

ON THE NON-UNIQUENESS OF ELASTIC
ROTATIONS FOR DEFORMATIONS
OF MATERIALS WITH ELASTIC RANGE

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ABSTRACT

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In this note precise definitions of the concepts of permanent deformation, annealed state, and non-softening material are used to establish non-uniqueness of elastic rotations for deformations of certain materials with elastic range.

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1. Introduction

A feature common to theories of elastic-plastic materials is a representation of deformations in terms of elastic and inelastic parts. The question of uniqueness of such a representation has been commented upon by many authors. In references [1], [2], and [3], the inherent non-uniqueness of such representations has been asserted. On the other hand, some writers (see [4], for example) have assumed uniqueness of such representations. The purpose of this note is to state precise conditions under which non-uniqueness (in the form of non-uniqueness of elastic rotations) arises in a mechanical theory proposed by the present author [1]. Roughly speaking, I show that in a material with elastic range which does not work soften. the larger the symmetry group the larger the number of possibilities for the elastic rotation. In particular, the elastic rotation is shown to be arbitrary for isotropic materials.

These results are useful when discussing approximate constitutive relations for the case of infinitesimal elastic

deformations. In fact, for isotropic materials it follows that one can exclude all elastic rotations which arise in such approximations.

2. Basic Concepts

In this section, I summarize the concepts presented in [1] which are relevant to the discussion which follows. In addition, a definition of the concept of a "non-softening" material is introduced.

A material with elastic range is a material whose stress response

$$T(t) = \Pi(F^t)$$

(with $T(t)$ the current stress tensor and F^t the history of deformation gradient up to time t) satisfies the additional restriction that continuations of the deformation gradient history F^t which remain in a set $\mathcal{IE}(F^t)$ produce a path independent response. Accordingly, one can write

$$\Pi(G^\tau) = \Pi^*(A, F^t)$$

whenever G^τ is a continuation of F^t which remains in $\mathcal{IE}(F^t)$ and ends at the point A in $\mathcal{IE}(F^t)$. The set $\mathcal{IE}(F^t)$ is called the elastic range corresponding to F^t and the function $\Pi^*(\cdot, F^t)$

is called the elastic response determined by F^t .

A history H of deformation gradient is said to determine an annealed state of the given material if three conditions on $\mathbb{E}(H)$ and $\Pi^*(\cdot, H)$ hold (see, [1], pp. 89-90); the condition needed here is:

($\alpha 3$) $\mathbb{E}(H)$ is invariant under both right and left multiplication of all tensors in this set by arbitrary orthogonal tensors.

This condition provides the "initial" elastic range, for deformation starting from an annealed state, with abundant symmetries.

Given a history F^t , a second history F_p^t is said to be a permanent deformation history corresponding to F^t if the following conditions hold:

P1. $F_p^t(\sigma)$ is in the elastic range determined by $F^{t-\sigma}$, for every $\sigma \geq 0$. ($F_p^t(\sigma)$ represents the permanent deformation at time $t - \sigma$.)

P2. As σ tends to zero, the points $F_p^t(\sigma)$ are bounded away from the boundary $\partial \mathbb{E}(F^t)$ of the elastic range $\mathbb{E}(F^t)$.

P3. A fixed function Π_0 determines the local elastic response through the relation

$$\Pi^*(A, F^{t-\sigma}) = \Pi_0(A(F_p^t(\sigma))^{-1})$$

for every $\sigma \geq 0$ and every A in $\mathbb{E}(F^{t-\sigma})$.

It should be noted that Condition P3 embodies the assumption that only the current permanent deformation and the current total deformation are necessary for the determination of the current stress at points in the current elastic range. In P3, $(F_p^t(\sigma))^{-1}$ denotes the inverse of $F_p^t(\sigma)$.

I assume in the following development that the history l^\dagger , corresponding to rest in a given reference configuration, determines an annealed state of the material and that l^\dagger is a permanent deformation history corresponding to itself (see [1], A3).

A final definition, not given in [1], is needed here. If the history l^\dagger is such that

$$\mathbb{E}(l^\dagger)_{F_p^t(\sigma)} \subset \mathbb{E}(F^{t-\sigma})$$

for every choice of histories F^t and F_p^t , with F_p^t a permanent deformation history corresponding to F^t , and for every $\sigma \geq 0$, then the given material is said to be non-softening. The set $\mathbb{E}(l^\dagger)_{F_p^t(\sigma)}$ can then be thought of as being a translated copy of the initial elastic range $\mathbb{E}(l^\dagger)$ which, according to the above condition, is a subset of $\mathbb{E}(F^{t-\sigma})$. Thus, the concept

of a non-softening material embodies the assumptions that, in a certain sense, the elastic range cannot shrink during continued deformation. (This concept would not be sensible were it not

for the fact that $\mathcal{F}^p(a)$ is in $\mathcal{E}(F^+)$ for every $t \in \mathcal{E}^+$ which implies that for all materials $\mathcal{E}(1) \cap \mathcal{F}^p(a) \cap \mathcal{E}(F^-)$ is not empty.)

3. Non-Uniqueness of Permanent Deformations

As the main step toward establishing the non-uniqueness of elastic rotations, a result is presented below which gives conditions under which permanent deformations are not unique. Before giving this result, a preliminary remark is needed: If an orthogonal tensor Q_0 is an element of the symmetry group of the response functional Π , then the same tensor is an element of the symmetry group of the function Π_0 , i.e. the condition

$$n_0(AQ_0) = n_0(A)$$

holds for all tensors A whenever the condition

$$n(F^t Q_0) = n(F^{tc})$$

holds for all histories F^t This result is proved as part of Proposition 1, p. 95, [1].

It is now possible to state and prove the following proposition: Given a non-softening material with elastic range, a history F^t , and a permanent deformation history F_p^t corresponding to F^t , it follows that $Q_o F_p^t$ is also a permanent deformation history F_p^t corresponding to F^t whenever Q_o is an orthogonal tensor which is in the symmetry group of the response functional Π . In particular, if the material is isotropic, then for every orthogonal Q_o the history $Q_o F_p^t$ is a permanent deformation history corresponding to F^t .

Thus, the existence of many symmetry transformations of the material leads to many possibilities for the permanent deformation.

The proposition can be established simply by showing that if the conditions P1, P2 and P3 are valid for F_p^t , then they are valid for $Q_o F_p^t$ whenever Q_o corresponds to a symmetry transformation.

P1. Since $Q_o F_p^t(\sigma) = Q_o l F_p^t(\sigma)$ and since the identity tensor l is in $\mathbb{E}(l^\dagger)$ (see [1], p. 89), it follows that $Q_o F_p^t(\sigma) \in Q_o \mathbb{E}(l^\dagger) F_p^t(\sigma)$. However, $Q_o \mathbb{E}(l^\dagger) = \mathbb{E}(l^\dagger)$, since l^\dagger determines an annealed state. Therefore, one concludes that $Q_o F_p^t(\sigma) \in \mathbb{E}(l^\dagger) F_p^t(\sigma) \subset \mathbb{E}(F^{t-\sigma})$; the last conclusion follows

from the fact that the material is non-softening. Thus, for every $a \geq 0$, $Q_{\circ F_p}^t(\alpha)$ is in the set $IE(F^{t-\sigma})$. (Note that this argument shows also that $Q_{\circ F_p}^t(a)$ is not in the boundary of $3E(F^{t-\sigma})$, since $3E(F^{t-\sigma})$ is open.

P2. Suppose that P2 is not satisfied. Then there exists points $A_1, A_2, \dots, A_n, \dots$ in $dIE(F^t)$ and times $a_1, a_2, \dots, a_n, \dots$, with $\lim_{n \rightarrow \infty} a_n = 0$, such that the distance between A_n and $Q_{\circ F_p}^t(a_n)$ tends to zero as n tends to infinity. However, $Q_{\circ F_p}^t(a_n)$ tends to $Q_{\circ F_p}^t(0)$ as n tends to infinity, so that the sequence $\{A_n\}$ must tend to the same limit. This is impossible, since the limit of the sequence $\{A_n\}$ must be in $dJE(F)$ whereas $Q_{\circ F_p}^t(0)$ is not in $d2E(F)$.

P3. For every $A \in IE(F^{t-\sigma})$, $\Pi^*(A, F^{t-\sigma}) = \Pi_{\circ}(A(F_p^t(a))^{-1})$ since F_p^t is a permanent deformation corresponding to F^t . However, for each Q_{\circ} in the symmetry group of the given material, the preliminary remark implies that

$$\Pi_{\circ}(A(F_p^t(\sigma))^{-1}) = \Pi_{\circ}(A(F_p^t(\sigma))^{-1} Q_{\circ}^T) = \Pi_{\circ}(A[Q_{\circ} F_p^t(\sigma)]^{-1}),$$

and this relation holds for all A in $JE(F^{t-\sigma})$. Hence, for all such A and for all $a \geq 0$

$$\Pi^*(A, F^{t-\sigma}) = \Pi_{\circ}(A[Q_{\circ} F_p^t(\sigma)]^{-1}).$$

This argument establishes P3.

4. Elastic Deformations and the Non-Uniqueness of Elastic Rotations

Given a history F^t and a corresponding permanent deformation history F_p^t , we define for every $\sigma \geq 0$

$$F_e^t(\sigma) = F^t(\sigma) (F_p^t(\sigma))^{-1};$$

F_e^t is called an elastic deformation history corresponding to F^t . Thus the relation

$$F^t(\sigma) = F_e^t(\sigma) F_p^t(\sigma)$$

holds for all $\sigma \geq 0$. Since for each Q_0 in the symmetry group of the material the history $Q_0 F_p^t$ also is a permanent deformation history corresponding to F^t , it follows that for each Q_0 the history $F_e^t Q_0^T$ is an elastic deformation history corresponding to F^t whenever F_e^t is an elastic deformation history. In particular, for an isotropic material the histories $F_e^t Q_0^T$ are, for every choice of orthogonal Q_0 , elastic deformation histories corresponding to F^t .

These remarks suffice to establish the non-uniqueness of elastic rotations. In fact, one can write the polar decomposition

$$F_e^t = V_e^t R_e^t$$

for any elastic deformation history F_e^t ; thus each orthogonal tensor Q_o in the symmetry group gives rise to an elastic deformation history

$$F_e^t Q_o = V_e^t (R_e^t Q_o^1)$$

having the same stretch V_e^t but different elastic rotation.

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