1.5) \qquad |u(\cdot,t) - v(\cdot,t)|^+_{L^1} \le |u(\cdot,0) - v(\cdot,0)|^+_{L^1}.$$

(We use the symbol $|h|^+_{L^1}$, to denote the $L^1$-norm of the positive part of $h$).

In the semigroup approach, the entropy condition plays a somewhat different role. Once the operator $A$ is known to generate a semigroup, the existence of weak solutions and the ordering property will follow from purely global considerations (§§4-6). However, the proof that $A$ is a generator involves a problem for an ordinary differential equation which exhibits many of the "hyperbolic" phenomena: discontinuous solutions, possible non-uniqueness, and a one-dimensional version of the entropy condition. The equation in question is $f + \lambda b(f)' = h$ $({}' = \frac{d}{dx})$, which is to have a solution $f$ in $L^1$ for each $h \in L^1$; this is the problem -- also encountered in the linear theory -- of showing that the resolvent $(I+\lambda A)^{-1}$ is everywhere defined.

The treatment by semigroup theory has the drawback that it makes a satisfactory uniqueness theorem hard to formulate. In a function space setting, where local properties are somewhat out of place, there may be no way to decide whether a given weak solution is the one generated by the semigroup. It is known from the work of Quinn [7], that the semigroup solutions are precisely the entropy solutions, but we have not attempted to include this result in the present study.

In summary, then, our approach offers something new insofar as it shows the theory of the conservation law (1.1) to be, to a considerable extent, a part of the theory of abstract evolution equations; in §7, we will also point out certain differences in the technical details. Finally, there may be some interest in this study as an *example in the theory of evolution equations*, since the problem (1.1-.2) is quite pathological in comparison with parabolic, or even semi-linear hyperbolic equations: the underlying Banach space almost certainly cannot be reflexive; the solution generated by the operator $A$ is not strong; and, because the operator $A$ is not monotone, the usual problem of the surjectivity of $I + \lambda A$ requires new techniques.

§2. We turn now to a preliminary description of the technical aspects of the paper. The fundamental theorem of M. G. Crandall and T. M. Liggett [1] was originally established for multi-valued operators defined on non-dense subsets of a Banach space; we require only a special case:

*THEOREM 0: Let* $X$ *be a Banach space,* $A$ *a (nonlinear) operator in* $X$ *with dense domain* $D(A)$. *Suppose that*

$(C)_1$ $A$ *is accretive,* i.e., *for all* $\lambda > 0$ *and all* $x, y \in D(A)$, (2.1) $$\|(I+\lambda A)x - (I+\lambda A)y\| \geq \|x-y\|;$$

(C)$_2$ _for all_ $\lambda > 0$, _the range of_ $I + \lambda A$ _is all of_ $X$. _Then the limit_

(2.2) $$S_t x \underset{\text{def}}{=} \lim_{N\to\infty} \left(I + \frac{t}{N} A\right)^{-N} x$$

_exists for all_ $x \in X$, _and defines a strongly continuous semi-group_ $S = \{S_t;\ t \geq 0\}$ _of contraction operators_:

$$\|S_t x - S_t y\| \leq \|x-y\|.$$

The three steps: defining $D(A)$, and verifying (C)$_1$ and (C)$_2$, are closely interdependent, more so than in the linear theory. We obtain the description of $D(A)$, and the estimate (2.1), as consequences of the surjectivity of $I + \lambda A$. The latter property means, concretely, that

(2.3) $$f + \lambda b(f)' = h$$

(where $' = \frac{d}{dx}$) has a solution $f \in D(A)$ for all $h \in L^1(\mathbb{R})$. By requiring that $f \in L^1$, we are in effect posing a two-point boundary value problem for a first-order equation. Attempts at a "direct" solution of (2.3) were unsuccessful, and so we approach the problem via an approximate second order equation over a finite interval,

(2.4) $$f + b_N(f)' - \varepsilon f'' = h, \quad f(\pm N_1) = 0.$$

($b_N$ is a cutoff version of $b$). (2.4) can be solved by the

Leray-Schauder degree theory; the necessary a-priori estimates of the solutions of (2.4) are obtained from a study of the associated parabolic equation

$$(2.5)\qquad \begin{cases} u_t = \varepsilon u_{xx} - b_N(u)_x \\ u(x,0) = f(x) \\ u(\pm N_i,t) = 0. \end{cases}$$

The reader will recognize the similarity of this development to the "method of vanishing viscosity", which provides solutions of (1.1-,2) as limits of solutions of the parabolic equations $u_t + b(u)_x = \varepsilon u_{xx}$. It should be noted, however — and this may be important for any extension of the present method to systems -- that we require only the <u>local solvability</u> of (2,5), whereas the viscosity method is based on the more difficult global solvability of the parabolic equation.

We arrive at a solution of (2.3), by letting $N \to \infty$ and $\varepsilon \to 0$ in (2.4). To show that the limiting function is independent of the manner in which $\varepsilon \to 0$, we must establish that all such limits satisfy a one-dimensional "entropy condition", and that any two solutions obeying this restriction are, in fact, identical. The details of this argument are carried out in §3.

Once the operator $A$ is known to generate a semigroup $S$, one must decide in what sense the function $t \mapsto S_t f$ is a solution of the Cauchy problem

$$(2.6)\qquad \frac{du(t)}{dt} + Au(t) = 0, \quad u(0) = f.$$

If the problem were set in a reflexive space, the results of [1] would yield the a.e.-differentiability of $t \longmapsto S_t f$, and this function could be shown to be the unique "strong" solution of (2.6). But $L^1$ is not reflexive, and the general theory cannot assert such regularity. One can, in fact, argue that the conservation law (1.1) will fail to have a strong solution for some data: as was mentioned earlier, the entropy solution must be expected to develop discontinuities, even when the initial value $f \in D(A)$. In general, $b(S_t f)$ will also become discontinuous, and as we shall see, this forces $S_t f$ outside of $D(A)$, so that (2.6) can no longer be satisfied -- by $S_t f$ -- in a pointwise sense.

In §4, we introduce generators in abstract divergence form: $A = LB$ (in the case of (1.1), $L = \frac{d}{dx}$, and $B(f) = b(f)$), and show that $t \mapsto S_t f$ can be interpreted as a weak solution of (2.6), whenever $f \in D(B)$. In §6, this abstract result will be shown to apply to equation (1.1); however, that final verification depends also on the availability of the ordering principle (1.5), which will be proved in §5 by means of an adaptation of K. Sato's theory of positive (linear) semigroups on Banach lattices ([8]).

I wish to take this occasion to express my deep gratitude to Professor V. J. Mizel, for his interest and encouragement. I also thank Professors C. V. Coffman and Z. Nehari, and Dr. D. D. M. Tong, for some helpful conversations. Finally, I want to acknowledge the influence of B. K. Quinn's paper [7], which stimulated my interest in this problem.

§3. As already stated, our object in this section is to show that Theorem 0 is applicable to the operator $A : f \mapsto b(f)'$. We assume that the function $b$ is four-times continuously differentiable, and (without any loss of generality) that $b(0) = 0$. First we solve the equation

$$f + b(f)' = h \tag{3.0}$$

by studying a fairly lengthy sequence of approximating equations; only after this has been accomplished, will we be able to describe $D(A)$ and verify accretiveness (condition $(C)_1$).

3.1 <u>A two-point boundary value problem, with</u> $h \in C_0^\infty$.

Let $h \in C_0^\infty(\mathbb{R})$ be fixed throughout this subsection, and choose $N > 0$ so that $\operatorname{supp} h \subset (-N,N)$. We want to solve the problem (for fixed $\varepsilon > 0$)

$$\begin{cases} f + [C_N b(f)]' - \varepsilon f'' = h, \\ f(\pm N_1) = 0. \end{cases} \tag{3.1}$$

Here $C_N$ is a "cutoff" function with these properties:

(i) $C_N \in C_0^\infty(\mathbb{R})$;

(ii) $\operatorname{supp} C_N \subset (-N_1, N_1)$;

(iii) $C_N \equiv 1$ on $[-N,N]$;

(iv) $|C_N(x)| \le 1$ for all $x$.

Furthermore, we require $\gamma =_{\text{def}} \sup|C_N'| + \sup|C_N''|$ to be small. To be precise, define

$$\begin{aligned} R &= \max\{|h|_1,\ 2|h'|_1 + 1\}, \\ K &= \sup\{|b'(\eta)|;\ |\eta| \le R\}. \end{aligned} \tag{3.2}$$

We wish to have

$$(3.3) \qquad \gamma K < \tfrac{1}{2}, \quad \gamma K|h|_1 < \tfrac{1}{2}.$$

The significance of these restrictions will become apparent later; for the moment, we only observe that a choice of $C_N$ consistent with (3.3) is possible, provided that $N_1$ is taken large enough. We do so, and now consider as fixed, once and for all, the quantities entering into (3.1).

A. Conversion to an integral equation.

We want to look at (3.1) as an operator equation in the space $C([-N_1,N_1]) =_{\text{def}} Y$, normed by the sup-norm $|\cdot|_0$. Let $K$ be the Green function for the operator $I - \varepsilon \frac{d^2}{dx^2}$ with zero boundary conditions at $x = \pm N_1$, and define an operator $T$ by

$$(3.4) \qquad (Tg)(x) = \int_{-N_1}^{N_1} \frac{\partial}{\partial y} K(x,y) \cdot C_N(y) b(g(y))\,dy + \int_{-N_1}^{N_1} K(x,y) h(y)\,dy.$$

LEMMA 3.1. *If* $f \in Y$ *satisfies* $f = Tf$, *then it is a twice-differentiable solution of* (3.1).

The proof is straightforward: it can be seen from the explicit formula for $K$ that $\frac{\partial^2 K}{\partial x \partial y}$ is continuous, so that $f$ is differentiable. One may then integrate by parts in the first integral in (3.4); the fact that $f$ satisfies (3.1) then follows from the definition of $K$.

We will use degree theory to establish the existence of a fixed point for $T$. The first step in this direction is

<u>LEMMA 3.2.</u> *T is a completely continuous operator from Y to Y.*

The proof is again standard. The fact that $T$ maps into $Y$, and is continuous, follows from elementary estimates. The compactness of $T$ is a consequence of the Arzela-Ascoli theorem.

B. *A-priori estimates on solutions of* $f = \mu Tf$.

We will prove

<u>LEMMA 3.3.</u> *If $R$ is defined as in (3.2), and if $|f|_0 = R$, then $f = \mu Tf$ cannot hold for any $\mu \in [0,1]$.*

First observe that if $f = \mu Tf$, then $f$ is a twice-differentiable solution of

$$(3.5)\qquad \begin{cases} f + \mu(C_N b(f))' - \varepsilon f'' = \mu h \\ f(\pm N_1) = 0. \end{cases}$$

(cf. Lemma 3.1).

We will estimate the sup-norm of a solution of (3.5) by getting an $L^1$-estimate of its derivative. The following is the basic step:

<u>LEMMA 3.4.</u> *Let $G(x,\eta)$ be $C^3$ in both variables $x \in [-N_1,N_1]$, $\eta \in \mathbb{R}$. Suppose that $G(\pm N_1,\eta) = 0$ for all $\eta$, and that $G(x,0) = 0$ for all $x$. Let $f \in C^3[-N_1,N_1]$, and assume that $f$ satisfies one of the end conditions*

$$(B)_1 \qquad f(\pm N_1) = 0$$

$$(B)_2 \qquad f'(\pm N_1) = 0.$$

*Define* $(G\circ f)(x) = G(x,f(x))$. *Then*

(3.6) $$|f+(G\circ f)' - \varepsilon f''|_1 \geq |f|_1.$$

*Proof.* We study the auxiliary parabolic problem

(3.7) $$u_t = \varepsilon u_{xx} - (G\circ u)x$$
$$u(x,0) = f(x)$$

with one of these two boundary conditions:

$(PB)_1$ $$u(\pm N_1,t) = 0, \quad t > 0.$$

$(PB)_2$ $$u_x(\pm N_1,t) = 0, \quad t > 0.$$

It is known (e.g., P. E. Sobolevskiĭ, Dok.A.N. 136,p.292) that there exists a solution $u \in C^2([-N_1,N_1]\times[0,t_0))$, for some $t_0 > 0$ depending on $f$.

Suppose we could prove, for this solution $u$:

$(3.8)_1$ $$|u(\cdot,t)|_1 \leq |f|_1, \quad 0 \leq t < t_0;$$

$(3.8)_2$ $$|t^{-1}[u(\cdot,t)-f] - [\varepsilon f''-(G\circ f)']|_1 \to 0 \quad \text{as} \quad t \to 0.$$

Then we could estimate as follows:

$$|f + t^{-1}[f-u(\cdot,t)]|_1 \geq (1+t^{-1})|f|_1 - t^{-1}|u(\cdot,t)|_1 \geq$$
$$\geq (1+t^{-1})|f|_1 - t^{-1}|f|_1 = |f|_1.$$

Letting $t \to 0$, and using $(3.8)_2$:

$$|f+(G\circ f)' - \varepsilon f''|_1 \geq |f|_1,$$

as desired.

Now $(3.8)_2$ is easy to verify: for each $x$,

$$t^{-1}[u(x,t)-f(x)] = t^{-1}[u(x,t)-u(x,0)] \to u_t(x,0) =$$

$$= \varepsilon f''(x) - (G\circ f)'(x).$$

This pointwise convergence is dominated, so that the $L^1$-convergence asserted by $(3.8)_2$ follows.

Lemma 3.4 will therefore be proved, once we establish $(3.8)_1$. We do so first for a function $f$ with <u>simple zeros</u> at points labelled $z_j$: $-N_1 = z_0 < \dots < z_{p+1} = N_1$, for condition $(B_1)$; $-N_1 < z_1 < \dots < z_p < N_1$, for condition $(B_2)$. Now solve $u(x,t) = 0$ for $x$ as a function of $t > 0$, with $x = z_j$ when $t = 0$. There will result a finite number of $C^1$-curves $x = x_j(t)$ (with $x_0 \equiv -N_1$, $x_{p+1} \equiv N_1$) all defined for $t \le$ some $t_1$. In case $(B)_1$ holds, $u(x_j(t),t) = 0$ for all $j$, $0 \le j \le p+1$; we may assume that the function $u$ changes sign across the "interior" curves (since it does so at $t = 0$). In case $(B)_2$ holds, we again have curves $x = x_j(t)$, $1 \le j \le p$, across which $u$ changes sign; to these we adjoin the boundary lines $x_0(t) \equiv -N_1$, $x_{p+1}(t) \equiv N_1$.

We will show that

$$\frac{d}{dt}|u(\cdot,t)|_1 \le 0;$$

$(3.8)_1$ immediately follows from this inequality. First we write

$$(3.9) \qquad |u(\cdot,t)|_1 = \sum_{j=0}^{p} \operatorname{sgn}(j) \int_{x_j(t)}^{x_{j+1}(t)} u(x,t)\,dx,$$

where sgn(j) is the sign of $u(x,t)$ for $x_j(t) < x < x_{j+1}(t)$. Now we differentiate (3.9); the right side will be a sum of terms of the form

$$\text{(3.10)} \qquad \operatorname{sgn}(j)\frac{d}{dt}\int_{x_j(t)}^{x_{j+1}(t)} u(x,t)\,dx.$$

Carrying out the differentiation, and omitting the sgn(j), we get

$$\{\dot{x}_{j+1}u(x_{j+1},t) - \dot{x}_j u(x_j,t)\} + \int_{x_j}^{x_{j+1}} u_t(x,t)\,dx$$

($x_j$ abbreviates $x_j(t)$, and $\dot{x}_j = \frac{d}{dt}x_j$). The bracketed term vanishes, since either $u(x_j,t) = 0$ or $\dot{x}_j = 0$. On using equation (3.7), we convert the second integral to

$$\text{(3.11)} \qquad \{\varepsilon u_x(x,t) - (G\circ u)(x,t)\}\Big|_{x=x_j}^{x_{j+1}}.$$

The term involving G vanishes for all j, by virtue of the assumptions on G. Hence, in (3.11) there only remains

$$\text{(3.12)} \qquad \varepsilon\{u_x(x_{j+1},t) - u_x(x_j,t)\};$$

this term is always opposite in sign to u on the interval $x_j < x < x_{j+1}$. Since (3.12) coincides with (3.10), less the factor sgn(j), we conclude that (3.10) is nonpositive, and -- on summing over j -- that

$$\frac{d}{dt}|u(\cdot,t)|_1 \leq 0.$$

Finally, let $f \in C^3$ be arbitrary. It can be approximated in $C^2([-N_1, N_1])$ by functions $f_n$ with simple zeros (which satisfy the appropriate end condition). For example: the Bernstein polynomials of $f$ converge to $f$ in $C^2$; a small correction term will remove any multiple roots. By dominated convergence, the approximating functions, and their first two derivatives, converge also in $L^1$. Since (3.6) holds for each $f_n$, it will still be true for the limit function. This proves Lemma 3.4.

(Ihe idea of studying the decomposition (3.9) is taken from Quinn [7]).

Proof of Lemma 3.3. Let $f$ be any solution of (3.5). According to Lemma 3.4, with $G(x,\eta) = \mu C_N(x) b(\eta)$, we have

(3.13) $|h|_1 = |f + \mu(C_N b(f))' - \epsilon f''|_1 \geq |f|_1$

Next, we differentiate (3.5) to get an equation for $f'$, and find that $g = f'$ is a solution of

(3.14) $g + \mu(C_N' b(f))' + \mu(C_N b'(f) g)' - \epsilon g'' = h'$.

Ihe appropriate end conditions are

(3.15) $g'(\pm N_1) = 0$;

to see this, observe that near $x = \pm N_1$, $C_N = h = 0$, so that equation (3.5) reduces to $f - \epsilon f'' = 0$. Since $f(\pm N_1) = 0$, we have $(f'')' = 0$ at $\pm N_1$, which suggests (3.15).

Choosing $G(x,\eta) = \mu C_N(x) b'(f(x))\eta$, we again apply Lemma (3.4) (for notational simplicity, we use the symbol $g$ instead of $f'$):

$$|g+\mu(C_N b'(f)g)' - \varepsilon g''|_1 \geq |g|_1.$$

From (3.14),

(3.16) $$|h'|_1 \geq |g|_1 - \mu|(C_N' b(f))'|.$$

Now

(3.17) $$|(C_N' b(f))'|_1 \leq |C_N'' b(f)|_1 + |C_N' b'(f)g|_1.$$

In terms of the constants introduced in (3.2), we continue to estimate in (3.17):

$$\leq \gamma K|f|_1 + \gamma K|g|_1,$$

which, by (3.13) and (3.3) is

$$< \frac{1}{2} + \frac{1}{2}|g|_1.$$

Inserting this into (3.16), we obtain, when $g = f'$:

$$|h'|_1 > |f'|_1 - \frac{\mu}{2}(1+|f'|_1) \geq \frac{1}{2}|f'|_1 - \frac{1}{2}.$$

Hence,

(3.18) $$|f'|_1 < 2|h'|_1 + 1 = R.$$

Finally, using the fact that $f(x) = \int_{-N_1}^{x} f'(y)\,dy$ we get from (3.18):

(3.19) $$|f|_0 < R.$$

Summarizing: the assumption that there is a solution $f$ of $f = \mu Tf$, for which $|f|_0 = R$, leads to the contradictory (3.19). QED.

C. The existence proof.

The results of A and B show that $T : Y \to Y$ is completely continuous, and that $(I-\mu T)f = 0$ has no solutions on the sphere $\{g \in Y; |g|_0 = R\} = Q$. By homotopy invariance of Leray-Schauder degree, we conclude that $\deg(I-T,Q,0) = \deg(I,Q,0) \neq 0$, and hence that $(I-T)f = 0$ has a solution $f$ with $|f|_0 < R$ (see, for example [2, Ch. 2]).

It is not hard to see that this solution is unique. If there were two solutions, $f$ and $g$, then their difference $f - g = F$ would satisfy

$$F' + (C_N BF)' - \varepsilon F'' = 0, \quad F(\pm N_1) = 0,$$

where

$$B(x) = \int_0^1 b'(tf(x)+(1-t)g(x))\,dt.$$

Another application of Lemma 3.4 then shows: $|F|_1 \leq 0$.

In summary:

LEMMA 3.5. *There exists a unique, twice-differentiable solution of problem* (3.1).

3.2. The second-order equation on the whole line.

As in §3.1, $h$ and $\varepsilon$ remain fixed. We want to let $N \to \infty$ in (3.5) to arrive at

LEMMA 3.6. *There exists a unique, twice-differentiable solution* $f$ *of*

$$f + b(f)' - \varepsilon f'' = h \tag{3.20}$$

*which satisfies*

(3.21) $$|f|_1 \leq |h|_1$$

(3.22) $$|f'|_1 \leq 2|h'|_1 + 1.$$

Proof. For each integer $N$, starting with a sufficiently large value, we choose a cutoff function $C_N$, and an integer $N_1 = N_1(N)$, as described in §3.1. The unique solution of (3.5) will be denoted by $f_N$. Using the estimates (3.13) and (3.19), we find

$$|f_N''|_1 = \frac{1}{\varepsilon}|f_N + (C_N b(f_N))' - h|_1 \leq B,$$

with $B$ independent of $N$. Since $f_N(\pm N_1) = 0$, $f_N'(x_0) = 0$ for some $x_0 \in [-N_1, N_1]$. Thus

$$f_N'(x) = \int_{x_0}^{x} f_N''(y)\,dy,$$

whence

(3.23) $$|f_N'|_0 \leq B, \text{ independently of } N.$$

By differentiating (3.5) twice, and imitating the above argument, we find: _There is a constant_ $B$, _independent of_ $N$, _such that_ $|f_N^{(i)}|_0 \leq B$, $i = 0,\ldots,3$. [The supremum is taken over the domain $[-N_1, N_1]$ of $f_N$].

By a diagonal argument, and repeated application of the Arzela-Ascoli theorem, we can assert the existence of a sequence $N_k \longrightarrow 0$, such that given any compact set $T \subset \mathbb{R}$, $\{f_{N_k}\}$ will converge in $C^2(T)$ to a function $f \in C^2(\mathbb{R})$. Furthermore,

$$f + b(f)' - \varepsilon f'' = h.$$

On any interval $[-M, M]$, $f_{N_k} \longrightarrow f$ uniformly, hence in $L^1$; since $|f_{N_k}|_1 \leq |h|_1$, we have

$$\int_{-M}^{M} |f(x)|\,dx \leq |h|_1,$$

whence $f \in L^1(\mathbb{R})$, with the estimate (3.21). (3.22) follows in a similar way from (3.18).

This proves existence. To prove that the solution is unique, we assume that (3.20), (3.21) and (3.22) are satisfied by two functions, $f$ and $g$. On any given interval $[-M,M]$ containing the support of $h$, their difference $f - g = F$ satisfies an equation of the form (see §3.1,C)

$$F + (B \cdot F)' - \varepsilon F'' = 0 \tag{3.24}$$

with non-homogeneous end conditions, say

$$F(-M) = \alpha, \quad F(M) = \beta.$$

One can now attempt to duplicate the argument which led to Lemma 3.3, in order to obtain the estimate $\int_{-M}^{M} |F| \leq 0$. This will not quite succeed, because in general $\alpha, \beta \neq 0$. There will be certain non-zero contributions from the endpoints in the parabolic estimates of Lemma 3.4, and one will obtain

$$\int_{-M}^{M} |F(x)|\,dx \leq k(M), \tag{3.25}$$

where $k(M) = O(|\alpha|+|\beta| + |F'(M)| + |F'(-M)|)$. This, however, is enough: (3.21), (3.22), and the $L^1$-estimate on $F''$ supplied by (3.24) show that $F(x)$ and $F'(x)$ tend to $0$ as $|x| \to \infty$; hence, $k(M) \to 0$ as $M \to \infty$, and so from (3.25),

$$|F|_1 \leq 0.$$

The details of this argument could easily be supplied by the interested reader.

3.3. Existence and uniqueness as $\varepsilon \to 0$; $h \in C_0^\infty$.

A. For each $\varepsilon > 0$, there is a unique solution $f_\varepsilon$ of (3.20), satisfying (3.21-.22). The uniform estimate $|f_\varepsilon|_{BV} \le R$ implies that the family $\{f_\varepsilon\}$ is of uniformly bounded variation. By Helly's theorem, there exists a subsequence $\{f_{\varepsilon_k}\}$ which converges pointwise to a function $f$ of bounded variation. Let us write $f_k = f_{\varepsilon_k}$.

On any interval $[-M,M]$, $f_k \to f$ boundedly, since $|f_k|_\infty \le R$. Hence, $f_k \to f$ in $L^1([-M,M])$, and from

$$\int_{-M}^{M} |f_k(x)|\,dx \le |h|_1 ,$$

we conclude (as above) that $f \in L^1(\mathbb{R})$, with $|f|_1 \le |h|_1$.

Now we verify that $f$ is a pointwise solution of the equation

$$f + b(f)' = h. \tag{3.0}$$

From the $L^1$-convergence of $f_k$ to $f$ on compact sets follows the weak convergence as well; in particular if $\varphi \in C_0^\infty(\mathbb{R})$, then

$$\int h\varphi\,dx = \int \{f_k + b(f_k)' - \varepsilon_k f_k''\}\varphi\,dx =$$

$$= \int \{f_k\varphi - b(f_k)\varphi' - \varepsilon_k f_k\varphi''\}\,dx,$$

and since $\varepsilon_k \int f_k\varphi''\,dx \to 0$ . $\int \varphi''\,dx = 0$ as $\varepsilon_k \to 0$, we obtain, in the limit:

$$(3.26) \qquad \int h\varphi dx = \int \{ f\varphi - b(f)\varphi' \} dx, \quad \varphi \in C_o^\infty(\mathbb{R}).$$

Now observe that (3.26) remains valid even for functions $\varphi$ which are approximated by $\varphi_n \in C_o^\infty$, in the sense that

$$|\varphi_n - \varphi|_o \longrightarrow 0, \quad |\varphi_n' - \varphi'|_1 \longrightarrow 0.$$

Fix an $x_o \in \mathbb{R}$, and $\Delta > 0$, and define

$$\varphi_{x_o,\Delta}(x) = \begin{cases} 1, & x \leq x_o - \Delta \\ \frac{1}{2\Delta}(x_o+\Delta-x), & |x-x_o| < \Delta \\ 0, & x \geq x_o + \Delta. \end{cases}$$

Then

$$\varphi_{x_o,\Delta}'(x) = \begin{cases} -\frac{1}{2\Delta}, & |x-x_o| < \Delta \\ 0, & \text{otherwise.} \end{cases}$$

The function $\varphi_{x_o,\Delta}$ can be approximated by $C_o^\infty$ functions in the manner just described; inserting it into (3.26), we get

$$(3.27) \qquad \int (f-h)\varphi_{x_o,\Delta} dx = \frac{1}{2\Delta}\int_{x_o-\Delta}^{x_o+\Delta} b(f)\,dx.$$

Now let $\Delta \longrightarrow 0$. Because $f - h \in L^1 \cap L^\infty$, the left side tends to

$$\int_{-\infty}^{x_o} f - h dx$$

for any choice of $x_o$. The right side converges to $-b(f(x_o))$ for a.e. $x_o$. Hence, redefining $f$ on a set of measure zero, if necessary, we find that

$$\int_{-\infty}^{x} f(y) - h(y)\,dy = -b(f(y)),$$

for all $x \in \mathbb{R}$. We have proved:

EXISTENCE RESULT. *There exists at least one function* $f \in L^1$ *with these properties:*

(3.28)
- i) $f$ *is of bounded variation, and bounded;*
- ii) $|f|_1 \leq |h|_1$;
- iii) $b(f)$ *is absolutely continuous;*
- iv) $b(f)' \in L^1$;
- v) $f + b(f)' = h$, *for* a.e. $x$.

B. The argument of (A) shows that, for any choice of $\varepsilon_k \longrightarrow 0$, a subsequence of the solutions $f_{\varepsilon_k}$ of (3.20) converges to a solution of (3.0), but it is not clear that two different sequences $\{\varepsilon_k\}$ determine the same limit function. Here we prove that this is in fact the case.

DEFINITION. *For* $\alpha, \beta \in \mathbb{R}$, *put*

$$T(\alpha,\beta) = \operatorname{sgn}(\alpha-\beta)(b(\alpha)-b(\beta)).$$

UNIQUENESS RESULT. *There exists exactly one* $f \in L^1$ *which satisfies* (3.28 (i),(iii)-(v)), *and the supplementary condition*

$$T(f(x^-),k) \geq T(f(x^+),k) \qquad (3.29)$$

*for all* $x$, *and all real* $k$. ($f(x^-)$, $f(x^+)$ are the left, resp. right, limits of $f$ at $x$).

Remark. (3.29) is similar to, though not the same as, the entropy condition on solutions of the conservation law (1.1). As an example, take b to be convex. If f has a discontinuity at x, then (3.29) requires $b(f(x^+) = b(f(x\sim))$ and $f(x\sim) > f(x^+)$. It is a familiar fact that for such a function b, the entropy solution of (1.1) can also have decreasing jump-discontinuities only. This relationship is to be expected, since the latter solution is a limit of solutions of ordinary differential equations of the form (3.0).

Proof. Suppose that both f and g have the stated properties. In particular,

(3.30) $$T(f(x''),k) \ \hat{}\, T(f(x^+),k)$$

(3.31) $$T(g(x''),k) \ \hat{}\, T(g(x^+),k),$$

for all x and k. Set $k = g(x\sim)$ in (3.30), and $k = g(x^+)$ in (3.31); using the symmetry of T, one finds

(3.32) $$T(f(x''),g(x'')) \ \hat{}\ T(f(x^+),g(x^+)).$$

Define $T(x) = T(f(x),g(x))$. It is known that r is of bounded variation, and that its distribution derivative 5r satisfies

(3.33) $$J \int_a^{\beta} STdx = T(P\sim) - r(a^+)$$

(see [9,equ. (5.22)]). Furthermore, the one-dimensional version of (5.29) in [9] (an integration-by-parts formula) gives

$$(3.34) \qquad \int_\alpha^\beta \partial\tau\, dx = \int_\alpha^\beta \{b(f)' - b(g)'\}\mathrm{sgn}(f-g)\, dx + \Sigma\, \tau(x^+) - \tau(x^-),$$

where the summation is taken over the points of discontinuity of $f - g$ in $(\alpha,\beta)$. We transform (3.34) by substituting $b(f)' - b(g)' = -(f-g)$, and using the fact that $\tau(x^+) \leq \tau(x^-)$ (cf.(3.32)):

$$\int_\alpha^\beta \partial\tau\, dx \leq -\int_\alpha^\beta |f-g|\, dx,$$

or, by (3.33)

$$(3.35) \qquad \int_\alpha^\beta |f-g|\, dx \leq \tau(\alpha^+) - \tau(\beta^-).$$

Since $b(f), b(g) \in L^1$, it follows from the definition of $\tau$, that the right side of (3.35) tends to zero as $\alpha \longrightarrow -\infty$, $\beta \longrightarrow +\infty$ through suitably chosen subsequences. Thus $|f-g|_1 \leq 0$, whence $f = g$ a.e. Q.E.D.

<u>LEMMA 3.7</u>. <u>Suppose</u> $f$ <u>is a limit of a sequence</u> $f_{\varepsilon_k}$ <u>of solutions of equation</u> (3.20) (<u>as described under</u> (A)). <u>Then</u> $f$ <u>satisfies</u> (3.29).

<u>Proof</u>. Let $\Phi$ be a convex, twice-differentiable function on $\mathbb{R}$, and let $\psi \in C_0^\infty(\mathbb{R})$, $\psi \geq 0$. Multiply the equation $f_{\varepsilon_k} + b(f_{\varepsilon_k})' - \varepsilon_k f''_{\varepsilon_k} - h = 0$ by $\Phi'(f_{\varepsilon_k})\psi$ and integrate by parts:

$$\int [\Phi'(f_{\varepsilon_k})(f_{\varepsilon_k} - h)\psi - \{\int_k^f b'(y)\Phi'(y)\, dy\}\psi' - \varepsilon_k \Phi(f_{\varepsilon_k})\psi'']\, dx =$$

$$= -\int \Phi''(f_{\varepsilon_k})(f'_{\varepsilon_k})^2 \psi \leq 0.$$

As $\varepsilon_k \to 0$, this results in

(3.36) $\int [\Phi'(f)(f-h)\psi - \{\int_k^f b'(y)\Phi'(y)dy\}\psi']dx \leq 0.$

Now (3.36) is still valid for functions $\Phi$ for which $\Phi'$ is approximable in $L^1$ by $C_0^\infty$-functions; with the choice $\Phi(\eta) = |\eta - k|$, we get

(3.37) $\int \psi(f-h)\,\mathrm{sgn}(f-k)\,dx - \int \mathrm{sgn}(f-k)\{b(f) - b(k)\}\cdot\psi' \; dx \leq 0.$

Let $x_0$ be fixed, and choose a sequence $\{\psi_n\}$ of triangular functions: $\psi_n(x_0) = 1$, $\psi_n(x) = 0$ for $|x-x_0| \geq \frac{1}{n}$, and piece-wise-linear for $|x-x_0| \leq \frac{1}{n}$. (3.37) is still valid for these $\psi_n$; as $n \to \infty$, the first integral tends to zero, and the second to $T(f(x^-),k) - T(f(x^+),k)$. (3.29) then follows. Q.E.D.

The proof of Lemma 3.7 is a one-dimensional version of an argument given by Kružkov [3].

### 3.4. Extension to general h.

<u>LEMMA 3.8.</u> <u>Let</u> $h_1, h_2 \in C_0^\infty$, <u>and let</u> $f_1, f_2$ <u>be the</u> (<u>unique</u>) <u>solutions of</u> $f + b(f)' = h_i$, $i = 1,2$, <u>obtained in</u> §3.3. <u>Then</u>

(3.38) $$|f_1 - f_2|_1 \leq |h_1 - h_2|_1.$$

<u>Proof</u>. $f_i$ is a limit of solutions of approximate equations $f + (C_N b(f))' - \varepsilon_k f'' = h_i$, $f(\pm N_1) = 0$; by virtue of the uniqueness result in §3.3, we may assume $\{C_N\}$, $\{N_1(N)\}$, and $\{\varepsilon_k\}$ to be independent of $i = 1,2$. In a by now familiar manner, Lemma 3.4 yields

$$|f_{1,N,\epsilon_k} - f_{2,N,\epsilon_k}|_1 \le |h_1 - h_2|_1$$

for the solutions of the approximating equations, and (3.38) follows by passage to the limit. Q.E.D.

Now let $h \in L^1$, and let $\{h_n\} \subset C_0^\infty$, $h_n \longrightarrow h$ in $L^1$. Let $f_n$ be the unique solution of $f + b(f)' = h_n$. From (3.38), we infer the existence of $f \in L^1$, for which $|f_n - f|_1 \longrightarrow 0$. Hence, $b(f_n)' = h_n - f_n \longrightarrow h - f$ in $L^1$. We may assume that $\{f_n\}$ and $\{h_n\}$ converge pointwise a.e.; then, for a.e. $x$,

$$b(f(x)) = \lim_{n\to\infty} b(f_n(x)) = \lim_{n\to\infty} \int_{-\infty}^{x} b(f_n(y))'dy =$$

$$= \lim_{n\to\infty} \int_{-\infty}^{x} h_n(y) - f_n(y)\,dy = \int_{-\infty}^{x} h(y) - f(y)\,dy.$$

Thus, after changing $f$ on a set of measure zero, if necessary, we conclude: $b(f)$ _is absolutely continuous, and_ $f + b(f)' = h.$

It is clear that this solution $f$ is independent of the sequence $\{h_n\}$. Following Kružkov, we could characterize $f$ as the unique solution of (3.0) which satisfies (3.37) (equ. (3.29) need no longer be meaningful), but this will not be necessary.

Of course, the results obtained in this paragraph hold equally well for the equation

$$f + \lambda b(f)' = h; \tag{3.39}$$

we summarize the above work in

THEOREM 1. _Let_ $\lambda > 0$, _and_ $h \in L^1$. _Then equation_ (3.39) _has a solution_ $f$ _with these properties_:

i) *$f \in L^1$, and $|f|_1 \leq |h|_1$;*

ii) *$b(f)$ is absolutely continuous;*

iii) *$b(f)' \in L^1$.*

*Furthermore, there is a unique solution which is either of bounded variation and satisfies (3.29), or (if no such solution exists) is the limit of solutions satisfying (3.29).*

Borrowing some terminology from hyperbolic equations, we will call this uniquely determined solution of (3.39) the *entropy solution*.

3.5. *Concerning Theorem O.*

It is now a simple matter to verify that the Crandall-Liggett theorem is applicable. $D(A)$ *is defined to be the set of entropy solutions of equation* (3.0), *as* $h$ *ranges over* $L^1$. Now:

1) $D(A)$ *is dense in* $L^1$. Indeed, $C_0^\infty \subset D(A)$, for if $g \in C_0^\infty$, then $g$ satisfies the equation (in $f$) $f + b(f)' = g + b(g)'$. Also, $g$ is continuous, so (3.29) holds trivially, and $g$ is an entropy solution.

2) $(C)_2$ *holds*. Let $h \in L^1$, and let $f_1$ be the entropy solution of $f(x) + b(f(x))' = h(\lambda x)$. Define $f_\lambda(x) = f_1(\frac{x}{\lambda})$. Then $f_\lambda$ is a solution of equation (3.39). It is easy to verify that it is actually the entropy solution.

3) $(C)_1$ *holds*. We know that $|f_1|_1 \leq \int |h(\lambda x)|dx = \frac{1}{\lambda}|h|_1$. The scaling under (2) then makes $|f_\lambda|_1 \leq |h|_1$, or $|(I+\lambda A)f_\lambda|_1 \geq |f_\lambda|_1$. (2.1) follows in a similar way from (3.38).

§4. Weak Solutions. In this paragraph, $X$ is a general Banach space, and $A$ a densely defined operator satisfying $(C)_1$ and $(C)_2$ of Theorem 0; thus, $A$ generates a semigroup $S$. We assume that $A$ contains a factorization $LB$, meaning that $D(LB) \subset D(A)$, where:

I. $L$ is a linear operator with non-empty resolvent set;

II. (for simplicity) $D(B)$ is closed;

III. $B : D(B) \to X$ is continuous from the strong topology to the weak $(X^*)$ topology;

IV. $J_\lambda(D(A) \cap D(B)) \subseteq D(A) \cap D(B)$ (together with II, this implies $S_t D(B) \subseteq D(B)$).

We have set

$$J_\lambda =_{def} (I+\lambda A)^{-1}.$$

DEFINITION. A function $u : \mathbb{R}^+ \to D(B)$ is a weak solution of the Cauchy problem

$$\frac{du(t)}{dt} + Au(t) = 0, \quad u(0) = x \in X, \tag{4.1}$$

if

$$\int_0^\infty \langle u(t), \varphi'(t)\rangle + \langle Bu(t), L^* \varphi(t)\rangle dt = 0$$

for all

$$\varphi \in C_0^2((0,\infty); D(L^*)), \text{ and if}$$

$$\|u(t)-x\| \to 0 \text{ as } t \to 0. \tag{4.3}$$

(We will use $x,y$ to denote elements of $X$; this should cause no confusion with our earlier notation $x \in H$. $\langle \cdot,\cdot \rangle$ is the pairing between $X$ and $X^*$, and $\|\cdot\|$ is the norm in $X$.)

THEOREM 2. *Let* $A$ *have the structure described above. Then if* $x \in D(B)$, *the function* $t \mapsto S_t x$ *is a weak solution of* (4.1).

*Proof*. It is enough to prove the theorem when $x \in D(A) \cap D(B)$. Assume this has been accomplished, and let $x_n \in D(A) \cap D(B)$ converge to $x \in D(B)$. For each $t \geq 0$, $S_t x_n \to S_t x$ (strongly), so by III and IV, $BS_t x_n \to BS_t x$ (weakly). If $\varphi \in C_0^2((0,\infty); D(L^*))$, then

(4.4) $$\langle S_t x_n, \varphi'(t)\rangle + \langle BS_t x_n, L^*\varphi(t)\rangle \to \langle S_t x, \varphi'(t)\rangle + \langle BS_t x, L^*\varphi(t)\rangle$$

for each $t$; since $\varphi$ has compact support in $t$, this convergence is dominated by an integrable function of $t$, and we obtain the desired conclusion by integrating (4.4).

Hence, let $x \in D(A) \cap D(B)$. Put $J_\lambda = (I+\lambda A)^{-1}$. With this notation,

$$S_t x = \lim_{n\to\infty} J_{\frac{t}{n}}^n x,$$

and

$$t^{-1}(S_t x - x) = \lim_{n\to\infty} t^{-1}(J_{\frac{t}{n}}^n x - x) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \left(\frac{t}{n}\right)^{-1} \left(J_{\frac{t}{n}}(J_{\frac{t}{n}}^i x) - J_{\frac{t}{n}}^i x\right)$$

$$= \lim_{n\to\infty} -\frac{1}{n} \sum_{i=0}^{n-1} \left(\frac{t}{n}\right)^{-1} (I - J_{\frac{t}{n}})(J_{\frac{t}{n}}^i x).$$

Since $AJ_\lambda = \lambda^{-1}(I - J_\lambda)$, the last sum may be written

(4.5) $$t^{-1}(S_t x - x) = \lim_{n\to\infty} -\frac{1}{n} \sum_{i=0}^{n-1} AJ_{\frac{t}{n}}^{i} x, \qquad x \in D(A).$$

Using (4.5), Crandall and Liggett show that if $A$ is strong-weak continuous, then

$$\operatorname{weak-lim}_{t\to 0} t^{-1}(S_t x - x) + Ax = 0.$$

We want to reach a related conclusion, but must first remove the unbounded factor $L$.

Let $w \in \rho(L)$, i.e., $(L-wI)^{-1}$ is bounded. Then, so is

$$L_w =_{\text{def}} (L-wI)^{-1}L = L(L-wI)^{-1}.$$

Now, observe that $L_w B$ is strong-weak continuous. Let $U$ be a weak convex neighborhood of $0$.

For $y \in D(A)$ ([1]),

$$\|y - J^i_{t/n} y\| \le t\|Ay\|, \quad 0 \le i \le n. \tag{4.6}$$

By weak continuity of $L_w B$, if $t$ is small enough, then

$$L_w By - L_w B(J^i_{t/n} y) \in U, \quad y \in D(A) \cap D(B). \tag{4.7}$$

From (4.7), we get -- using convexity of $U$ --

$$L_w By - L_w\Big\{\frac{1}{n}\sum_{i=0}^{n-1} B(J^i_{t/n} y)\Big\} \in U, \tag{4.8}$$

for $t$ small enough, and all $n$.

Now by (4.5),

$$(L-wI)^{-1}[t^{-1}(S_t y - y)] = -\lim_{n\to\infty} L_w\Big\{\frac{1}{n}\sum_{i=0}^{n-1} B(J^i_{t/n} y)\Big\},$$

and so from (4.8)

$$(L-wI)^{-1}[t^{-1}(S_t y - y)] + L_w By \in U,$$

for $t$ small enough.

Hence

(4.9) $$\text{weak-}\lim_{t\to 0} (L-wI)^{-1}[t^{-1}(S_t y-y)] = -L_w By, \; y \in D(A) \cap D(B).$$

We will use (4.9) to establish the connection with weak solutions.

Let $\varphi \in C_0^2((0,\infty); D(L^{*2}))$. We wish to show

(4.10) $$\int_0^\infty \langle S_t x, \varphi'(t)\rangle dt = -\int_0^\infty \langle BS_t x, L^* \varphi(t)\rangle dt$$

(condition (4.3) is guaranteed to hold by strong continuity of $S_t$).

Write

(4.11) $$\varphi'(t) = -\varepsilon^{-1}(\varphi(t)-\varphi(t-\varepsilon))+O(\varepsilon),$$

taking $\varepsilon$ so small that $\varphi(\tau) \equiv 0$ for $0 \leq \tau \leq \varepsilon$. $O(\varepsilon)$ is independent of $t$, since $\varphi$ has compact support. Inserting (4.11) into the left side of (4.10), and making the substitution $t - \varepsilon \to t$ in one of the resulting integrals, we get

(4.12) $$\int_0^\infty \langle S_t x, \varphi'(t)\rangle = \int_0^\infty \langle \varepsilon^{-1}(S_{t+\varepsilon}x - S_t x), \varphi(t)\rangle dt + O(\varepsilon).$$

We write the second integrand as

$$\langle (L-wI)^{-1} \varepsilon^{-1}(S_\varepsilon - I) S_t x, \quad (L-wI)^* \varphi(t)\rangle,$$

and note that according to (4.9), it converges to

(4.13) $$-\langle L_w BS_t x, \; (L-wI)^* \varphi(t)\rangle = -\langle BS_t x, L^* \varphi(t)\rangle$$

as $\varepsilon \to 0$.

If we let $\varepsilon \longrightarrow 0$ in (4.12), and use (4.13), we obtain the desired conclusion (4.10). Q.E.D.

REMARK. The hypothesis III is weaker than necessary for our application; perhaps it will be appropriate for some other problem. On the other hand, II and IV could probably be weakened. Finally, $D(L^{*2})$ may be very small (unless $L$ is suitably restricted). The theorem would then be true, but not very interesting.

§5. The Ordering Principle. In this section, we adapt a characterization, due to K. Sato [8], of (linear) generators of positive semigroups on a Banach lattice. It will actually be more convenient to work in this general setting for a while, but we will not try to find hypotheses which are either precise or weak. The subject of nonlinear, order-preserving semigroups seems to be of sufficient interest to warrant a more thorough, separate study.

So, let $X$ be a Banach lattice (as defined, for example, in [8]), with cone $K$ of non-negative elements. We write $\|x^+\| = \|x\|^+$, and assume that $\|x_n - x\|^+ \longrightarrow 0$ whenever $\|x_n - x\| \longrightarrow 0$ (this certainly holds in $L^1$). Following Sato, we define

$$\tau(x,y) = \lim_{\varepsilon \to 0^+} \varepsilon^{-1}(\|x+\varepsilon y\| - \|y\|),$$

$$\sigma(x,y) = \inf \tau(x,(y+z) \vee (-\beta x)),$$

where $x \in K$, and the inf is taken over all $\beta > 0$, and $z$ satisfying $|z| \wedge x = 0$.

THEOREM 3. *Let* $A$ *satisfy the hypotheses of Theorem* 0, *and suppose that*

$$\sigma((x-y)^+, Ax-Ay) \geq 0$$

*for all* $x,y \in D(A)$, $\lambda > 0$. *Then*

$$\|S_t x - S_t y\|^+ \leq \|x-y\|^+.$$

*Proof*. We record the following properties of $\sigma$ (see [8, Prop. 3.1]):

i) $\sigma(x,y) \leq \|y\|^+$,

ii) $\sigma(x,ax+y) = a\|x\| + \sigma(x,y)$ for all $a \in \mathbb{R}$,

iii) if $x \wedge |z| = 0$, then $\sigma(x,y) = \sigma(x,y+z)$.

iv) for $a > 0$, $\sigma(x,y) \geq 0 \Rightarrow \sigma(x,ay) \geq 0$.

(Always, $x \geq 0$.)

Now let $u,v \in D(A)$, and $(I+\lambda A)u = x$, $(I+\lambda A)v = y$. Then

$$\begin{aligned}\|x-y\|^+ &\geq \sigma((u-v)^+, x-y) = \sigma((u-v)^+, (u-v)+\lambda(Au-Av)) = \\ &= \sigma((u-v)^+, (u-v)^+ + \lambda(Au-Av)) = \\ &= \|u-v\|^+ + \sigma((u-v)^+, \lambda(Au-Av)) \geq \\ &\geq \|u-v\|^+.\end{aligned}$$

Alternatively,

$$\|J_\lambda x - J_\lambda y\|^+ \leq \|x-y\|^+. \tag{5.1}$$

By iteration of (5.1),

$$\|J^n_{\frac{t}{n}}x - J^n_{\frac{t}{n}}y\|^+ \leq \|x-y\|^+,$$

and so, in the limit $n \to \infty$,

$$\|S_t x - S_t y\|^+ \leq \|x-y\|^+. \qquad \text{Q.E.D.}$$

If $X = L^1$, we have the following explicit expression for $\sigma$ (See [8]):

$$\sigma(f^+,g) = \int_{\{f>0\}} g(x)\,dx.$$

In order to apply Theorem 3 to the conservation law, we need only verify that

$$(5.2) \qquad \int_{\{f>g\}} b(f)' - b(g)'dx \geq 0,$$

whenever $f,g \in D(A)$. If $f,g$ are of bounded variation, then the one-dimensional Green formula ([9,(5.22)]) shows that the integral in (5.2) is actually zero, since $b(f) - b(g) = 0$ on the boundary of the set $\{f>g\}$. It was established in §3.4 that every $f \in D(A)$ is the limit, in $L^1$, of functions $f_n$ of bounded variation, for which (5.2) is valid; because $b(f_n)' \to b(f)'$ in $L^1$, (5.2) follows easily for $f$.

(A somewhat longer proof, using approximation by smooth solution of the second-order equation (3.20), but not requiring Green's formula, would also be possible).

§6. We wish to apply Theorem 2, or more precisely, the inequality

$$\text{(5.1)} \qquad \|J_\lambda x - J_\lambda y\|^+ \le \|x-y\|^+$$

obtained during the proof of Theorem 2, to connect the semigroup solutions of (1.1-.2) with the usual weak solutions. To this end, we observe that $A$ contains a factorization $LB$, where $L = -\frac{d}{dx}$, $Bf = b(f)$; the domain of $B$ is taken as

$$D(B) = L^1 \cap \{f ; |f| \le M\}$$

for some fixed $M > 0$, and $D(L)$ is the usual domain of $\frac{d}{dx}$. Of the conditions I - IV of §5, only IV requires a proof, which we now supply.

<u>Recall IV</u>; <u>For any</u> $\lambda > 0$, $J_\lambda(D(A) \cap D(B)) \subset D(A) \cap D(B)$.

<u>Proof</u>. Certainly, $J_\lambda(D(A) \cap D(B)) \subset D(A)$. Now let $f \in D(A) \cap D(B)$, and let $\eta > 0$ be given. Choose $K$ so that $\int_{|x|>K} |f(x)|\,dx < \eta$. Now define the function $\varphi$ as follows:

$$\varphi(x) = \begin{cases} M, & \text{for } |x| \le K \\ M \exp-(x-K)/\lambda B, & x > K \\ M \exp(x+K)/\lambda B, & x < -K, \end{cases}$$

where $B = \sup\{|b'(y)| ; |y| \le M\}$.

Observe that $\varphi \in D(A)$: clearly, $\varphi$ and $b(\varphi)'$ are integrable, and $b(\varphi)$ is absolutely continuous; moreover, $\varphi$ is an entropy solution because it has no jumps (see §3.5).

Define $\psi = \varphi + \lambda b(\varphi)'$. For $|x| < K$, $\psi = \varphi = M$, and since $|f|_\infty \leq M$, we have

$$(f-\psi)^+ = 0 \quad \text{on } [-K,K].$$

For $|x| > K$,

$$\psi = \varphi + \lambda b(\varphi)' \geq \varphi - \lambda B|\varphi'| = 0;$$

hence

$$(f-\psi)^+ \leq f^+ \quad \text{for } |x| > K.$$

From (5.1) we now find

$$|J_\lambda f-\varphi|_1^+ = |J_\lambda f-J_\lambda\psi|_1^+ \leq |f-\psi|_1^+ \leq \int_{|x|>K} |f^+(x)|dx < \eta.$$

In particular, since $\varphi = M$ on $[-K,K]$,

$$\int_{-K}^{K} (J_\lambda f(x)-M)^+ dx < \eta.$$

We conclude that $(J_\lambda f)(x) \leq M$ for a.e. $x$.

In a similar fashion, one shows that $(J_\lambda f)(x) \geq -M$, and hence that $|J_\lambda f|_\infty \leq M$, so that $J_\lambda f \in D(B)$. Q.E.D.

By the results of §4, the function $t \longmapsto S_t$ is an abstract weak solution for $f \in D(B)$; given our choice of $L$, it is clearly a weak solution in the sense of the definition in §1.

§7. <u>Concluding remarks.</u> Although the semigroup approach to the conservation law (1.1) is rather different, conceptually, from the other treatments, the technical details have proved to be less independent of the known theory; specifically, our uniqueness argument in §3.4 makes use of some of the most important ideas in the definitive papers by Vol'pert and Kružkov. Nonetheless, there are some differences of degree, which I want to point out again: (1) the present approach requires only the local, not the global, solvability of a parabolic initial value problem; (2) the study of entropy solutions takes place in one dimension, not two.

Without much current justification, I entertain the hope that these differences might facilitate the study of hyperbolic systems. There are other difficulties in the way of an extension in that direction, and the first of these is the choice of an appropriate function space. Once this step has been taken, even for a specific class of systems, the other details should be manageable. One might try to begin with a "simple" nonlinear wave-equation, for which arbitrary Riemann problems are known to be solvable. There are some problems which may be more accessible, whose solution would be of interest (to the author, at least):

It seems desirable to streamline some of our arguments. For instance: in §3, p.d.e.'s were used to study an ordinary differential equation. Can this methodological detour be avoided?

Can the uniqueness proof of §3 be simplified? The tools we used may be more powerful than necessary for the one-dimensional problem.

Can the abstract weak-solution theorem (§4) be applied to (1.1) without use of the ordering principle?

Some possible generalizations also suggest themselves:

In §6, we showed (although this was not pointed out explicitly) that A generates a semigroup on the set $\{f \in L^{\infty};\ |f|_{\infty} \leq M\}$. Problem: find conditions on operators L and B, supplementing those of §4, which guarantee that A generates a semigroup on D(B).

In many concrete evolution equations, the operator A is monotone and coercive, and surjectivity of $I + \lambda A$ follows from a general theory. Is there a more abstract point of view, of which our result of §3 is a particular example?

Define a functional $\Phi$ on X to be <u>invariant</u> for the semigroup S if $\Phi(S_t x) = \Phi(x)$ for all $t \geq 0$. One expects that if the semigroup generator A is restricted to the set $D(A) \cap \Phi^{-1}(\alpha)$, it should again generate a semigroup, one on whose orbits $\Phi$ is constant $= \alpha$. One can show that $\Phi(f) = \int f dx$ on $L^1$ has precisely that property, relative to the semigroup generated by $b(\cdot)'$. There may be some interest in a general study of such invariant functionals (see [5]).

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