EXISTENCE AND UNIQUENESS OF POSITIVE EIGENFUNCTIONS FOR A CLASS OF QUASILINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS OF SUBLINEAR TYPE

by

Charles V. Coffman

Report 72-6

February 1972

T*his research was supported by National Science Foundation Grant GP-21512.

EXISTENCE AND UNIQUENESS OF POSITIVE EIGENFUNCTIONS FOR A CLASS OF QUASILINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS OF SUBLINEAR TYPE

by

Charles V. Coffman

Abstract

We consider the Euler equations associated with a functional of the form

$$\underline{h}_\lambda(u) = -\int_\Omega F_\lambda(x,u,\text{grad } u)\,dx$$

where

$$F_\lambda(x,u,p) = a(x,u^2,p\otimes p) - \lambda b(x,u^2),$$

and F_λ is convex in $(u^2,p\otimes p)$ for $\lambda > 0$. This class of equations contains the class of semi-linear second order elliptic problems generally referred to as sublinear. Under appropriate conditions of continuity, ellipticity, and "strict sublinearity" we prove existence and uniqueness of a positive u for which $\underline{h}_\lambda$ attains a maximum.

EXISTENCE AND UNIQUENESS OF POSITIVE EIGENFUNCTIONS FOR A CLASS OF QUASILINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS OF SUBLINEAR TYPE

by

Charles V. Coffman

1. Following the classification which appears to have been introduced by Pimbley [14], [15], one calls the semi-linear boundary value problem

(1.1) $$Lu = f(x,u), \qquad x \in \Omega$$

(1.2) $$Mu = 0 \qquad x \in \partial\Omega$$

<u>sublinear</u> if the linear operator $\mathcal{L}$ determined by L and the boundary conditions (1.2) has, in some sense, a positive inverse and $0 < uf(x,tu) \leq tuf(x,u)$ for $x \in \Omega$, $u \neq 0$ and $t \geq 1$; the problem is called <u>superlinear</u> if $\mathcal{L}$ is the same and $0 < f(x,tu) \leq tuf(x,u)$ for $x \in \Omega$, $u \neq 0$, and $0 < t < 1$.

If the operator $\mathcal{L}$ has a positive Green's function, so that (1.1)-(1.2) is equivalent to an integral equation

$$u(x) = \int_\Omega G(x,t)f(t,u(t))dt,$$

then one can expect uniqueness of the positive solution of (1.1)-(1.2), i.e. that this problem can have at most one solution which is positive in Ω. For the second order ordinary differential equation with zero end conditions such a result was

---

This research was supported by National Science Foundation Grant GP-21512.

proved by Picard [13]. (For this case the uniqueness proof reduces to showing that there is never a conjugate point in the interior of the interval. For the superlinear case, as considered in [3], the question of the uniqueness of a positive solution is a more subtle one.) In [17] Urysohn proved uniqueness of a positive solution for a sublinear case of the equation bearing his name, i.e.

$$u(x) = \int_{\Omega} K(x,t;u(t))\,dt.$$

Urysohn's result was abstracted and generalized by Krasnosel'skii and Ladyzhenskii [8]; see also [6] or [7]. Similar uniqueness results are found, for second order ordinary differential equations in [1], [4] and [5], for integral equations in [9] and [12] and for second order elliptic partial differential equations in [16]. Uniqueness results for elliptic problems with non-homogeneous boundary conditions are given in [10].

In this paper we show how the notion of sublinearity can be extended to a class of quasi-linear second order elliptic boundary value problems of variational type. For such equations we give a uniqueness theorem for positive weak solutions. The proof uses elementary variational inequalities; by contrast the proofs of the theorems quoted above depend mainly on order theoretic methods, which in fact are not entirely absent here either.

The problem which we treat here can be written formally as

(1.3) $\sum_{i,j=1}^{N}$ ∂x J*- ½(j(^r- + T T∂-) a (x , u,²ξ(x)) ^—)
j ^ij 5 ji i

+ ||(x,u²(x),?(x))u(x) = -|^ b(x,u²(x)), in Ω

(1.4) u(x) = 0 on ∂Ω,

where %(x) is the matrix (∂u/-> *∂u/r) , and b(x,rj) is concave in T] while a(x,r),?) is convex in (rj,?). It is consistent with the classification referred to earlier to call this problem a sublinear boundary value problem. We observe that, formally at least, it includes the semi-linear problems studied in [4] and [16].

2. Let Q be a bounded region in R^N with a C^2 boundary and let Q_o be a subset of £1 of full measure. We will denote by u the set of all triples (x.,rjj,5) where x c Q_o, r\ is a non-negative real number and 5 = (?• •) is a real N x N matrix.

Throughout this paper p will denote a fixed real number with p ^ 2 and s will denote a fixed element in the extended real number system with

(2.1) p £ s, and s < Np/(N-p) if p £ N.

By E_{p*s} we denote the Banach space of measurable functions x —> (tfj?) y T) a real number, 5 a real N x N matrix, which are defined on Q, and for which the following norm is finite

(2.2) 8(77,5) $(\cdot)\|_{p,s} = \|\eta\|_{\frac{s}{2}} + \|\xi\|_{\frac{p}{2}}$,

where U»*«$\|_{\frac{s}{2}}$ i-^{s} *the* L$^{\frac{s}{2}}$ (flJ norm

$$\|\eta\|_{\frac{s}{2}} = (\int_Q |\eta|^{\frac{s}{2}} dx)^{\frac{2}{s}}, \quad s < \infty$$

Ihlloo = ess sup $|\eta|$,

and

$$\|\xi\|_{\frac{p}{2}} = (\int |\xi|^{\frac{p}{2}} dx)^{\frac{2}{p}},$$

where

$$|\xi| = (\sum_{i,j=1}^{N} \xi_{ij}^2)^{\frac{1}{2}},$$

We denote by $E^+_{p,s}$ the "half-space"

[(i],?)eE$_{n\,o}$: TJ(X) ^ 0, X€ Q$_{-}$}.

Let a = afx,^,?,) be a function defined on u and let

a_η (x,T},§) = čia(x,T),§)/d?)

$$a_{ij}(x,\eta,\xi) = \frac{1}{2}(\partial a(x,\eta,\xi)/\partial\xi_{ij} + \partial a(x,\eta,\xi)/\partial\xi_{ji}),$$

we shall assume that a satisfies the following conditions.

(A.I) For each fixed pair *{r\,%),* a(«,rj,§) is measurable, a(x,',-) is of class C^1 and convex in (r?,?) for each fixed $x \in Q_o$ and

(2.3) a(x,O,O) = 0, $x \in O_o$ *

There exists an $m > 0$ such that for all $(x,\eta,\xi) \in U$

$$\sum_{i,j=1}^{N} a_{ij}(x,\eta,\xi)\, t_i t_j \geq m \sum_{i=1}^{N} t_i^2$$

for all real vectors t. Finally

$$a_\eta(x,\eta,\xi) \geq 0.$$

(A.2) The mapping

$$(\eta,\xi)(\cdot) \longrightarrow a_\eta(\cdot,\eta(\cdot),\xi(\cdot))$$

takes $E^+_{p,s}$ into $L^r(\Omega)$ and is bounded on bounded sets and continuous, where

(2.4) $$r = s/(s-2), \qquad (r = 1 \text{ if } s = \infty).$$

For each $i,j = 1,\ldots,N$, the mapping

(2.5) $$(\eta,\xi)(\cdot) \longrightarrow a_{ij}(\cdot,\eta(\cdot),\xi(\cdot))$$

takes $E^+_{p,s}$ into $L^{r_1}(\Omega)$ and is bounded on bounded sets and continuous with respect to the strong topology on $E^+_{p,s}$ and the weak topology on $L^{r_1}(\Omega)$, where

(2.6) $$r_1 = p/(p-2), \qquad (r_1 = \infty \text{ if } p = 2).$$

We shall say that a satisfies condition (A.2+) if it satisfies (A.2) and the mapping (2.5) from $E^+_{p,s}$ to $L^{r_1}(\Omega)$ is continuous; in the case $p = 2$, $r_1 = \infty$, this becomes overly restrictive, when on the other hand $r_1 < \infty$ then (A.2) implies (A.2+).

As a consequence of the convexity assumption in (A.1) we have for symmetric matrices %,, §' the inequality

(2.7) a(x,t?,5) - a(x,Tj',?') 1 $\underset{i,j=l}{\overset{N}{Z}}$ a. .$_{lJ}$(x,TJ,§) (?. $_{3}$-Sj$_{3}$.)

+ a_{η}(x,r»,5) (77-TjM .

With the aid of this inequality we prove the following result.

LEMMA 1. Let (A.I) and (A. 2) hold. Then

(2.8) a(u) = f a(x,u^2(x),5(x))dx

where

defines a continuous functional a., bounded on bounded sets, on $W_o^{1,p}$(« and

(2.9) a(0) = 0.

Moreover, a ^s Gateaux differentiable with derivative A defined by

$$(2.10)\qquad (Au,v) = 2\int_{\Omega}\Big(\sum_{i,j=1}^{N} a_{ij}(x,u^2(x),\xi(x))\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}$$

$$+ a_{\eta}(x,u^2(x),5(x))u(x)v(x)\Big)dx,$$

for $v \in W_Q^{1,P}$(J|, and where § is as above; if (A. 2+) holds, then a is Frechet differentiable.

Proof. Making use of the Sobolev imbedding theorem, we first observe that the mapping $u \longrightarrow (u^2, (\frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}))$ is continuous from $W_o^{1,p}(\Omega)$ to $E_{p,s}^+$, maps bounded sets onto bounded sets, and is Frechet differentiable; its Frechet derivative at $u \in W_o^{1,p}(\Omega)$ to $E_{p,s}$ defined by

$$v \rightarrow (2uv, (\frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} + \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j})).$$

We now consider the functional

$$\int_\Omega a(x,\eta(x),\xi(x))dx \tag{2.11}$$

on $E_{p,s}^+$. From the inequality (2.7), used twice, we have for $(\eta,\xi)(\cdot)$, $(\eta',\xi')(\cdot) \in E_{p,s}^+$

$$\begin{aligned}\int_\Omega (\sum_{i,j=1}^N a_{ij}(x,\eta',\xi')(\xi_{ij}-\xi'_{ij}) + a_\eta(x,\eta',\xi')(\eta-\eta'))dx \\ \leq \int_\Omega (a(x,\eta,\xi) - a(x,\eta',\xi'))dx \\ \leq \int_\Omega (\sum_{i,j=1}^N a_{ij}(x,\eta,\xi)(\xi_{ij}-\xi'_{ij}) + a_\eta(x,\eta,\xi)(\eta-\eta'))dx.\end{aligned} \tag{2.12}$$

The continuity and boundedness on bounded sets of (2.11) on $E_{p,s}^+$ follows from (A.2) and (2.12). If (A.2+) holds then the left-most and right-most terms in (2.12) differ by $o(\|(\eta-\eta',\xi-\xi')\|_{p,s})$ as $(\eta',\xi') \longrightarrow (\eta,\xi)$ in $E_{p,s}^+$. Otherwise we put $\xi' = \xi + th$, $\eta' = \eta + tk$, t real, and then for h,k fixed these same two terms differ by $o(|t|)$ as $t \longrightarrow 0$, and the Gateaux differentiability follows.

The lemma now follows by composition and application of the chain rule; (2.9) follows immediately from (2.3).

We note that Au is of the form $L_u u$ where L_u is the linear operator from $W_o^{1,p}(\Omega)$ to $(W_o^{1,p}(\Omega))^*$ defined by

(2.13) $(L_u w,v) =$

$$2\int_\Omega \Big(\sum_{i,j=1}^{N} a_{ij}(x,u^2(x),\xi(x))\frac{\partial w}{\partial x_i}\frac{\partial v}{\partial x_j} + a_\eta(x,u^2(x),\xi(x))w(x)v(x)\Big)dx.$$

In fact we have the following.

LEMMA 2. *The operator* A *satisfies*

$$Au = L_u u$$

where L_u *is defined by* (2.13). *The mapping*

$$u \longrightarrow L_u$$

is continuous with respect to the strong topology on $W_o^{1,p}(\Omega)$ *and the weak operator topology*.

We now let $b(x,\eta)$ denote a real-valued function defined on $\Omega_o \times \overline{R}_+$ and assume the following.

(B.1) For each fixed $x \in \Omega_o$, $b(x,\cdot)$ is of class C^1 and concave in η,

(2.14) $$b(x,0) = 0, \qquad x \in \Omega_o.$$

(B.2) The mapping

(2.15) $$\eta(\cdot) \longrightarrow b_\eta(\cdot,\eta(\cdot))$$

takes $L^{\frac{s}{2}}(\Omega)$ into $L^r(\Omega)$, and is bounded on bounded sets and continuous, where r is given by (2.4).

The concavity assumption implies

(2.16) $$b(x,\eta) - b(x,\eta') \leq (\eta-\eta')b_\eta(x,\eta'), \quad x \in \Omega_o.$$

LEMMA 3. *The functional*

$$\underline{b}(u) = \int_\Omega b(x,u^2(x))\,dx$$

is bounded on bounded sets and continuous with respect to the weak topology on bounded subsets of $W_o^{1,p}(\Omega)$;

(2.17) $$\underline{b}(0) = 0.$$

Moreover, $\underline{b}$ *is Frechet differentiable with derivative*

$$Bu = M_u u$$

where $M_u : W_o^{1,p}(\Omega) \to (W_o^{1,p}(\Omega))^*$ *is the multiplication operator*

$$w \to 2b_\eta(\cdot,u^2(\cdot))w.$$

The linear operator M_u *and the operator* B *are both compact and*

$$u \to M_u$$

is continuous on $W_o^{1,p}(\Omega)$ *with respect to the uniform operator topology*.

Proof. The boundedness assumption concerning (2.15) is equivalent to

$$|b_\eta(x,\eta)| \leq \sigma(x) + c\eta^{\frac{s}{2r}},$$

for some non-negative $\sigma \in L^r(\Omega)$ and positive constant c. This implies that

$$|b(x,\eta)| \leq \eta(\sigma(x) + 2cr(s+2r)^{-1}\eta^{\frac{s}{2r}}),$$

which in turn implies the boundedness assertion concerning $\underline{b}$. To complete the proof one uses the compactness of the imbedding $W_o^{1,p}(\Omega) \longrightarrow L^s(\Omega)$, the argument is similar to that in the proof of Lemma 1.

We set

$$h(x,\eta,\xi) = b(x,\eta) - a(x,\eta,\xi)$$

and

$$\underline{h} = \underline{b} - \underline{a}$$

and assume the following condition of strict sublinearity.

(H.1) For $\eta \neq \eta'$

$$h(x,\tfrac{1}{2}(\eta+\eta'),\tfrac{1}{2}(\xi+\xi')) > \tfrac{1}{2}(h(x,\eta,\xi) + h(x,\eta',\xi')).$$

Note that the convexity assumption on a and the concavity assumption on b, which were made respectively in (A.1) and (B.1) above imply

$$h(x,\tfrac{1}{2}(\eta+\eta'),\tfrac{1}{2}(\xi+\xi')) \geq \tfrac{1}{2}(h(x,\eta,\xi) + h(x,\eta',\xi')).$$

LEMMA 4. *Let* (A.1), (A.2), (B.1), (B.2) *and* (H.1) *hold*. *Then for* $u,v \in W_o^{1,p}(\Omega)$

(2.18) $$\underline{h}(v) - \underline{h}(u) \leq (L_u u,u) - (M_u u,u) - (L_u v,v) + (M_u v,v),$$

with equality only if $|u| = |v|$.

Proof, We use the inequalities (2.7) and (2.16) to estimate

$$h(x, v^2, \frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}) \le h(x, u^2, (\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j})),$$

and then integrate to obtain (2.18). From (H.I) it follows that the inequality in the indicated estimate will be strict at points $x \in \Omega_o$ where $|u(x)| \neq |v(x)|$, it follows that (2.18) is strict unless $|u(x)| = |v(x)|$ almost everywhere in Ω.

3. Throughout this section we shall assume the conditions (A.I), (A.2), (B.I), (B.2) and (H.I).

THEOREM 1. If

(3.1) $\qquad b(x,\eta) > 0$ for all $x \in \Omega_o$ and all η

then the weak problem (1.3)-(1.4), i.e.

(3.2) $\qquad L_u u = M_u u$

has at most one non-trivial non-negative solution. If such a solution exists then it must be positive almost everywhere.

Proof. Let u be a non-trivial non-negative solution of (3.2), then from (2.18) and (3.2)

$$\underline{h}(u) > \underline{h}(v) + (L_u v,v) - (M_u v,v)$$

for all $v \in W_o^{1,2}(\Omega)$ with $|v| \le u$, the inequality being strict by Lemma 4. Since u is an eigenfunction corresponding to the eigenvalue 1 of

$$L_u w = \lambda M_u w,$$

it follows from Theorem 3, in the appendix , that

$$(L_u v, v) - (M_u v, v) > 0$$

unless v is proportional to u. We conclude that $\underline{h}$ assumes a maximum in $W_o^{1,p}(\Omega)$ at and only at u and $-u$. It follows that (3.2) cannot have another non-trivial non-negative solution.

Remarks. The non-negativity condition on $a_\eta(x,\eta,\xi)$ in assumption (A.1) can be relaxed to

$$a_\eta(x,\eta,\xi) \geq -m', \qquad (x,\eta,\xi) \in U$$

where m' depends only on m in (5.2) and on Ω, and is chosen so as to guarantee that L_u is strictly definite on $W_o^{1,p}(\Omega)$. This is particularly relevant for the semi-linear case, where we obtain a theorem similar to that of Rabinowitz [16]. The proof in [16] makes essential use of the maximum principle; here also we use this principle (see the proof of Lemma 8) but only for harmonic functions.

(2) The variational uniqueness principle for sublinear problems which is developed here is in a sense dual to the variational existence principles used by Nehari in [11] and subsequent work are further exploited by the author in [2].

4. In this section we shall prove the existence of an almost everywhere positive solution in $W_o^{1,p}(\Omega)$ of (3.2). In order to do this we must make the following assumption.

(H.2) For p as above there exists a positive constant $e > 0$ and there exist functions $f \in L^r(\Omega)$, $g \in L^1(\Omega)$, (where r is given by (2.4), and s satisfies (2.1)) such that for $x \in \Omega_o$, $\eta \geq 0$, $t \in R^N$,

(4.1) $$b(x,\eta) - a(x,\eta,(t_i t_j)) \leq -e|t|^p + f(x)\eta + g(x).$$

If $p = 2$ then for each $x \in \Omega_o$, $a(x,u^2,(t_i t_j))$ is convex in (u,t), and, given any $\varepsilon > 0$, f and g can be chosen so that

$$\|f\|_r < \varepsilon \quad .$$

<u>THEOREM 2</u>. <u>Let</u> (A.1), (A.2), (B.1), (B.2) <u>and</u> (H.2) <u>hold</u>. <u>Assume also that the spectrum of the linear eigenvalue problem</u>

(4.2) $$L_o w = \lambda M_o w$$

<u>contains a number</u> < 1. <u>Then</u> (3.2) <u>has a non-negative non-trivial solution</u> u.

<u>Proof</u>. From (H.2) it follows that for $u \in W_o^{1,p}(\Omega)$

$$\underline{h}(u) \leq -e\|u\|_{1,p}^p + \|f\|_r \|u\|_s^2 + \|g\|.$$

For such a u,

$$\|u\|_s \leq C\|u\|_{1,p},$$

thus, by making a suitable choice of f,g if $p = 2$, we have

$$\underline{h}(u) \le -\varepsilon' \|u\|^{p}_{1,p} + \gamma \qquad (4.3)$$

where $\varepsilon' > 0$, γ is a constant.

From (2.18), since $\underline{a}$ and $\underline{b}$ vanish at 0

$$\underline{b}(u) - \underline{a}(u) \ge (M_u u, u) - (L_u u, u). \qquad (4.4)$$

From the hypothesis concerning the spectrum of (4.2) and the continuity assertions in Lemmas 2 and 3 concerning the operators L_u and M_u it follows that for some $u \in W^{1,p}_{0}(\Omega)$

$$(M_u u, u) - (L_u u, u) > 0. \qquad (4.5)$$

indeed by hypothesis there is a v with $(M_0 v,v) - (L_0 v,v) > 0$, then as $\alpha \to 0$,

$$(M_{\alpha v} v, v) - (L_{\alpha v} v, v) \longrightarrow (M_0 v, v) - (L_0 v, v),$$

putting $u = \alpha v$, for some small α, we have 4.5. From (4.4) and (4.5) it follows that

$$\sup(\underline{h}(u) : u \in W^{1,p}_{0}(\Omega)) > 0. \qquad (4.6)$$

From (4.3) we conclude that $\underline{h}$ is bounded above and $\underline{h}(u) \to -\infty$ as $\|u\|_{1,p} \to \infty$. Since $\underline{b}$ is weakly continuous, in order to prove the existence of a maximum for $\underline{h}$ it suffices to show that $\underline{a}$ is weakly upper semi-continuous; this is proved in Lemma 5 below, if $\underline{h}$ attains a maximum at u then u must be a solution of (3.2) since the Gateaux derivative of $\underline{h}$ at u must be zero.

Since u can be replaced by |u| without changing the value of h (this is proved in much the same way as Lemma 6, below, is proved), we can assume that u is non-negative. Up to the proof of Lemma 5 this completes the proof of Theorem 2.

LEMMA 5. The functional a. is^ weakly upper semi-continuous on $W_o^{*,p}(\Omega)$.

Proof, If p = 2 then by (H. 2) a. is convex on $W^{\wedge,2}(\Omega)$ and the weak upper semi-continuity follows. If p > 2 then we consider the functional

r

on E^+_{p*s}. The weak upper semi-continuity of this functional follows from the convexity assumption in (A. 1)9 that of a. follows directly.

Remark, The proof of Theorem 2 is essentially the argument given by Hempel [4] in the proof of the same result for the semi-linear case. We note that Hempel's hypothesis, in our notation

(4.7) $$\lim_{\eta \to \infty} b_r(x,\eta) \leq 0,$$

together with (B.I) and (B.2) implies that given £ > 0

(4.8) $$b(x,\eta) \leq f(x) |\eta| + g(x)$$

with $\|f\|_r \wedge £$, $g \in L^1(\Omega)$. Indeed

$$\wedge(XjT?) \; 1 \; b_\eta (x,0)$$

so let δ be chosen so that

$$\int_A b_\eta(x,0)\,dx \le \frac{1}{2}\varepsilon$$

when $A \subseteq \Omega$ is measurable and $\mu(A) \le \delta$ (μ denotes Lebesgue measure). Choose $T_\varepsilon > 0$ (by Egoroff's theorem) so that on a measurable subset B of Ω with $\mu(B) > \mu(\Omega) - \delta$,

$$b_\eta(x,\eta) \le \varepsilon/2\mu(\Omega) \text{ when } \eta \ge T_\varepsilon .$$

Thus for $x \in B$

$$b(x,\eta) = \int_0^\eta b_\eta(x,t)\,dt \le \eta\,\varepsilon/2\mu(\Omega) + T_\varepsilon\, b_\eta(x,0) .$$

Thus (4.8) holds with

$$f = \chi_A b_\eta(\cdot\,,0) + \varepsilon/2\mu(\Omega) , \qquad g = T_\varepsilon b(x,0)$$

where χ_A denotes the characteristic function of $A = \Omega \setminus B$. Thus, when $p = 2$ the condition

$$(4.9) \qquad a(x,\eta,(t_i t_j)) \ge e|t|^2 - \gamma'$$

for all $x \in \Omega_o$, $\eta \ge 0$, $t \in R^N$, together with (4.7) implies (4.1), with f,g as indicated. In particular Theorem 2 contains the corresponding result for the semi-linear case as obtained in [4].

5. Appendix. Here we treat certain elements, required above, from the theory of eigenvalue problems for Dirichlet forms with integrable coefficients. As above, we consider only real-valued functions.

Let Ω be a bounded, smoothly bounded connected open set in R^N and let $a_{ij}(x)$, $i,j = 1,\dots,N$, $b(x)$, $c(x)$ belong to $L^1(\Omega)$ with

(5.1) $$Mb(x) \geq c(x) > 0 \qquad \text{a.e. on } \Omega,$$

for some positive constant M. Let there exist $m > 0$ such that for almost all $x \in \Omega$

(5.2) $$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq m|\xi|^2$$

for all real vectors ξ.

We denote by X the space obtained by completing $C_o^\infty(\Omega)$ with respect to the inner product

(5.3) $$\langle u,v\rangle = \int_\Omega \left(\sum_{i,j=1}^{N} a_{ij}(x)u_{x_i}v_{x_j} + b(x)uv\right) dx.$$

From (5.1) and (5.2) it is clear that

$$X \subseteq H_o^1(\Omega) = W_o^{1,2}(\Omega).$$

LEMMA 6. *If $u \in X$ then $|u| \in X$ and u and $|u|$ have the same norm.*

Proof. Let $u \in C_o^\infty(\Omega)$, then there is a sequence of functions $\{w_n\}$ in $C_o^\infty(\Omega)$ such that: $w_n \to |u|$ uniformly on Ω and the sequence $\{\text{grad } w_n\}$ remains bounded in the sup norm and tends uniformly to $(\text{sgn } u)\text{grad } u$ on any closed set in Ω not intersecting the zero set of u. Since this zero set is contained in the union of a set of small measure and a set where $|\text{grad } u|$ is small, the assertion follows for $u \in C_0^\infty(\Omega)$. It clearly follows then for arbitrary $u \in X$.

COROLLARY. *If* $u,v \in X$ *are such that* $uv = 0$ *a.e. in* Ω *then*

$$\langle u,v \rangle = 0.$$

Proof. By considering positive and negative parts we can reduce to the case where u and v are non-negative. Then u and v can be regarded as positive and negative parts of a function $w \in X$, i.e. $w = u - v$. By the lemma above we then have

$$\|u-v\|_X = \|u+v\|_X$$

from which follows the orthogonality of u and v.

By Y we denote the Hilbert space of real-valued measurable functions f on Ω for which

$$\int_\Omega |f(x)|^2 c(x)\,dx < \infty;$$

the inner product on Y is

$$(f,g) = \int_\Omega f(x)\,g(x)\,c(x)\,dx.$$

We denote by i the imbedding $X \to Y$; by (5.1) this is bounded. The adjoint i^* is the operator of the Lax-Milgram theorem and

$u = i^*f$ is the solution of the weak problem

$$\langle u,v\rangle = (f,iv), \qquad v \in X.$$

LEMMA 7. *The operator* i^* *is non-negative i.e. if* $f(x) \geq 0$ *a.e. in* Ω, $f \in Y$, *and* $u = i^*f$ *then* $u(x) \geq 0$ *a.e. in* Ω.

Proof. For $f \in Y$, $u = i^*f$ maximizes (f,iu) subject to $u \in X$, $\langle u,u\rangle \leq 1$. From Lemma 6 and the properties of $(.,.)$ it follows that this maximum is attained (and by uniqueness only attained) for $u \geq 0$ if $f \geq 0$.

LEMMA 8. *Let* $u \in Y$ *be a non-negative eigenfunction of the positive definite self-adjoint operator* $k = ii^*$. *Then* u *is positive almost everywhere on* Ω.

Proof, The operator k is clearly self-adjoint, that it is positive definite follows from the relation

$$(kf,f) = \langle i^*f,i^*f\rangle.$$

Since i clearly preserves non-negativity so, by Lemma 7, does k.

Suppose now that u is an eigenfunction of k and $u(x) \geq 0$ a.e. on Ω. Let $\Omega = A \cup B$ where A,B are measurable, $A \cap B$ has measure zero, $u(x) > 0$ a.e. on A and $u(x) = 0$ a.e. on B. We shall show that B must have measure zero. An elementary argument shows that $f \in Y$ and

$$f(x) = 0 \qquad \text{a.e. on } A \tag{5.7}$$

implies $(kf)(x) = 0$ a.e. on A and that if $g \in Y$ and

(5.8) $$g(x) = 0 \qquad \text{a.e. on } B$$

then $(kg)(x) = 0$ a.e. on B. (Notice that since k is positive definite and preserves non-negativity then in fact $(kh)(x) > 0$ a.e. on any measurable set in Ω on which $h(x) > 0$ a.e., $h \in Y$.)

We can write

$$Y = M \oplus N$$

where $f \in M$ if and only if (5.7) holds and $g \in N$ if and only if (5.8) holds. Since i is an imbedding, and $k = ii^*$, it follows from what we have just shown concerning k that $(i^*f)(x) = 0$ a.e. on A for $f \in M$ and $(i^*g)(x) = 0$ a.e. on B for $g \in N$. Thus, by the corollary to Lemma 6, $i^*(M)$ and $i^*(N)$ are orthogonal in X. Since $i^*(Y)$ is dense in X it follows that $X = U \oplus V$ where the functions in U vanish a.e. on A and those in V vanish a.e. on B. Finally, if j denotes the inclusion $X \subset H_0^1(\Omega)$, then since $j(X)$ is dense in $H_0^1(\Omega)$ and $j(U)$ and $j(V)$ are orthogonal in $H_0^1(\Omega)$ we obtain a similar decomposition of that space

$$H_0^1(\Omega) = U_1 \oplus V_1.$$

This is impossible unless B has measure zero since otherwise the problem

$$\Delta u = f \text{ in } \Omega, \; u = 0 \text{ on } \partial\Omega,$$

for $f \in L^2(\Omega)$, $f(x) = 0$ a.e. on B, would have a weak solution in $H_o^1(\Omega)$ vanishing a.e. on B, and this cannot happen for non-negative $f \not\equiv 0$. This completes the proof of Lemma 8.

The following result, which is trivial if k is compact, seems of some interest for its own sake. The proof was provided by Professor R. J. Duffin.

LEMMA 9. *If* $ku = \lambda u$ *and* $u(x) > 0$ *a.e. on* Ω *then*

$$\lambda = \|k\|. \tag{5.9}$$

Proof. Let

$$\Omega = E_1 \cup E_2 \cup \ldots \cup E_n \tag{5.10}$$

be a partitioning of Ω into measurable sets; put

$$f_i = \sigma_i \chi_{E_i} u, \qquad i = 1,\ldots,n \tag{5.11}$$

where χ_{E_i} is the characteristic function of E_i, and $\sigma_i > 0$ is chosen so that

$$(f_i, f_j) = \delta_{ij}, \tag{5.12}$$

δ_{ij} is the Kronecker delta. For some positive constants $c_i, i = 1,\ldots,n$,

$$u = c_1 f_1 + \ldots + c_n f_n.$$

It follows that the vector $c = (c_1,\ldots,c_n)$ is an eigenvector of the non-negative symmetric matrix $K = ((f_i, kf_j))$ corresponding to the eigenvalue λ. Since the components of c are positive, λ is the largest in absolute value of the eigenvalues of K.

If P denotes the orthogonal projection of Y onto the subspace spanned by $f_1,\dots,f_n$, then κ is the matrix of PkP relative to the basis $f_1,\dots,f_n$. By choosing a sequence of finer and finer partitions (5.10) we obtain a corresponding sequence of projections $\{P_\mu\}$ such that, because of (5.11) and the fact that $u(x) > 0$ a.e. on Ω, P_μ tends strongly to I. Thus also $P_\mu k P_\mu$ tends strongly to k. Since $\|P_\mu k P_\mu\| = \lambda$ for each μ it follows that $\|k\| = \lambda$. This completes the proof.

THEOREM 3. *Let the Dirichlet form* (5.3) *have integrable coefficients satisfying* (5.2) *and*

$$b(x) \geq 0 \qquad \text{a.e. on } \Omega.$$

Let $c \in L^1(\Omega)$ *and*

$$c(x) > 0, \qquad \text{a.e. on } \Omega.$$

If the weak eigenvalue problem

$$(5.13) \quad \int_\Omega \Big(\sum_{i,j=1}^{N} a_{ij}(x) u_{x_i} v_{x_j} + b(x) u v dx \quad = \lambda \int_\Omega u v c(x)\, dx, \quad v \in C_0^\infty(\Omega),$$

has a non-negative eigenfunction u_1 *corresponding to an eigenvalue* λ_1 *then* $u_1(x) > 0$ *a.e. on* Ω *and for all* $u \in C_0^\infty(\Omega)$,

$$(5.14) \quad \int_\Omega \Big(\sum_{i,j=1}^{N} a_{ij}(x) u_{x_i} u_{x_j} + b(x) u^2 \Big) dx \geq \lambda_1 \int_\Omega u^2 c(x)\, dx,$$

and λ_1 *is a simple eigenvalue, so that* (5.14) *is strict unless* u *is proportional to* u_1.

Proof. When (5.1) holds the first part of the theorem is an immediate consequence of the preceding lemmas. The general case is trivially reduced to this by adding $\int_0 uvc(x)dx$ to both sides of (5.13).

To prove simplicity of A_1 we observe that if $7\backslash_1$ has multiplicity > 1 then there must be an eigenfunction a) orthogonal to u_1 and hence not essentially of one sign. Because k is non-negative a simple variational argument shows that $a)_+$ and cu_- are also in the eigenspace corresponding to A_1. Since uu_+ and a)_ vanish on sets of positive measure, the existence of u) contradicts Lemma 8.

<u>Remark</u>. It is clear that in the preceding discussion the condition (5.1) could have been weakened to

$$Mb(x) \wedge c(x) - M^1$$

provided that $M^{!}M^{-1}$ were sufficiently small so that the form <.,.> on $C_0^{\infty}(U)$ still dominates the $H_0^1(\&l)$ inner product.

<u>ACKNOWLEDGEMENT</u>

I wish to express my indebtedness to my colleagues R. J. Duffin, V. J. Mizel and H, Flaschka for several very helpful conversations concerning this work.

References

[1] Coffman, C. V., "On the uniqueness of solutions of a non-linear boundary value problem", J. Math. Mech. 13 (1964), 751-764.

[2] _______________, "A minimum-maximum principle for a class of non-linear integral equations", J. d'Analyse Math. 22 (1969), 391-419.

[3] _______________, "On the positive solutions of boundary value problems for a class of non-linear differential equations", J. Diff. Eq. 3(1967), 92-111.

[4] Hempel, J. A., "Multiple solutions for a class of non-linear boundary value problems", Indiana University Math. J. 20(1971), 983-996.

[5] Kolodner, I. I., "Heavy rotating string -- a non-linear eigenvalue problem", Comm. Pure and Appl. Math. 8(1955), 395-408.

[6] Krasnosel'skii, M. A., Topological Methods in the Theory of Non-linear Integral Equations, MacMillan, New York, 1964.

[7] _______________________, Positive Solutions of Operator Equations, Noordhoff, Groningen, 1964.

[8] _______________________ and L. A. Ladyzhenskii, "The structure of the spectrum of positive non-homogeneous operators", Trud. Mosk. Mat. Obsch. 3(1954), 321-346.

[9] _______________________ and V. Ja. Stecenko, "On the theory of concave operator equations", Sibirsk. Mat. Z. 10(1969), 565-572.

[10] Laetsch, T., "Uniqueness for sublinear boundary value problems," to appear.

[11] Nehari, Z., "On a class of non-linear second order differential equations", Trans. Amer. Math. Soc. 93(1959), 30-52.

[12] Parter, S. V., "A note on the eigenvalue problem for sublinear Hammerstein operators", J. Math. Anal. and Appl. 32 (1970), 104-117.

[13] Picard, E., Traité d'Analyse, 2nd ed., vol.III, Ch. 7, Gauthier-Villars, Paris, 1908.

[14] Pimbley, G. H., "A sublinear Sturm-Liouville problem", J. Math. Mech. 11(1962), 121-138.

[15] ______________, "A superlinear Sturm-Liouville problem", Trans. Amer. Math. Soc. 103(1962), 229-248.

[16] Rabinowitz, P. H., "A note on a non-linear eigenvalue problem for a class of differential equations", J. Diff. Eq. 9(1971), 536-548.

[17] Urysohn, P. S., "On a type of non-linear integral equations", Math. Sb. 31(1924), 236-251.