ON THE NONLINEAR METHOD OF WILKINS
FOR COOLING FIN OPTIMIZATION

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Abstract

Of concern is the nonlinear differential equation $(k(u)p(x)u')' = p^{\eta}(x)Q(u) \quad 0 \leq x \leq b$ subject to the boundary conditions: $u = 1$ at $x = b$ and (kpu') takes the values $y_0 > 0$ and 0 at $x = b$ and $x = 0$ respectively. Here $0 \leq \eta < 1$ is a given constant and k and Q are known functions and the question posed is to find a positive constant $b > 0$, a function $p(x) > 0$ in $0 < x \leq b$ and a solution u of the differential equation such that the norm $(\int_0^b p^{\rho} dx)^{\frac{1}{\rho}}$, $\rho \geq 1$ is minimized. A special transformation of variables together with Hölder's inequality leads to the solution in terms of explicit quadrature formulas.

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ON THE NONLINEAR METHOD OF WILKINS FOR COOLING FIN OPTIMIZATION

1. Introduction.

This is a study of a minimizing problem suggested by the works of J. E. Wilkins, Jr., concerning minimizing the mass of cooling fins [5,6,7].

Under consideration is the differential equation

$$\text{(I)} \qquad (k(u)p(x)u')' = p^{\eta}(x)Q(u) \quad \text{in} \quad 0 \leq x \leq b, \quad (\ )' = \frac{d}{dx}$$

subject to the boundary conditions: $u = 1$ at $x = b$, $(kpu') = y_0$ (>0) at $x = b$ and $(kpu') = 0$ at $x = 0$. Here $0 \leq \eta < 1$ is a given constant and k and Q are known functions and the question posed is to find a positive constant $b > 0$, a function $p(x) > 0$ in $0 < x \leq b$ and a solution u of the differential equation such that the L_p-norm

$$\text{(II)} \qquad \left(\int_0^b p^{\rho}dx\right)^{\frac{1}{\rho}}, \quad \rho \geq 1$$

is minimized. It is assumed that k is a positive function of u. Also $Q > 0$ for $u > 0$ and $Q \leq 0$ for $u \leq 0$.

By a heuristic analysis we are led to the following transformation:

$$\text{(III)} \qquad \begin{cases} \psi = \int_0^u k(s)\,[Q(s)]^{\zeta}ds, & \zeta = \frac{\rho+1}{\rho-\eta} \geq 1 \\ \varphi = \frac{1}{\delta}\,y^{\delta}, & \delta = 1 + \frac{1}{\zeta} \end{cases}$$

where $y(x) = k(u(x))p(x)u'(x)$. This transforms the norm II into

(IV) $$\int_{\psi_1}^{\psi_2} \left(\frac{d\varphi}{d\psi}\right)^{\xi} d\psi, \quad \frac{1}{\xi} + \frac{1}{\zeta} = 1$$

and this is minimized by applying the Hölder's inequality to $\int_{\psi_1}^{\psi_2} \left(\frac{d\varphi}{d\psi}\right) . 1 d\psi$. The condition for equality in Hölder's inequality leads to a key relation, namely

(V) $$\left(\frac{y}{y_0}\right)^{\delta} = \left(\frac{\psi}{\psi_2}\right).$$

Here $\psi_2 = \int_0^1 k(s)\,[Q(s)]^{\zeta} ds$. This yields expressions for $u(x)$, $p(x)$ and b satisfying I and the above mentioned boundary conditions and minimizing the norm II.

The papers [5,6,7] by Wilkins, mentioned above, concern the design of the profile of minimum mass cooling fins and spines. He has studies two cases of the equation I with $\eta = 0$ (rectangular cooling fin) and $\eta = \frac{1}{2}$ (cooling spine), separately. The equation I is then the steady state heat flow equation. $u(x)$ is the temperature, $p(x)$ the thickness (radius) of the cooling fin (spine). Also $k(u)$ is proportional to specific conductivity of the material. He minimizes the weight of the cooling fin (spine), that is to say $\int p dx$. Thus $\rho = 1$ in his case. He uses different algebraic identities, instead of Holder's inequality, for the cases $\eta = 0$ and $\eta = \frac{1}{2}$ to minimize $\int p dx$. It is assumed that the cooling coefficient of the fin (spine) is a constant and ambient temperature is $0^{\circ}A$. For the Newton's linear law of cooling $Q(u) = u$ and for the Stefan-Boltzmann law of cooling $Q(u) = u^4$.

Our results coincide with those of Wilkins [5,6,7] on taking $\eta = 0$, $p = 1$ and $\eta = \frac{1}{2}$, $p = 1$. Moreover if in these cases we take $k(u) =$ constant and $Q(u) = u$ we obtain the results of E. Schmidt [3] and R. Focke [4] giving the profiles of minimum mass rectangular cooling fin and minimum mass circular spine.

Certain results complementing the results obtained here have been treated by R. J. Duffin, D. K. McLain and S. Bhargava. In reference [2], R. J, Duffin solves the maximizing of the heat dissipation in rectangular and circular fins subject to the constraint that the weight be a constant. In reference [1] R. J. Duffin and D. K. McLain treat the same problem for arbitrary convex fins. Their approach permits a variable cooling coefficient in contrast to Wilkin's Method given in [5] and generalized here. Further the Duffin-McLain approach has been further developed in reference [9] by S. Bhargava and R. J. Duffin to give dual extremum principles for the maximum heat dissipation under more general non-linear weight constraint of the L_p-norm type: $(\int \rho^p dx)^{\frac{1}{p}} \le K$, $p \ge 1$. However these principles are limited to the case of Newton's linear law of cooling. For example, the approach does not permit the Stefan-Boltzmann u^4-law of cooling. In references [8] and [10] they obtain analogous dual extremum principles for optimum Network and Beam designs.

The method developed by Wilkins in [5] for minimizing the L_1-norm $\int \rho dx$ and generalized here for minimizing the L_p-norm

$(\int p^{\rho} dx)^{\frac{1}{\rho}}$ permits a class of cooling laws Q(u) including Newton's linear law and Stefan-Boltzmann's u^4-law. But his approach does not seem to lead to dual extremum principles for the minimum weight nor does it seem to extend to Q being a function of X. Thus it does not apply to the circular cooling fins treated in [2].

2, The Minimizing Problem and Heuristic Analysis.

Problem JL. To find two functions p(x) and u(x) and a positive number b having the following properties; u(x) JLS a differentiable function in 0 £ x £ b satisfying the differential equation

(1) (k(u)p(x)u«) • = p^CxXMu) in 0 1 x £ b, ( ) • = $\frac{d}{dx}$

and the boundary conditions

(2) u = 1 at x = b,

(3) (kpu') =0 at x = 0,

(4) (kpu') = Y_0(>0) at x = b;

p(x) jus continuous in 0 £ x £ b and satisfies the condition

(5) p(x) > 0 in 0 < x £ b.

Under these conditions we seek to minimize the functional

(6) $W = \int^{b} p^{p} dx.$

Here 0 £ r) < 1 and *p* ^> 1 are given constants and Q(u) and k(u) are known functions such that Q(u) > 0 for u > 0, Q(u) ^ 0 for u £ 0 and k(u) > 0 for all u. Further Q jj3 continuous and k JLS^ differentiable in every finite interval.

It is shown in section 5 that if for a b > 0, u(x) and p(x) satisfy (l)-(5) then, under the assumptions on k(u) and Q(u) made above, u(x) and y(x) where

(7) y(x) = k(u(x))p(x)u» (x) in O ^ x ^ b

are positive and increasing for $x > 0$ and hence $Q > 0$ in (1). Further the positivity of u and y show, by (1) and (7), that

(8) $$\left(\frac{du}{dx}\right) \div \left(\frac{dy}{dx}\right) > 0 \quad \text{for} \quad x > 0.$$

We now derive by a heuristic analysis a transformation which will enable us to obtain rigorous solution of Problem 1. We introduce the following quantities for convenience:

(9) $$\xi = \frac{\rho + 1}{\eta + 1} \geq 1$$

(10) $$\frac{1}{\xi} + \frac{1}{\zeta} = 1$$

(11) $$\delta = 1 + \frac{1}{\zeta}.$$

On using (7) equation (1) becomes

(12) $$y' = p^{\eta} Q \quad \text{in} \quad 0 \leq x \leq b.$$

Equating p^{η} obtained from (7) and (12) gives

(13) $$\left(\frac{y'}{Q}\right) = \left(\frac{y}{ku'}\right)^{\eta}.$$

Using (7) and (13) $p^{\rho}dx$ can be expressed in terms of u, y, k and Q as

$$p^{\rho}dx = \left(\frac{y}{kdu}\right)^{\rho}(dx)^{\rho+1} = \left(\frac{y}{kdu}\right)^{\frac{\xi}{\zeta}}\left(\frac{dy}{Q}\right)^{\xi} = \frac{\left(y^{\frac{1}{\zeta}}dy\right)^{\xi}}{[(kdu)^{\xi-1}\, Q^{\xi\zeta-\zeta}]}.$$

Thus

(14) $$p^\rho dx = [\frac{y^{\frac{1}{\zeta}} dy}{kQ^{\zeta} du}]^{\xi} (kQ^{\zeta} du).$$

This suggests the transformation yi—»(p and u»-» J0 given by

(15) $$\phi = \int_0^u k(s)\,[Q(s)]^{\zeta} ds$$

(16) $$\psi = \int y^{6}.$$

Using these in (14) gives

(17) $$w = \int_0^b p^\rho dx = \int_{\psi_1}^{\psi_2} \ \xi d\psi$$

where

(18) $$\theta_1 = \theta|_{x=0} = \int_0^{u(0)} k(s)Q^{\zeta}(s)ds$$

and

(19) $$0 < \theta_2 = \theta|_{x=b} = \int_0^1 k(s)Q^{\zeta}(s)ds \wedge \wedge_{x}.$$

Relation (2) has been used in (19). Also, on using (3) and (4) we introduce

(20) $$\psi_1 = \psi|_{x=0} = 0$$

and

(21) $$0 < \psi_2 = \psi|_{x=b} = \wedge Y \hat{Q}.$$

It is shown in Section (3) that on applying Hölder's inequality to (17) that $w = \int_0^b p^\rho dx$ is minimized if and only if $u(x)$ and $p(x)$ and the constant b satisfy

$$\left(\frac{\varphi}{\cdot}\right)_2 = \left(\frac{\psi}{\cdot}\right)_2, \quad \psi_1 = 0$$

besides (1)-(5). These relations yield unique solutions $p(x) = p_0(x)$, $u(x) = u_0(x)$ and $b = b_0$ of Problem 1.

3. Solution of the Minimizing Problem.

On introducing $y(x)$ defined by (7), equation (1) is equivalent to the two first order equations (7) and (12). If u, y and p satisfy (7) and (12) then, by the transformation given by (15) and (16), (6) is equivalent to (17) and the boundary conditions (2), (3) and (4) are equivalent to (19), (20) and (21) respectively. Now, consider the identity

$$(\varphi_2-\varphi_1) = \int_{\psi_1}^{\psi_2} \left(\frac{d\varphi}{d\psi}\right)\cdot 1 d\psi.$$

Applying Hölder's inequality to this and using (18)-(21) gives

(22) $$\varphi_2 \leq \left(\int_{\psi_1}^{\psi_2} \left(\frac{d\varphi}{d\psi}\right)^{\xi} d\psi\right)^{\frac{1}{\xi}} \left(\int_{\psi_1}^{\psi_2} 1^{\zeta} d\psi\right)^{\frac{1}{\zeta}}.$$

Rewriting this and using (17) gives

(23) $$\int_0^b p^{\rho} dx \geq \frac{\varphi_2^{\xi}}{(\psi_2-\psi_1)^{\frac{\xi}{\zeta}}} \geq \left(\frac{\varphi_2}{\psi_2^{\frac{1}{\zeta}}}\right)^{\xi}.$$

There is equality in (22) if and only if

(24) $$\frac{d\varphi}{d\psi} = \lambda \quad (\geq 0 \text{ by } (8),\ (15) \text{ and } (16)).$$

Using this, there is equality throughout (23) if and only if (24) holds and

(25) $$\psi_1 = 0.$$

Using (20) and (25) and integrating (24) gives

(26) $$\left(\frac{\varphi}{\varphi_2}\right) = \left(\frac{\psi}{\psi_2}\right) \text{ in } 0 \leq x \leq b.$$

Here (19) and (21) have been used. Relation (26) can be written as

$$(27) \qquad \left(\frac{y}{y_0}\right)^{\delta} = \left(\frac{\psi}{\psi_2}\right) \text{ in } 0 \leq x \leq b.$$

Condition (25), on using (18) becomes

$$(28) \qquad u(0) = 0.$$

Thus a necessary and sufficient condition that $u(x)$, $y(x)$, $p(x)$ and b satisfying (9) and (12) and the side conditions (2)-(5) minimize (6) is that (27) and (28) be satisfied. Now, eliminating p and y between (7), (12) and (27) gives

$$(29) \qquad \left(\frac{du}{dx}\right) = \left(\frac{\delta\psi_2^{\frac{1-\eta}{\delta}}}{y_0^{1-\eta}}\right)^{\frac{1}{1+\eta}} \cdot \left[\frac{(\psi(u))^{\frac{\rho}{1-\eta+2\rho}}}{k(u)\,(Q(u))^{\frac{1}{\rho-\eta}}}\right] \text{ in } 0 \leq x \leq b.$$

Integrating (29) and using (28) gives $u(x) = u_0(x)$ and then using (2) gives $b = b_0$:

$$(30) \qquad x = \left(\frac{y_0^{1-\eta}}{\delta\psi_2^{\frac{1-\eta}{\delta}}}\right)^{\frac{1}{1+\eta}} \left(\int_0^{u_0} \left[\frac{k(s)\,(Q(s))^{\frac{1}{\rho-\eta}}}{(\psi(s))^{\frac{\rho}{1-\eta+2\rho}}}\right] ds\right.$$

$$(31) \qquad b_0 = \left(\frac{y_0^{1-\eta}}{\delta\psi_2^{\frac{1-\eta}{\delta}}}\right)^{\frac{1}{1+\eta}} \left(\int_0^{1} \left[\frac{k(s)\,(Q(s))^{\frac{1}{\rho-\eta}}}{(\psi(s))^{\frac{\rho}{1-\eta+2\rho}}}\right] ds\right)$$

Using (27) and (29) gives $p(x) = p_0(x)$

$$(32) \qquad p_0(x) = \left(\frac{y_0^2}{\delta\psi_2^{\frac{2}{\delta}}}\right)^{\frac{1}{1+\eta}} [\psi(u_0(x))]^{\frac{1}{1-\eta+2\rho}} [Q(u_0(x))]^{\frac{1}{\rho-\eta}}.$$

For these $u_0(x)$, $p_0(x)$ and b_0, (27) and (28) are satisfied besides (1)-(5) and therefore (23) gives

$$(33) \qquad W = \min \int_0^b p^\rho dx = \left(\frac{Y_0^\delta}{\delta \psi_2^{\frac{1}{\zeta}}}\right)^\xi.$$

We can formulate the above discussion as follows:

<u>Theorem 1</u>. <u>Problem 1 has unique solution given by</u> (30), (31) <u>and</u> (32) <u>and the minimum</u> W <u>of the functional</u> (6) <u>is given by</u> (33). <u>In these formulae</u> (30)-(33), ξ, ζ, δ; ψ <u>and</u> ψ_2 <u>are given by</u> (9)-(11), (15) <u>and</u> (19).

<u>Proof</u>: See the discussion immediately preceding the statement.

4. Special Cases and Application to Design of Minimum Mass Cooling Fins.

In this section we demonstrate that our results obtained in Section 3 contain earlier results of J. E. Wilkins [5], E. Schmidt [3] and R. Focke [4] concerning the profile of minimum mass cooling fins and spines. By applying Theorem 1 we obtain the results tabulated in Table 1:

TABLE 1

Fin description	$Q(u)$ Law of Cooling	$k(u)$ Specific conductivity	$p_0(x)$ thickness or $[\text{radius}]^2$	$u_0(x)$ temperature	b_0 length	W Minimum Weight	Reference and Remark
Rectangular Cooling Fin. $\rho = 1$ $\eta = 0$	$Q(u)$	$k(u)$	thickness: $\dfrac{2y_0^2[\Psi(u_0(x))]^{1/3} Q(u_0(x))}{3\Psi_2^{4/3}}$	$x = \left(\dfrac{2y_0}{3\Psi_2^{2/3}}\right)\displaystyle\int_0^{u_0} \dfrac{kQ\,du}{\Psi^{1/3}}$	Put $x = b_0$ and $u_0 = 1$ in the previous column	$\dfrac{4y_0^3}{9\Psi_2}$	Wilkins [5] $\Psi = \int_0^u k\,Q^2\,du$ $\Psi_2 = \Psi\vert_{u=1}$
	u (Newton)	k = Constant	thickness: $\dfrac{x^2}{2k}$	$\dfrac{x}{b_0}$	$2y_0$	$\dfrac{4y_0^3}{3k}$	— Schmidt [3]
Circular Cooling Spine $\rho = 1$ $\eta = \frac{1}{2}$	u^4 (Stefan–Boltzman)	k = Constant	$[\text{radius}]^2$: $\left[\dfrac{68y_0^2}{5k}\right]^{2/3}\left(\dfrac{x}{b_0}\right)^{\frac{74}{11}}$	$\left(\dfrac{x}{b_0}\right)^{5/11}$	$\left[\dfrac{23120\,k y_0}{1331}\right]^{1/3}$	$\left[\dfrac{256 \times 17 \times y_0^5}{625k}\right]^{1/3}$	Wilkins [5]
	u (Newton)	k = constant	$[\text{radius}]^2$: $\left[\dfrac{16y_0^4}{k^2}\right]^{\frac{1}{3}}\left(\dfrac{x}{b_0}\right)^4$	$\left(\dfrac{x}{b_0}\right)$	$(16y_0 k)^{1/3}$	$\left[\dfrac{256\,y_0^5}{125k}\right]^{1/3}$	— Focke [4]

Rate of heat supplied at the foot $= y_0$ (or proportional to)
Temperature of the foot $= 1$
Temperature of surroundings $= 0$
Coefficient of cooling $= 1$

5. A Lemma Showing the Temperature is Positive,

In this section we show that if p, u satisfy (1)-(5) and y is given by (7) and k and Q satisfy the conditions stated in Problem 1, then u and y are positive and increasing in $x > 0$.

Lemma 1. Let $k(u)$ and $Q(u)$ satisfy the conditions stated in Problem 1. Let $p(x)$ and $u(x)$ satisfy (1)-(5) for some $b > 0$ and let y be given by (7). Then $u(x)$ and $y(x)$ are positive and increasing in $x > 0$.

Proof; Integrating (1) and using (3) gives

$$(34) \qquad \frac{du}{dx} = \frac{1}{k(u)p(x)} \int_0^x p(x)Q(u)\,dx, \quad x > 0.$$

Suppose $u < 0$ and $x = 0$, so, from (34), $\left(\frac{du}{dx}\right) \leq 0$ in the right neighborhood of $x = 0$. Hence u is a decreasing function and never positive contradicting (1). A similar argument holds if u has a negative maximum at a point $x = b'$ inside $(0,b)$. Then (34) holds with b' replacing b. Thus $u(x) > 0$ for $x > 0$ and by (34), $\frac{du}{dx} > 0$ in $x > 0$; hence $y(x) = k(u)p(x)u'(x) > 0$ in $x > 0$, $Q(u) > 0$ in $x > 0$ and $\frac{dy}{dx} = pQ(u) > 0$ in $x > 0$. This completes the proof of the lemma.

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