DUAL EXTREMUM PRINCIPLES
RELATING TO
OPTIMUM BEAM DESIGN*

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Abstract

Of concern is a cantilever beam resting on an elastic foundation and supporting a load at the free end. The beam is of rectangular cross section and of constant height but variable width. It is required to taper the beam for maximum strength, or what is the same, for supporting maximum vertical load W at the free end when the free end is given unit deflection. The constraint is that the weight of the beam should not exceed a given bound K. It is shown that the optimum taper should be chosen so that the curvature of the beam is constant. This yields the solution of the problem in terms of explicit formulas. Turning to more general constraints, a duality inequality is found which gives upper and lower bounds for the maximum load W.

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DUAL EXTREMUM PRINCIPLES RELATING TO OPTIMUM BEAM DESIGN

1. Introduction.

Beams are used in structures to support loads imparted by other members of the structure. A beam can be supported by an elastic wall or sheet along its entire length. A loading condition on such a beam of rectangular cross section is shown in Figure 1 where a vertical load is acting at one end while the other end is clamped.

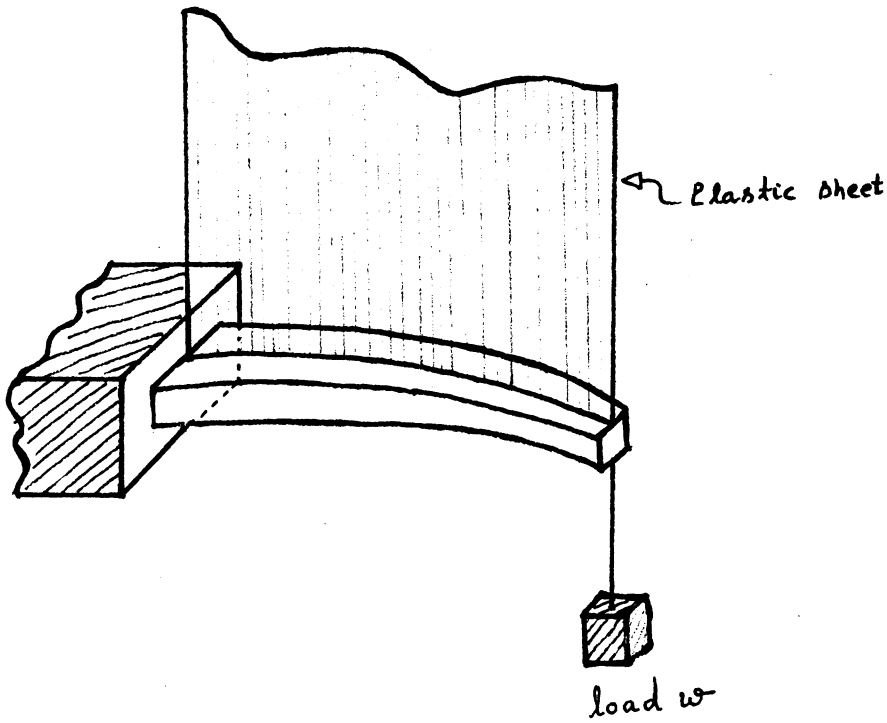

Figure 1. Cantilever beam on an elastic foundation

A design question for such a cantilever beam would be — how should the width of a. beam of limited weight K b£ tapered so as to have maximum strength or what is the same, so as to support ja maximum end load W while this end is to have unit deflection?

By making certain tacit assumptions the physical problem is given a mathematical setting. The maximizing question is transformed into a minimax problem of the Calculus of Variations and then given a rigorous analysis.

This study leads to a simple criterion for the optimum tapering of a rectangular beam of fixed height -- the curvature of the deflection curve or what is the same, the second derivative of the deflection function, should be a, constant. This leads to explicit expression for the flexural rigidity function, and hence for the tapering of the width of the optimal beam.

Analogous design problems in Networks and Heat Transfer were treated in references [l]-[4]. For example in [4] we considered a lumped network having a finite number of conducting branches. Certain branches, termed set B, are allowed to vary their conductance but the total conductance is limited by an l_p norm type constraint

$$(\Sigma_B g_s^{\rho})^{1/\rho} \leq K \qquad \rho \geq 1.$$

Here g_s is the conductance o£ branch s and K and p are positive constants. Then the design problem is to maximize the joint conductance of the network between two specified input points. The network question may be characterized as a maximizing problem of mathematical programming, thus suggesting that there is a dual minimizing problem. Pursuing this idea we were lead to the duality inequality, namely

$$\|v\|_{2,\alpha} \geq \Gamma^{1/2} \geq 1/\|y\|_{2,\beta}.$$

Here $\| \ \|_{2,\alpha}$ and $\| \ \|_{2,\beta}$ are certain dual norms. The vector v is an arbitrary normalized voltage distribution satisfying Kirchhoff's voltage law. The vector y is an arbitrary normalized current vector satisfying Kirchoff's current law. Γ is the optimum conductance. There is no "duality gap"; in other words the duality inequality could be used to give a sharp estimate of Γ.

This network model suggested to us what duality inequality should hold for the elasticity problem on hand. Moreover it suggested replacing the weight constraint $\int_0^1 p\,dx \leq K$ on the width p by the more general constraint $\int_0^1 p^\rho dx \leq K^\rho$. Thus we prove in this work the following duality inequality

$$\|u\|_{2,\alpha} \geq W^{1/2} \geq 1/\|y\|_{2,\beta}.$$

Here $\| \ \|_{2,\alpha}$ and $\| \ \|_{2,\beta}$ are certain dual norms. The functions $u(x)$ and $y(x)$ are arbitrary smooth functions normalized at the boundary. Moreover $u(x)$ is a deflection function and $y(x)$ is a moment function. W is the maximum strength of the beam. This duality inequality could be used to give upper and lower bounds for W.

Various problems concerning the design of beams and columns for maximum strength were solved by H. Blasius [5]. Recently interesting developments along similar lines have been made by I. Tadjbaksh and J. B. Keller [6]. A comprehensive treatment of beams supported by elastic foundations has been given by M. Hetenyi [7]. None of these authors treat the problem posed here.

Certain modifications of the basic problem are treated. Presumably several other modifications are feasible.

2. Formulation of The Problem and Heuristic Analysis.

Consider a beam of rectangular cross section of fixed height elastically supported by an elastic sheet or wall; it is clamped at one end and a load is acting at the other end so as to produce a desired deflection of this end. In Figure 1 is shown such a beam. In Figure 2 is sketched the deflection curve.

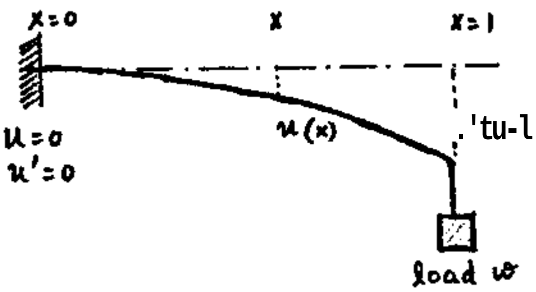

Figure 2. Deflection Curve

Let $u(x)$ be the deflection function and $q(x) > 0$ the elastic coefficient of the supporting material. Assuming that the reaction due to the elastic support is a linear function of the deflection $u(x)$, equilibrium state of bending is given by the differential equation

$$(1) \qquad (pu'')'' + qu = 0 \quad \text{in} \quad 0 \le x \le 1, \quad (\ )' = d/dx$$

where $p(x)$ is the flexural rigidity of the beam. The geometrical boundary conditions are

(2) $u = 0$ at $x = 0$.

(3) $u^1 = O$ at $x = 0$.

(4) $u = -1$ at $x = 1$.

The mechanical boundary conditions on torque and force are

(5) $pu'' = 0$ at $x = 1$

and

(6) (pu^M) » $= w$ at $x = 1$.

Of course

(7) p ^ 0 in 0 £ x £ 1.

•The load w in (6) can be regarded as the strength of the beam in the following sense: The beam being clamped at $x = 0$ is capable of supporting a vertical load w at $x = 1$ While maintaining a deflection -1 of the end $x = 1$.

Now, the flexural rigidity function is proportional to the width of the beam since the height is a constant throughout the length of the beam. Thus the integral of the width function or that of the flexural rigidity function is a measure of the weight of the beam. It is assumed that the density and Young[1]s modulus of the material of the beam are constants. In view of this, the inequality (8) below can be regarded as weight constraint.

With the deflection determined by (l)-(5) we can formulate the following optimization problem:

Problem 1. Find the maximum strength W oE ci beam subject to the constraint that the weight is bounded by ei given constant:

(8) $$\int_0^1 p\,dx \leq K, \quad K > 0.$$

We now develop a solution of Problem 1 providing K exceeds a positive bound. This bound on K shall be established in the course of the solution. It is desirable to recast the problem into a minimax problem in terms of the functional

(9) $$E(p,u) = \int_0^1 (pu''^2 + qu^2)\,dx.$$

This change is motivated by the following heuristic analysis.

Let p, q and u be sufficiently smooth in $0 \leq x \leq 1$ so that integration by parts below is valid. Let v be an arbitrary smooth function. Then

(10) $$E(p,u+v) = E(p,u) + E(p,v) + 2\int_0^1 (pu''v'' + quv)\,dx.$$

Integrating by parts gives

(11) $$\int_0^1 (pu''v''+quv)\,dx = \int_0^1 [(pu'')''+qu]v\,dx + [(pu'')v']_0^1 - [(pu'')'v]_0^1 .$$

The first integral on the right side vanishes by (1). If we impose $v = 0$ at $x = 0$ and $x = 1$ and $v' = 0$ at $x = 0$ then $u + v$ satisfies the geometrical boundary conditions (2)-(4) and the boundary terms in (11) reduce to zero. Here (5) is also used. Thus (10) becomes

(12) $$E(p,u+v) = E(p,u) + E(p,v) \geq E(p,u).$$

It follows that $E(p,\tilde{u})$ is minimized for the class of functions satisfying the boundary conditions (2)-(4) by u satisfying (1) and (5). This is a standard result of the calculus of variations. In the calculus of variations (5) is termed a natural boundary condition because it is necessarily satisfied by the minimizing function.

Now letting $v = u$ in (11) and using (1)-(6) gives

$$E(p,u) = [(pu'')']_{x=1} = w. \tag{13}$$

Lemma 1. If the function u satisfies the Euler differential equation (1) corresponding to the saddle functional $E(p,u)$ and if p and u satisfy the boundary conditions (2)-(5) then

$$w = E(p,u) \tag{14}$$

where w is the strength of the beam.

Proof: This immediately follows from (13).

In view of relations (12) and (14) we pose an equivalent problem:

Problem 2. Find

$$W = \max_{p} \min_{u} E(p,u). \tag{15}$$

Here u is not necessarily subject to the differential equation (1) but satisfies the boundary conditions (2)-(4) and $p(x)$ is subject to the constraints (7) and (8).

Thus the original maximizing problem has been replaced by a minimax problem. We continue the heuristic analysis and investigate this minimax problem.

Lemma 2. If $p(x)$ is continuous in $0 \leq x \leq 1$ satisfying the conditions (7) and (8) and $u(x)$ is continuous with continuous first derivative and piecewise continuous second derivative in $0 \leq x \leq 1$, then

$$(16)\qquad E(p,u) \le K \sup_{0\le x\le 1}(u'')^2 + \int_0^1 qu^2dx.$$

This is an equality if

$$(17)\qquad \int_0^1 pdx = K$$

and

$$(18)\qquad u'' = \lambda, \text{ a constant in } 0 \le x \le 1.$$

Proof: It is enough to see that

$$(19)\qquad \int_0^1 pu''^2dx \le \left(\int_0^1 pdx\right)\cdot \sup_{0\le x\le 1}(u'')^2 \le K \sup_{0\le x\le 1}(u'')^2.$$

Because (19) becomes an equality throughout when (17) and (18) hold, the case of equality is also shown.

The relation (18) enables us to solve for a $p(x)$ and a $u(x)$ satisfying (1)-(5), (7) and (8). This we do below in Lemma 3. That this solution for p and u is the unique optimal solution of Problems 1 and 2 will be established in Section 3.

Lemma 3a. *The solution of the differential equation* (18) *subject to the boundary conditions* (2)-(4) *and corresponding* λ *are given by*

$$(20)\qquad u_0(x) = -x^2 \quad \text{in } 0 \le x \le 1.$$

$$(21)\qquad \lambda = -2.$$

Proof: Integrating (18) gives

$$u_0 = \frac{\lambda x^2}{2} + c_1x + c_2.$$

Using (2)-(4) gives $c_1 = 0 = c_2$, $\lambda = -2$.

Lemma 3b. For u _given by_ (20) _the differential equation_ (1) _reduces to one for_ p _namely_

$$p'' = -\frac{1}{2} x^2 q(x) \quad \text{in} \quad 0 \leq x \leq 1 \tag{22}$$

and the boundary condition (5) _becomes_

$$p = 0 \quad \text{at} \quad x = 1. \tag{23}$$

Subjecting the constant K _to the limitation_

$$K \geq \frac{1}{4} \int_0^1 (1-x) x^3 q(x)\,dx \tag{24}$$

and assuming $q(x) > 0$ _is continuous in_ $0 \leq x \leq 1$, _the unique solution of_ (22) _satisfying_ (7), (17) _and_ (23) _is given by_

$$p_0(x) = 2K(1-x) + \frac{(1-x)}{2} \left[\int_0^1 x^4 q(x)\,dx\right] - \frac{1}{2}\left[\int_x^1 x^3 q(x)\,dx\right] + \frac{x}{2}\left[\int_x^1 x^2 q(x)\,dx\right] \quad \text{in} \quad 0 \leq x \leq 1. \tag{25}$$

Moreover

$$p_0(x) > 0 \quad \text{in} \quad 0 < x < 1. \tag{7a}$$

Proof: Substituting (20) in (1) and (5) immediately gives (22) and (23). Then integrating (22) twice and using (23) gives

$$2p_0'(x) = -\left(\alpha + \int_0^x x^2 q(x)\,dx\right) \tag{25a}$$

and

$$2p_0(x) = \alpha(1-x) + \int_0^1 x^2 q(x)\,dx - x\int_0^x x^2 q(x)\,dx - \int_x^1 x^3 q(x)\,dx \tag{26}$$

where α is the integration constant uniquely determined by (17) namely

(27) $$a = 4K - \int_0^1 (1-x^2)x^2 q(x)\,dx.$$

Substituting (27) in (26) gives (25). If p_0 is to satisfy (7), in particular $p_0(0) \geq 0$ and this and (25) give (24). The condition (24) is sufficient for p_0 to satisfy (7) and (7a) for, substituting (24) in (25) gives

$$2p_0(x) \geq (1-x)\int_0^x x^3 q(x)\,dx + x\int_x^1 x^2(1-x)q(x)\,dx.$$

Here, because $q(x) \geq 0$ in $0 \leq x \leq 1$ the right side is non-negative in $0 \leq x \leq 1$ and positive in $0 < x < 1$.

This completes the proof of the lemma.

Lemma 3. The functions u_0 and p_0 given by (20) *and* (25) *satisfy the differential equation* (1), *the boundary conditions* (2)-(5) *and* the constraints (7) and (8). Moreover p_0 satisfies

(7a) $$p_0(x) > 0 \text{ in } 0 < x < 1.$$

and (17).

Proof; This is an immediate consequence of Lemma (3a) and Lemma (3b).

3. Comparison Relations and the Main Proof.

In this section we obtain rigorous results to show that the solution $p = p_0$, $u = u_0$ of Lemma 3 is the optimal beam and is the unique optimal beam. Because $u_0'' =$ constant, this beam $[p = p_0,\ u = u_0]$ will be referred to hereafter as the "Constant Curvature" beam or CC-beam.

It is desirable to relax some of the restrictions of Problem 2. Thus we formulate the following problem:

Problem 3. Let u be a continuous function in $0 \leq x \leq 1$ with continuous first derivative and piecewise continuous second derivative there. Let $p \geq 0$ be a continuous function in $0 \leq x \leq 1$. Let u satisfy the boundary conditions (2)-(4) and p the weight constraint (8). Then find

$$W = \sup_p \inf_u E(p,u) \tag{28}$$

the constant K being given to satisfy the bounding inequality (24).

Lemma 4. Let p_0 and u_0 be the flexural rigidity function and deflection function for the CC-beam given by (20) and (25). Let p be an arbitrary admissible flexural rigidity function. Then the saddle functional satisfies

$$E(p_0,u_0) \geq E(p,u_0). \tag{29}$$

Proof: Since $u_0''^2 = c^2 = 4$ and $\int_0^1 p_0 dx = K$ we have

$$E(p_0,u_0) = c^2K + \int_0^1 q(x)\,u_0^2(x)\,dx \geq$$

$$c^2 \int_0^1 p\,dx + \int_0^1 q(x)\,u_0^2(x)\,dx =$$

$$\int_0^1 [p{u_0''}^2 + qu_0^2]\,dx = E(p,u_0).$$

<u>Theorem 1</u>. <u>The CC-beam</u> $[p = p_0,\ u = u_0]$ <u>is optimal</u>.

<u>Proof</u>: Let p and u refer to arbitrary admissible flexural rigidity function and deflection function respectively. Then

$$\inf_u E(p,u) \leq E(p,u_0) \leq E(p_0,u_0) \text{ by (29)},$$

and therefore

$$\text{(30)} \qquad \sup_p \inf_u E(p,u) \leq E(p_0,u_0).$$

Also,

$$\text{(31)} \qquad \sup_p \inf_u E(p,u) \geq \inf_u E(p_0,u) = E(p_0,u_0).$$

The last equality in (31) is true because u_0 satisfies the Euler differential equation, (1) with $p = p_0$, of the positive definite quadratic functional $E(p_0,u)$ and the boundary conditions (2)-(5). The relations (30) and (31) now imply the desired result that

$$\text{(32)} \qquad E(p_0,u_0) = \sup_p \inf_u E(p,u).$$

The relation (32) is the same as (28) with $W = E(p_0,u_0)$, the strength of the CC-beam.

<u>Theorem 2</u>. The CC-beam is the only optimal beam.

<u>Proof</u>: The uniqueness of the optimal beam is also a consequence of Lemma 4. It can be treated in essentially the same manner as the uniqueness proof given by Duffin and McLain in Theorem 2, reference [3], In order not to interrupt the main line of thought the uniqueness proof is postponed until Section 7.

4- <u>Formulae for the Optimum Taper</u>,

It is a direct consequence of the results of preceding sections that the optimum beam has flexural rigidity given by (25) and that the deflection of the beam is given by (20) .

The infimum and supremum in the formulation of Problem 3 are actually assumed for the flexural rigidity and deflection functions of the CC-beam. Hence the solution of Problem 3 is also a solution of Problem 2. The flexural rigidity and deflection functions of the CC-beam are sufficiently smooth (p_0 has two continuous derivatives and u_0 has four continuous derivatives -- in fact, u_0 has continuous derivatives of all order -- in $0 \le x \le 1$) so that integration by parts in Section 1 is valid. It follows that the solution of the minimax problem is the maximum strength or, what is the same, the maximum vertical end load which can be supported by a cantilever beam having a unit deflection at the loaded end, when the beam is subject to the constraint that its weight cannot exceed a constant -- this constraint being expressed by

$$(8) \qquad \int_0^1 p\,dx \le K, \quad K > 0,$$

and the maximum strength is given by

$$(33) \qquad W = (p_0 u_0'')'\Big|_{x=1} = 4K + \int_0^1 x^4 q(x)\,dx.$$

This formula is a simple consequence of substituting (20) and (27) in (25a) .

If $q(x)$ is a constant the integrations can actually be carried out and more explicit formulae obtained for p^0 and W.

The results for constant and non-constant q are summarized in the following theorem.

Theorem 3. The flexural rigidity function p_0 for the beam of maximum strength or, what is the same, for the beam which supports the maximum load in the class of all beams which are clamped at the end $x = 0$, which have a deflection -1 at the other end, which have continuous flexural rigidity functions $p(x)$ in $0 \leq x \leq 1$ satisfying

$$\int_0^1 p\,dx \leq K, \quad K > 0 \tag{8}$$

in which K satisfies

$$K \geq \frac{1}{4}\int_0^1 (1-x)x^3 q(x)\,dx, \tag{24}$$

and which are supported throughout the length by an elastic sheet of elastic coefficient $q(x) > 0$, is given by

$$p_0(x) = 2K(1-x) + \frac{(1-x)}{2}\left[\int_0^1 x^4 q(x)\,dx\right] - \frac{1}{2}\left[\int_x^1 x^3 q(x)\,dx\right] + \frac{x}{2}\left[\int_x^1 x^2 q(x)\,dx\right] \tag{25}$$

in $0 \leq x \leq 1$.

The deflection of this optimum beam is given by

$$u_0(x) = -x^2 \quad \text{in} \quad 0 \leq x \leq 1. \tag{20}$$

The end load that the beam can support at $x = 1$, or the maximum strength W is given by

$$W = (p_0 u_0'')'\Big|_{x=1} = 4K + \int_0^1 x^4 q(x)\,dx. \tag{33}$$

If q i^ a constant these become

(34) $$p_Q(x) = (2K-\frac{q}{40}) - (2K-\frac{q}{15})x - \frac{q}{24}x^4$$

(35) $$W a 4K + \frac{q}{5}$$

when K > 0 .is. limited by

(36) $$K \geq \frac{q}{80}.$$

Remark JL. rf the beam is an ordinary cantilever and not elastically supported then^

(37) $$q = 0.$$

In this case (36) j-s, automatically satisfied and it is ci simple matter to show that the optimum taper and maximum strength are given by (34) and (35) with q = 0. Namely,

(38) $$p_Q(x) = 2K(l-x) \qquad 0 £ x £ l_f$$

and

(39) $$W = 4K.$$

5. A Duality Inequality for the Maximum Strength.

This section concerns a different approach to the optimization problem and an interesting generalization. The generalization consists in replacing the linear constraint $\int_0^1 p\,dx \leq K$ by a nonlinear constraint of the form $\int_0^1 p^\rho dx \leq K^\rho$. The new approach is to obtain a duality inequality giving upper bounds and lower bounds for the maximum strength W. The following derivation of the duality inequality does not depend on the previous theorems.

Theorem 4. Let u and y be continuous functions in $0 \leq x \leq 1$ with continuous first derivatives and piecewise continuous second derivatives in that interval. Let u and y satisfy the boundary conditions

(40) $$u(0) = u'(0) = 0, \quad u(1) = -1 \quad \text{and}$$

(41) $$y(1) = 0, \quad y'(1) = 1.$$

Let

(42) $$\|u''\|_\alpha = \left(\int_0^1 |u''|^\alpha dx\right)^{1/\alpha}, \quad \|y\|_\beta = \left(\int_0^1 |y|^\beta dx\right)^{1/\beta}.$$

Then define dual norms as

$$\|u\|_{2,\alpha} = [K\|u''\|_\alpha^2 + \int_0^1 qu^2 dx]^{1/2}$$

$$\|y\|_{2,\beta} = [K^{-1}\|y\|_\beta^2 + \int_0^1 \frac{y''^2}{q}\,dx]^{1/2}$$

where α, β are positive constants satisfying

(43) $$\alpha^{-1} + \beta^{-1} = 1,$$

and K is another positive constant and $q(x) > 0$ is continuous in $0 \leq x \leq 1$. Then the following inequality holds for some constant W* independent of u and y:

(44) $$\|u\|_{2,\alpha} \leq W^{*1/2} \|y\|^{-1}_{2,\beta}.$$

Proof; **Substituting the boundary conditions (40) and (41) in the identity**

$$[(u'y-uy')]_0^1 = \int_0^1 (u''y-uy'')dx$$

gives

$$1 = \int_0^1 (u''y-uy'')dx$$

and therefore

$$1 \leq \left|\int_0^1 u''y\,dx\right| + \left|\int_0^1 uy''dx\right|.$$

Here applying Holder's inequality to the first term and weighted Schwarz's inequality to the second gives

$$1 \leq \|u''\|_\alpha \|y\|_\beta + \left(\int_0^1 qu^2 dx\right)^{1/2} \left(\int_0^1 \frac{y''^2}{q} dx\right)^{1/2}$$

$$= (K^{1/2}\|u''\|_\alpha)(K^{-1/2}\|y\|_\beta) + \left(\int_0^1 qu^2 dx\right)^{1/2} \left(\int_0^1 \frac{y''^2}{q} dx\right)^{1/2}.$$

Applying Cauchy inequality to this gives

$$1 \leq \|u\|_{2,\alpha}\|y\|_{2,\beta}.$$

The factors on the right side cannot vanish and therefore division gives

$$\|u\|_{2,\alpha} \leq W^{*1/2} \|y\|^{-1}_{2,\beta}$$

for some W^*. This is the same as (44). W^* is independent of u and y because in (44), $\|u\|_{2,\alpha}$ does not depend on y and $\|y\|_{2,\beta}$ does not depend on u.

This completes the proof of Theorem 4. The relation (44) furnishes upper and lower bounds for the quantity W^* and so we term (44) a duality inequality. It may be that the duality inequality gives a non-unique W^*. However it would seem that under the constraint

$$\int_0^1 p^\rho dx \leq K^\rho, \ \frac{1}{\rho} + \frac{2}{\alpha} = 1 \quad \rho \geq 1$$

the relation (44) determines a unique W^* equal to the maximum value of Problem 1a where Problem 1a is stated as Problem 1 with the linear constraint $\int_0^1 p dx \leq K$ replaced by the more general L_ρ type constraint $\int_0^1 p^\rho dx \leq K^\rho$, $\rho \geq 1$. We do not pursue in this work either this question or the solution of Problem 1a.

However we now show that for $\alpha = \infty$ and $\rho = 1$ and the usual meaning for $q(x)$ and K -- as in Problem 3, for instance -- the relation (44) determines a unique W^* equal to the minimax value W of Problem 3. In such a case when W^* is unique we say there is no gap in the duality inequality.

Theorem 5a. Let W be the maximum strength -- the minimax value of Problem 3 -- and let u be a continuous function in $0 \leq x \leq 1$ with continuous first derivative and piecewise continuous second derivative in that interval. Let u satisfy the boundary conditions (40). Then

(45) $$W \leq \|u\|^2_{2,\infty}$$

or what is the same

(46) $$W \leq K \sup_{0\leq x\leq 1} u''^2 + \int_0^1 qu^2 dx.$$

<u>Moreover these become equalities for</u> $u = u_0$ <u>the deflection of the CC-beam.</u>

<u>Proof</u>: Let p_0 and u_0 be the flexural rigidity function and the deflection function of the CC-beam. Then

$$W = E(p_0,u_0) = \min_u E(p_0,u) \le E(p_0,u) =$$
$$\int_0^1 p_0 u''^2 dx + \int_0^1 qu^2 dx \le \left(\int_0^1 p_0 dx\right)\cdot \sup_{0\le x\le 1} u''^2 + \int_0^1 qu^2 dx =$$
$$K \sup_{0\le x\le 1} u''^2 + \int_0^1 qu^2 dx.$$

This proves (45) and (46).

Now if $u = u_0$, since $u_0'' =$ constant, the right side of (45) or (46) becomes

$$\int_0^1 p_0 u_0''^2 dx + \int_0^1 qu_0^2 dx = E(p_0,u_0) = W.$$

The proof is complete.

It follows from Theorem 5a that we can take the constant W^* of the duality inequality equal to W. This proves the inequalities (47) and (48) of Theorem 5b below. Theorem 5b is the dual of Theorem 5a and shows that there is no duality gap.

<u>Theorem 5b</u>. <u>Let</u> W <u>be the maximum strength -- the minimax value of Problem 3. Then</u>

$$W^{-1} \le \|y\|_{2,1}^2 \tag{47}$$

<u>or what is the same</u>

$$W^{-1} \le K^{-1}\left(\int_0^1 |y|\,dx\right)^2 + \int_0^1 \frac{y''^2}{q}\,dx, \tag{48}$$

<u>where y is a continuous function in $0 \le x \le 1$ with continuous first derivative and piecewise continuous second derivative there and satisfies the boundary conditions</u> (41). <u>Moreover these become equalities for $y = y_0$ where y_0 is defined by</u>

(49)
$$y_0 = \frac{1}{W}(p_0 u_0'') .$$

<u>Here p_0 and u_0 are the flexural rigidity function and deflection function of the CC-beam.</u>

<u>Proof</u>: Clearly y_0 is an admissible y-function for $(p_0 u_0'') = 0$ at $x = 1$ and $(p_0 u_0'')' = W$ at $x = 1$.

Since $u_0''^2 = c^2 = 4$ and $\int_0^1 p_0 dx = K$, (49) gives

(50)
$$\left(\int_0^1 |y_0| dx\right)^2 = \frac{K^2}{W^2} c^2.$$

On the other hand $p_0 u_0''^2 = c^2 p_0$ so

(51)
$$K \sup_{0 \le x \le 1} u_0''^2 = \int_0^1 p_0 u_0''^2 dx = c^2 K.$$

Further, since p_0, u_0 satisfy (1), (49) gives

(52)
$$y_0'' = - \frac{q u_0}{W} .$$

This gives

(53)
$$\int_0^1 \frac{y_0''^2}{q} dx = \frac{1}{W^2} \int_0^1 q u_0^2 dx.$$

Then (50), (51) and (53) show that

$$K \sup_{0 \le x \le 1} u_0''^2 + \int_0^1 q u_0^2 dx = W^2 \left[K^{-1}\left(\int_0^1 |y_0| dx\right)^2 + \int_0^1 \frac{y_0''^2}{q} dx\right].$$

Left side of this being W by Theorem (5a), we have

$$W^{-1} = K^{-1}\left(\int_0^1 |y_O|\,dx\right)^2 + \int_0^1 \wedge \frac{2}{\wedge}\,dx$$

and the proof is complete.

Theorem 5a and Theorem 5b are the formulations of the dual extremum principles promised in the title of this study.

6. A Design Problem for a Different Set of Boundary Conditions.

We now consider a beam of rectangular cross section with fixed height and variable width which is hinged at both ends, $x = 0$ and $x = 1$ (Figure 3). The beam is subject to external moments M at the ends and is supported throughout its length by an elastic sheet of elastic coefficient 1. The slope of the beam at the ends $x = 0$ and $x = 1$ is required to have values -1 and 1 respectively and it is desired to taper the beam subject to the weight constraint $\int_0^1 p\,dx \leq K$ so as to support maximum M.

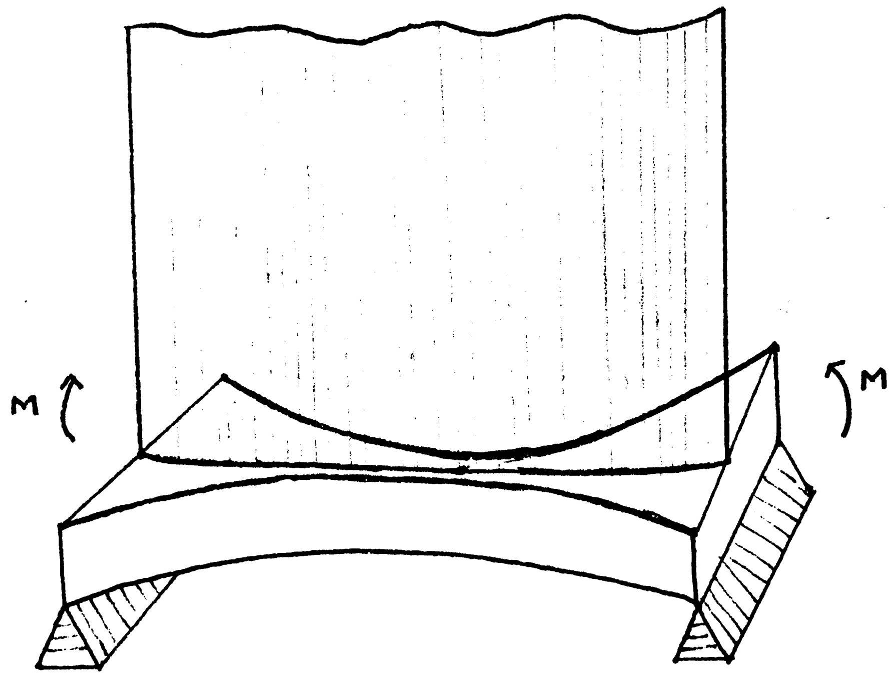

Figure 3. Hinged Beam on An Elastic Foundation.

Here $p(x)$ is the flexural rigidity function, symmetric about the point $x = \frac{1}{2}$ and $K > 0$ is a given constant. We call this Problem 4. In this problem, the quantity M can be called the strength of the beam and thus we have set out to find a beam of maximum strength.

The problem can be solved under a suitable bound for K. We merely give the formulae for the optimum taper and omit the details.since the treatment is similar to that of Problem 1, with obvious modifications.

Theorem 6. The flexural rigidity function p for the beam of maximum strength, or what is the same, for the beam which supports maximum external moments M at: the ends x = 0, x = 1 in the class of all beams which are simply supported at the ends x = 0, x = 1 where they have slopes -1 and 1 respectively, which have continuous flexural rigidity functions p(x) in 0 £ x £ 1 symmetric about x = 1/2 an(^ satisfying

(54) $\int_0 pdx$ £ K, K > 0

in which K satisfies

(55) K > - 2 -

640

and which are supported throughout the length by an elastic sheet of elastic coefficient 1 jLs, given by

(56) $p_0(x)$ = K + i § o - ^ + 1 $\frac{x^3}{2}$ - f $\frac{4}{4}$ iS ° £ x < l l.

1*he deflection of this optimum beam is given by

(57) $u_Q = x^2 - x$ in 0 £ x £ 1.

The maximum strength M is given by

(58) $$M = (p_0 u_0'')_{x=0} = (p_0 u_0'')_{x=1} = (2K + \frac{1}{60}).$$

The duality inequality for the maximum strength M is

(59) $$\frac{1}{2}[K \operatorname{Sup}(u'')^2 + \int_0^1 u^2 dx] \geq M \geq 2[K^{-1}(\int_0^1 |y|\, dx)^2 + \int_0^1 y''^2 dx]^{-1}.$$

Here u and y are arbitrary continuous functions with continuous first derivatives and piecewise continuous second derivatives satisfying the boundary conditions

(60) $$u(0) = u(1) = 0, \; u'(0) = -1, \; u'(1) = 1$$

(61) $$y(0) = 1 = y(1).$$

There is equality throughout (59) for $u = u_0$ given by (57) and $y = \frac{p_0 u_0''}{M}$ where p_0 and M are given by (56) and (58).

7. Uniqueness Proof.

Given here is a detailed proof of the statement in Section 3.

Theorem 2. The CC-beam is the only optimal beam.

Proof: Let p_0 and u_0 be the flexural rigidity function and deflection function for the CC-beam. Assume that P is another admissible flexural rigidity function, that U is the deflection function which minimizes $E(P,u)$ and that

$$E(P,U) = E(p_0,u_0). \tag{62}$$

Then

$$E(P,u_0) \geq E(P,U). \tag{63}$$

But by Lemma 4

$$E(p_0,u_0) \geq E(P,u_0). \tag{64}$$

Now, (62), (63), and (64) imply that (63) is actually an equality. Thus

$$E(P,u_0) = E(P,U) = E(p_0,u_0). \tag{65}$$

Considering $E(P,u)$ as a quadratic functional in u, by parallelogram law for quadratic functionals we have

$$0 \leq E(P,U-u_0) = 2E(P,U) + 2E(P,u_0) - 4E(P,Z) \tag{66}$$

where $Z = (U+u_0)/2$. Because Z satisfies the boundary conditions (2)-(4)

$$E(P,Z) \geq E(P,U). \tag{67}$$

Now, by (65), (66) and (67) it follows that

(68) $$E(P, U - u_0) = 0$$

and hence

(69) $$U \equiv u_0 .$$

That $P \equiv p_0$ in $0 \leq x \leq 1$ now follows is carried out in Lemma 5 and Lemma 6 below. There, the "Second Lemma" of the Calculus of Variations [8] is employed to show that the continuous function P has in fact two continuous derivatives and then the relation (65) and the property $E(P,u_0) = \min_u E(P,u)$ are used to conclude that $P \equiv p_0$.

Lemma 5. Let P be the flexural rigidity function appearing in the proof of Theorem 2. Then

(70) $$P u_0'' = c_0 + c_1 x - \int_0^x \left[\int_0^x q u_0 \, dx\right] dx \quad \text{in} \quad 0 \leq x \leq 1$$

for some constants c_0 and c_1.

Proof: The proof is essentially an application of the well-known "Second Lemma" of the Calculus of Variations [8].

Let us define the following continuous function which appears frequently in this proof:

(71) $$M(x) = P u_0'' + \int_0^x \left[\int_0^x q u_0 \, dx\right] dx - c_0 - c_1 x \quad \text{in} \quad 0 \leq x \leq 1.$$

The system of equations

(72) $$\int_0^1 M(x)\,dx = 0$$
$$\int_0^1 x M(x)\,dx = 0$$

in c_0 and c_1 has a unique solution for constants c_0 and c_1 since its determinant is non-zero.

$$\begin{vmatrix} \int_0^1 dx & \int_0^1 x dx \\ \int_0^1 x dx & \int_0^1 x^2 dx \end{vmatrix} = \frac{1}{12} .$$

Thus the constants c_0 and c_1 in (71) be defined by (72). Then $\tilde{v}(x)$ defined in $0 \le x \le 1$ by

(73) $$\tilde{v}(x) = \int_0^x (x-t) M(t) dt$$

has continuous first and second derivatives in $0 \le x \le 1$. Indeed,

(74) $$\tilde{v}'(x) = \int_0^x M(t) dt$$

and

(75) $$\tilde{v}''(x) = M(x).$$

Moreover by (71), (72) and (74), $\tilde{v}$ satisfies the boundary conditions

(76) $$\tilde{v}(0) = \tilde{v}'(0) = 0 = \tilde{v}(1) = \tilde{v}'(1).$$

Now, because

$$E(P,u_0) = \min_u E(P,u)$$

for all u continuous in $0 \le x \le 1$ and having continuous first derivatives and piecewise continuous second derivatives there and satisfying the boundary conditions (2)-(4), the first variation of the functional $E(P,u_0)$ should necessarily vanish:

(77) $$\int_0^1 [Pu_0''v'' + qu_0 v] dx = 0$$

for all v continuous in $0 \le x \le 1$ having continuous first derivatives and piecewise continuous second derivatives there and satisfying $v(0) = v'(0) = 0 = v(1)$. Substituting $v = \tilde{v}$ in (77) and integrating by parts gives

$$\int_0^1 [M(x) + c_0 + c_1 x] M(x)\,dx = 0. \tag{78}$$

Here (71), (75) and (76) have been used. Using (72) in (78) gives

$$\int_0^1 M^2 dx = 0$$

which proves $M(x) = 0$ in $0 \leq x \leq 1$ and hence (70).

Lemma 6. The flexural rigidity function $P(x)$ appearing in the proof of Theorem 2 coincides with the flexural rigidity function $p_0(x)$ of the CC-beam.

Proof: By virtue of Lemma 3b, it is enough to show that $P(x)$ satisfies the differential equation (22) and the boundary condition (23) and the integral equality (17). This is because it was proved in Lemma 3b that the solution for $P(x)$ is unique and equals $p_0(x)$ in $0 \leq x \leq 1$.

Now the right side of (70) being continuously differentiable twice, it immediately follows that $P(x)$ satisfies the differential equation (22). Since P has two continuous derivatives and u_0 has four continuous derivatives in $0 \leq x \leq 1$ and since

$$E(P,u_0) = \min_u E(P,u)$$

for all u continuous in $0 \leq x \leq 1$ and having continuous first derivatives and piecewise continuous second derivatives there and satisfying the boundary conditions (2)-(4), the natural boundary condition $Pu_0'' = 0$ is satisfied at $x = 1$ and this means $P(x)$ satisfies (23). Lastly, we have

$$E(P,u_0) = E(p_0,u_0). \tag{65}$$

But this reduces to

$$\int_0^1 P\,dx = \int_0^1 p_0\,dx = K$$

proving that $P(x)$ satisfies (17).

The proof of the Lemma is complete and hence the proof of Theorem 2.

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