A SHORT PROOF OF ALEXANDROFF'S THEOREM

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ABSTRACT

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A. D. Alexandroff proved that there is a linear isometry between C (T) and the space of regular, bounded, additive set functions defined on a field 3 of subsets of T. Here C (T) is the dual of the space of bounded, continuous functions on a topological space T. 3 is the field generated by the zero sets of T.

Dunford and Schwartz have given a simple proof of this duality theorem in the case when the underlying topological space is a normal Hausdorff space. In this note we use the methods of Dunford and Schwartz to give an elementary proof of Alexandroff's result.

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1. Introduction

In his paper, "Additive Set Functions in Abstract Spaces", Alexandroff [1] has given a concrete representation of the dual of the space of bounded continuous functions on a topological space T. Dunford and Schwartz [2] give a shorter and more elementary proof of this theorem in the special case when T is a normal Hausdorff space (Theorem 2, p. 262 of [2]). In this note we provide a simple proof of Alexandroff's theorem using the techniques applied by Dunford and Schwartz. The authors are indebted to Professor K. Sundaresan for suggesting this method of proof.

2. Preliminaries

In this section we give a few definitions and mention certain basic results concerning zero and cozero sets. Let T be an arbitrary topological space.

$$C(T) = \{f \mid f : T \longrightarrow R,\ f \text{ is bounded and continuous}\}.$$

Then $C(T)$ is a normed linear space with $\|f\| = \sup_{t \in T} |f(t)|$.

$$C^*(T) = \{L \mid L : C(T) \longrightarrow R,\ L \text{ is continuous and linear}\}.$$

Then $C^*(T)$ is the <u>normed conjugate</u> of $C(T)$ with $\|L\| = \sup_{\|f\| \leq 1} |L(f)|$.

A set $Z \subset T$ is a <u>zero set</u> if $Z = f^{-1}(0)$ for some $f \in C(T)$. Let $\mathcal{Z} = \{Z \subset T | Z \text{ is a zero set}\}$ and let $\mathfrak{F}$ be the field generated by $\mathcal{Z}$. A set $G \subset T$ is a <u>cozero set</u> if $T \setminus G \in \mathcal{Z}$. Throughout this paper Z and Z_i will always denote zero sets, G and G_i will always denote cozero sets.

We say that a set function $m : \mathfrak{F} \longrightarrow R$ is <u>regular</u> if given $E \in \mathfrak{F}$ and $\varepsilon > 0$ there are Z and G such that $Z \subset E \subset G$ and $C \subset G \setminus Z$, $C \in \mathfrak{F}$ implies $|m(C)| < \varepsilon$. Let

$$M = \{m | m : \mathfrak{F} \longrightarrow R, \ m \text{ is regular, bounded and additive}\}.$$

If $\overline{m}$ denotes the variation of m, then we define $\|m\| = \overline{m}(T)$. We order M by defining $m \geq 0$ if $m(E) \geq 0$ for all $E \in \mathfrak{F}$. $C^*(T)$ has the usual ordering, i.e., $L \geq 0$ if $L(f) \geq 0$ for all $f \geq 0$, $f \in C(T)$.

In the following lemma, we collect some results which are easily verified from the definitions.

<u>LEMMA</u>. (1) The intersection or union of two zero sets is a zero set.

(2) If E is a closed set (open set) in R and $f \in C(T)$ then $f^{-1}(E)$ is a zero set (cozero set) in T.

(3) If $\{Z_i\}_{i=1}^n$ are pairwise disjoint (p.w.d.) then there are $\{G_i\}_{i=1}^n$ p.w.d. such that $Z_i \subset G_i$.

(4) If $Z \subset G$ then there exists $f \in C(T)$ such that $f(t) = 1$ for $t \in Z$ and $f(t) = 0$ for $t \in T \setminus G$.

3. Proof of the Duality Theorem

THEOREM. Let T, $C(T)$, $C^*(T)$ and M as above. Then there is an isometric isomorphism between $C^*(T)$ and M such that corresponding elements L and m satisfy

$$L(f) = \int_T f\,dm$$

for all $f \in C(T)$. Further, this isomorphism preserves order.

Proof: We first note that if $f \in C(T)$ and $m \in M$ then f is integrable with respect to m. For let $\varepsilon > 0$. Cover $f(T)$ with open sets $U_1, \dots, U_n$ such that $\mathrm{diam}(U_i) < \varepsilon$. Let $A_1 = U_1$, $A_j = U_j \setminus \bigcup_{i=1}^{j-1} U_i$ for $j = 2, \dots, n$. If $A_j \neq \emptyset$, choose $a_j \in A_j$ and if $A_j = \emptyset$ let $a_j = 0$. Then if $B_j = f^{-1}(A_j)$ and $g = \sum_{i=1}^{n} a_i \chi_{B_i}$, g is an m-simple function and clearly $\|g - f\| \le \varepsilon$. Thus f is the uniform limit of m-simple functions and since $\bar{m}(T) < \infty$, f is m integrable.

Since

$$\left|\int_T f\,dm\right| \le \sup_{t \in T} |f(t)| \cdot \bar{m}(T)$$

if $L(f) = \int_T f\,dm$ then clearly $L \in C^*(T)$ and $\|L\| \le \|m\|$. To show $\|L\| = \|m\|$, let $\varepsilon > 0$ be given and let $\{E_i\}_{i=1}^{n}$ be p.w.d. sets in 3 such that $\sum_{i=1}^{n} |m(E_i)| \ge \|m\| - \varepsilon$. Noting that $\bar{m}$ is regular since m is regular, we may choose $Z_i \subset E_i$ so that $\bar{m}(E_i \setminus Z_i) < \varepsilon/n$. Then choose $\{G_i\}_{i=1}^{n}$ p.w.d. such that $Z_i \subset G_i$ and $\bar{m}(G_i \setminus Z_i) < \varepsilon/n$. Define $a_i = \pm 1$ according as $m(E_i) > 0$ or $m(E_i) < 0$ and let $f_i \in C(T)$, $0 \le f_i \le 1$ such that $f_i(t) = 0$

if $t \in T \setminus G_i$ and $f_i(t) = 1$ if $t \in Z_i$. Defining $f_o = \sum_{i=1}^{n} \alpha_i f_i$ we have that $\|f_o\| \leq 1$ and

$$|L(f_o)| = |\int_{\bigcup_{i=1}^{n} G_i} f_o dm| = |\sum_{i=1}^{n} \int_{Z_i} \alpha_i f_i dm + \sum_{i=1}^{n} \int_{G_i \setminus Z_i} \alpha_i f_i dm|$$

$$\geq \sum_{i=1}^{n} |m(E_i)| - 2\varepsilon \geq \|m\| - 3\varepsilon .$$

Thus $\|L\| = \|m\|$.

Since it is clear that our correspondence represents a linear map, we need only show that given $L \in C^*(T)$ there is $m \in M$ such that $L(f) = \int_T f\, dm$ for all $f \in C(T)$. Therefore let $L \in C^*(T)$. Then L has a continuous extension $\hat{L} : B(T) \longrightarrow R$ where

$$B(T) = \{f : T \longrightarrow R | f \text{ is a bounded function}\}.$$

$B(T)$ is equipped with the sup norm. By Corollary 5.3, p. 259 of Dunford and Schwartz [2] there is an isometry between $B^*(T)$ and

$$ba(T) = \{m | m : 2^T \longrightarrow R, M \text{ is a bounded additive set function}\}.$$

Therefore let $\lambda \in ba(T)$ be such that $\hat{L}(f) = \int_T f\, d\lambda$ for all $f \in B(T)$. By the Jordan Decomposition Theorem we may assume that $\lambda \geq 0$. We must find $m \in M$ such that $\int_T f\, dm = \int_T f\, d\lambda$ for all $f \in C(T)$. Define $\mu_1 : Z \longrightarrow R$ by $\mu_1(Z) = \inf_{Z \subseteq G} \lambda(G)$ for all $Z \in \mathcal{Z}$ and define $\mu_2 : 2^T \longrightarrow R$ by $\mu_2(E) = \sup_{Z \subseteq E} \mu_1(Z)$ for all $E \subseteq T$. It is obvious that both μ_1 and μ_2 are non-negative and non-decreasing.

Now if Z_1, G_1, and G are such that $Z_1 \backslash G_1 \subset G$ then $Z_1 \subset G \cup G_1$ and since $\lambda(G \cup G_1) \leq \lambda(G) + \lambda(G_1)$ we have that $\mu_1(Z_1) \leq \lambda(G_1) + \lambda(G)$. Therefore $\mu_1(Z_1) \leq \lambda(G_1) + \mu_1(Z_1 \backslash G_1)$. Allowing G_1 to range over all cozero sets containing $Z \cap Z_1$ we have $\mu_1(Z_1) \leq \mu_1(Z \cap Z_1) + \mu_2(Z_1 \backslash Z)$. If $E \subset T$ and Z_1 ranges over all zero sets which are subsets of E then $\mu_2(E) \leq \mu_2(E \cap Z) + \mu_2(E \backslash Z)$.

Let Z_1 and Z_2 be disjoint. Choose disjoint cozero sets G_1 and G_2 such that $Z_1 \subset G_1$, $Z_2 \subset G_2$. If $G \supset Z_1 \cup Z_2$ then $\lambda(G) \geq \lambda(G \cap G_1) + \lambda(G \cap G_2)$ so $\mu_1(Z_1 \cup Z_2) \geq \mu_1(Z_1) + \mu_1(Z_2)$. Now let $E \subset T$ and $Z \in \mathcal{Z}$. If Z_1 ranges over all zero sets which are subsets of $E \cap Z$ while Z_2 ranges over all zero sets which are subsets of $E \backslash Z$, we therefore have that $\mu_2(E) \geq \mu_2(E \cap Z) + \mu_2(E \backslash Z)$. Thus we have proven that $\mu_2(E) = \mu_2(E \cap Z) + \mu_2(E \backslash Z)$ for any $E \subset T$ and $Z \in \mathcal{Z}$. By Lemma 5.2, p. 133 of Dunford and Schwartz [2], if m is defined to be the restriction of μ_2 to $\mathfrak{F}$, then m is an additive set function on $\mathfrak{F}$. From their definitions it is clear that $\mu_1(Z) = \mu_2(Z) = m(Z)$ if $Z \in \mathcal{Z}$. Therefore $m(E) = \sup_{Z \subseteq E} m(Z)$ if $E \in \mathfrak{F}$ so that m is regular and since $m(T) < \infty$ we have that $m \in M$.

We need only show that $\int_T f \, dm = \int_T f \, d\lambda$ for all $f \in C(T)$. We can assume $0 \leq f \leq 1$. Let $\varepsilon > 0$ and partition T by a family $\{E_i\}_{i=1}^n$ of p.w.d. sets in $\mathfrak{F}$ such that

$$\sum_{i=1}^{n} a_i m(E_i) + \varepsilon \geq \int_T f \, dm$$

where $a_i = \inf_{t \in E_i} f(t)$. There exist sets $Z_i \subset E_i$ such that $m(E_i \setminus Z_i) < \varepsilon/n$ which implies that

$$\sum_{i=1}^{n} a_i m(Z_i) + 2\varepsilon \geq \int_T f\, dm.$$

Now choose $\{G_i\}_{i=1}^{n}$ p.w.d. such that $Z_i \subset G_i$ and

$$b_i = \inf_{t \in G_i} f(t) \geq a_i - \frac{\varepsilon}{n\|m\|},$$

so that $\sum_{i=1}^{n} b_i m(G_i) + 3\varepsilon \geq \int_T f\, dm$. If $Z \subset G$ we have $m(Z) \leq \lambda(G)$ so that $m(G) \leq \lambda(G)$. Therefore $\sum_{i=1}^{n} b_i m(G_i) \leq \sum_{i=1}^{n} b_i \lambda(G_i) \leq \int_T f\, d\lambda$ and thus $\int_T f\, dm \leq \int_T f\, d\lambda$. Since $m(T) = \lambda(T)$ we also have $\int_T (1-f)\, d\lambda \leq \int_T (1-f)\, dm$ and we can conclude $\int_T (1-f)\, dm = \int_T (1-f)\, d\lambda$. Therefore, replacing f by $1-f$ we have $\int_T f\, dm = \int_T f\, d\lambda$ for all $f \in C(T)$.

To complete the proof we must show that this isometry is order-preserving. Clearly $\int_T f\, dm \geq 0$ if $m \geq 0$ and $f \in C(T)$, $f \geq 0$. Conversely let $\int_T f\, dm \geq 0$ for each $f \in C(T)$ such that $f \geq 0$ and suppose that there is $E \in \mathfrak{F}$ such that $m(E) < -\varepsilon < 0$. Since $\overline{m}$ is regular there are sets Z and G such that $Z \subseteq E \subseteq G$ and $\overline{m}(G \setminus Z) \leq \varepsilon/4$. Let $g \in C(T)$, $0 \leq g \leq 1$, such that $g(t) = 1$ if $t \in Z$ and $g(t) = 0$ if $t \in T \setminus G$. Then $\left|\int_T g\, dm - m(E)\right| \leq \varepsilon/2$ contradicting $\int_T g\, dm \geq 0$. Therefore the mapping is order-preserving. |

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