DUAL EXTREMUM PRINCIPLES

RELATING TO COOLING FINS[*]

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Carnegie-Mellon University

ABSTRACT

Under consideration is a differential equation $(pu^r)^r = qu$ of the Sturm-Liouvilie type where the function $q(x) > 0$ is given. The problem is to find a function $p(x) > 0$ in $0 £ x < b$, a constant b and a solution $u(x)$ of the corresponding **differential** equation such that the energy functional

$$\int_o^b [P(u^!)^2 + qu^2]dx$$

is maximized when $p(x)$ is subject to the constraint $\int_o^b p^P dx < £ K$ and u is subject to the boundary conditions $u = 1$ at $x = 0$ and $p \dfrac{du}{dx} = 0$ at $x = b$. Here $K > 0$ and $p]>_ 1$ are constants*

A key relation $j \dfrac{du}{dx}| = Ap^2$, where A is a positive constant, is found. This criterion leads to explicit solution of the problem. A further consequence of this criterion together with a pair of dual extremum principles is a "duality inequality" giving sharp upper and lower estimates of the maximum value of the energy functional.

This study is a sequel to previous studies on the optimization of cooling fins by R. J. Duffin and D. K. McLain [Jour. Math. and Mech. 8(1959), 47-56 and 17(1968), 769-784.] In those papers $\rho = 1$. However, the latter deals with fins on convex cylinders and thus the Sturm-Liouville equation they study is a partial differential equation.

DUAL EXTREMUM PRINCIPLES RELATING TO COOLING FINS

1. Introduction

In reference [1], R. J. Duffin studied the maximization of the functional

$$\mathcal{H} = \int_0^b q(x) u(x) dx$$

where $q(x) > 0$ is a given continuous function in $x \geq 0$ and $u(x)$ is a differentiable function satisfying the Sturm-Liouville differential equation

$$\frac{d}{dx}\left(p(x) \frac{du}{dx}\right) = q(x) u \quad \text{in} \quad 0 \leq x \leq b$$

and satisfying the boundary conditions

$$u = 1 \quad \text{at} \quad x = 0 \quad \text{and} \quad p \frac{du}{dx} = 0 \quad \text{at} \quad x = b$$

and where $p(x)$ is a continuous function subject to the constraints

$$p(x) > 0 \quad \text{in} \quad 0 \leq x < b$$

and

$$\int_0^b p(x) dx = K > 0, \quad \text{a given constant.}$$

In order to find the unknown constant $b > 0$ and the unknown functions $p(x)$ and $u(x)$ having the above properties, he eliminated the differential equation and the boundary conditions and studied an equivalent maxmax problem. By a variational argument it was shown that -- the optimal function $u(x)$ should

satisfy the condition

(I) $\left|\dfrac{du}{dx}\right| = $ constant in $0 < x < b$.

This criterion led to explicit solution for the unknowns b, $p(x)$, and $u(x)$ maximizing the functional &.

In this study we have considered the problem of maximizing the functional &(u) subjecting $p(x)$ to a more general type of constraint, namely an inequality constraint of the L_p norm type

$$\int_o^b p^P dx \pounds K^P, \quad p \wedge 1, \quad K > 0.$$

With a view to find upper and lower estimates for the maximum value H, a "duality inequality" is obtained, namely:

(II) $\|u\|_{2ja} k H^{1/2} * \|Y\|_2 "^.$

Here $\|\cdot\|$ and $\|\cdot\|_Q$ are certain dual norms, $u(x)$ and $y(x)$ are arbitrary functions except that they satisfy certain smoothness and normalized boundary conditions.

The problem of maximizing the functional Ji is recast into a minimax or saddle point problem rather than a maxmax problem. This together with (II) facilitates the formulation of a pair of dual extremum principles for the maximum value H, namely

(III) $\qquad \min_{u} \|u\|_{2,\alpha} = H^{1/2} = \max_{y} \|y\|_{2,\beta}^{-1}.$

This is proved taking q(x) as a constant and in that case it is

shown that the optimal functions p(x) and u(x) satisfy the

condition

(IV) $\qquad |\frac{du}{dx}| = \lambda(p(x))^{\frac{\rho-1}{2}} \qquad$ in $0 \leq x \leq b$

where λ is a positive constant. This criterion leads to

explicit solution for the unknowns b, p(x), and u(x) which

maximize $H(u)$.

The paper [1] by Duffin, mentioned above, concerns the optimum

design of a cooling fin of limited weight K so as to dissipate

the most heat H. The thickness of the fin, p(x), of the

optimum fin is determined by the criterion (I). Previously,

E. Schmidt [5] had proposed the criterion (I) but he based his

argument on a fallacious physical assumption.

Previous authors, among which R. J. Duffin and D. K. McLain

[2], J. E. Wilkins, Jr. [6,7], C. Y. Liu [8,9], F. C. Appl and

H. M. Hung [10] and R. Focke [11] are some, have given several

developments of the cooling fin problem. But they have not

considered the generalization treated here. Moreover, the duality

inequality (II) can be suitably employed to obtain upper and lower

bounds for the maximum conductance H of the cooling fin.
Several ramifications of duality inequality are given in
references [3,4].

This study is a sequel to a study on heat transfer in
networks [13]. There a network model for the cooling fin was
treated. These studies complement each other; the present
treatment is more specific and the network treatment is more
general. A later study [14] will treat, in an analogous way,
a problem in the design of an elastically supported beam for
maximum strength.

2. Statement of the Maximization Problem and an Equivalent Formulation

In this section we state a maximization problem and analyze it heuristically to obtain an equivalent minimax problem. Rigorous developments are given in later sections.

PROBLEM JL. To find two functions $u(x)$ and $p(x)$ and a positive number b which have the following properties; $u(x)$ is differentiable in the interval $0 \leq x \leq b$ and $p(x)$ is continuous in the same interval and the two satisfy the Sturm-Liouville differential equation for u:

(1)
$$\frac{d}{dx}[p(x) \frac{d}{dx}] = S(x)u(x) \quad \text{in} \quad 0 \leq x \leq b$$

where $q(x)$ is positive and continuous in $x \geq 0$; and u is to satisfy the boundary conditions

(2)
$$u = 1 \quad \text{at} \quad x = 0$$

and

(3)
$$p(x) \frac{\partial}{\partial x} = 0 \quad \text{at} \quad x = b.$$

Moreover $p(x)$ has to satisfy the conditions

(4)
$$p(x) > 0 \quad \text{in} \quad 0 \leq x < b$$

and

$$(5) \qquad \int_0^b p^\rho dx \leq K^\rho$$

where $\rho \geq 1$ and $K > 0$ <u>are given constants</u>. <u>Under these conditions on</u> $p(x)$ <u>and</u> $u(x)$ <u>we seek to maximize the functional</u>

$$(6) \qquad \mathcal{H}(u) = \int_0^b q(x) u(x) dx.$$

Introducing the functional

$$(7) \qquad E(p,u) = \int_0^b [pu'^2 + qu^2]dx, \qquad (\ )' = d/dx$$

enables us to recast the problem into a minimax problem. This change is motivated by the following heuristic analysis.

Let $p(x)$ and $u(x)$ be functions satisfying conditions (1)-(4) for some positive number b and be sufficiently smooth in $0 \leq x \leq b$ so that integration by parts below is valid. Let v be an arbitrary smooth function in $0 \leq x \leq b$. Then

$$(8) \qquad E(p,u+v) = E(p,u) + E(p,v) + 2\int_0^b [pu'v' + quv]dx.$$

Integration by parts gives

$$(9) \qquad \int_0^b [pu'v' + quv]dx = \int_0^b v[qu - (pu')']dx + [(pu')v]_0^b.$$

The first integral on the right side vanishes by (1). If we impose $v = 0$ at $x = 0$ then the boundary term at $x = 0$

vanishes and the other boundary term vanishes because of (3).

Moreover u + v satisfies the boundary condition (2). Thus

(8) becomes

(10) $E(p,u+v) = E(p,u) + E(p,v) \hat{} E(p,u).$

It follows that if $E(p,\tilde{u})$ is minimized for the class of functions

$\tilde{u}$ satisfying (2) but not necessarily (1) then the minimum is

achieved by a function u satisfying (1) and (3). This is a

standard result of the Calculus of Variations where (3) is termed

a natural boundary condition because it is necessarily satisfied

by the minimizing function.

    Now, choosing v = u in (9) gives

(ID $E(p,u) = - [pu\ast]_Q.$

On the other hand, choosing v = 1 in (9) gives

(12) $J qudx = - [pu\ast]_o.$
                        o

    LEMMA JL. If the function u satisfies the Euler differential

equation (1) corresponding to the saddle functional E(p,u) and

if p and u satisfy the boundary conditions (2) and (3) and

the condition (4), then-

(13) $H(u) = E(p,u)$

where H and E are given by (6) and (7).

Proof: The lemma is an immediate consequence of (11) and (12).

In view of relations (10) and (13) the following problem is presumably equivalent to Problem 1.

PROBLEM 2. Find

$$(14) \qquad\qquad H = \max_{p} \min_{u} E(p,u)$$

subject to the following conditions: $p \geq 0$ and p has a finite support which is an interval with 0 for left-hand end point, p is continuous in the closure of its support; the region of integration is the support of p, and the integral of p^ρ over the support is not to exceed a given positive constant K^ρ; the function u is continuous, has a piecewise continuous first derivative and takes value 1 at $x = 0$. Here $q(x)$ occuring in the expression for E is a positive continuous function in $x \geq 0$ and $\rho \geq 1$ is a given constant.

We continue with the heuristic analysis and investigate this problem.

LEMMA 2. If $p(x)$ is non-negative and continuous in $0 \leq x \leq b$ and satisfies (5) and $u(x)$ is continuous with piecewise continuous

derivative in $0 \leq x \leq b$ then, for E given by (7)

$$(15) \qquad E(p,u) \leq K(\int_0^b |u'|^\alpha dx)^{\frac{2}{\alpha}} + \int_0^b qu^2 dx$$

where the positive constant α is given by

$$(16) \qquad \rho^{-1} + 2\alpha^{-1} = 1.$$

There is equality in (15) if

$$(17) \qquad \int_0^b p^\rho dx = K^\rho$$

and

$$(18) \qquad u' = -\lambda p^{\frac{\rho}{\alpha}}$$

where λ is a positive constant.

Proof: In

$$(7) \qquad E(p,u) = \int_0^b [pu'^2 + qu^2] dx$$

applying the Hölder's inequality gives

$$(19) \qquad \int_0^b pu'^2 dx \leq (\int_0^b p^\rho dx)^{\frac{1}{\rho}} (\int_0^b |u'|^\alpha dx)^{\frac{2}{\alpha}} \leq K(\int_0^b |u'|^\alpha dx)^{\frac{2}{\alpha}},$$

where α is given by (16). Here (5) has been used. Substituting (19) in (7) gives (15).

If p and u satisfy (17) and (18), there is equality

throughout (19) and hence (15) also becomes an equality. This completes the proof of the lemma. We regard the relation (18) as a key result because it enables us to obtain explicit expressions for the solution.

Problem 1a where Problem 1a is stated as Problem 1 with $q(x) \equiv q > 0$, a constant, and this particular case is our main interest from now on. Similarly, Problem 2a is stated as Problem 2 with $q(x) \equiv q$.

LEMMA 3. A pair $b = b_0$ and $\lambda = \lambda_0$ of positive constants and a solution $u = u_0$, $p = p_0$ of the system of equations

(1) $$\frac{d}{dx}[p \frac{du}{dx}] = qu(x) \quad \underline{in} \quad 0 \le x \le b$$

(18) $$u' = -\lambda p^{\alpha} \quad \underline{in} \quad 0 \le x \le b$$

under the boundary conditions

(2) $$u = 1 \quad \underline{at} \quad x = 0$$

(3) $$p \frac{du}{dx} = 0 \quad \underline{at} \quad x = b$$

and satisfying

(4) $$p > 0 \quad \underline{in} \quad 0 \le x < b$$

(17)
$$\int_0^b p^P dx = K^\rho$$

are given by

(20)
$$u_o = (1 - \frac{x}{b_o})^\rho$$

(21)
$$p_o = \delta_o (1 - \frac{x}{b_o})^2$$

where

(22)
$$\delta_o = \frac{qb_o^2}{\rho(\rho+1)}$$

(23)
$$b_o = \left[\frac{K\rho(\rho+1)(2p+1)^P}{q} \right]^{\frac{1}{p/(1+2p)}}$$

and

(24)
$$A_o = \frac{\rho}{b_o 6_o^{P/<x}} \quad .$$

    Proof; Multiplying (1) by $2u'$ and integrating gives

(25)
$$-\int_x^b p' u^1{}^2 dx + pu'^2 = qu^2 .$$

Here the boundary condition (3) has been used and an additional condition $u = 0$ at $x = b$ has been imposed. Substituting (18) in (25) and integrating gives

$$(26) \qquad \frac{\lambda^2}{\rho} (\rho+1) p^{\rho} = qu^2.$$

Here $p(x)$ has been required to satisfy $p = 0$ at $x = b$. Eliminating $p(x)$ between (18) and (26) and integrating the result and using the initial condition (2) gives $u(x)$ and then substituting this in (26) gives $p(x)$, namely

$$(27) \qquad u = (1 - \frac{x}{b})^{\rho}$$

and

$$(28) \qquad p = \delta (1 - \frac{x}{b})^2$$

where δ, λ and b satisfy

$$(29) \qquad \frac{1}{b} = \frac{\lambda^{\frac{1}{\rho}}}{\rho^{(\rho+1)/2\rho}} (\frac{q}{1+\rho})^{\frac{\rho-1}{2\rho}}$$

$$(30) \qquad \delta^{\rho} = \frac{q\rho}{\lambda^2 (\rho+1)} \qquad .$$

Requiring p to satisfy (17) gives

$$(31) \qquad \frac{q\rho b}{\lambda^2 (\rho+1)(2\rho+1)} = K^{\rho}.$$

Eliminating λ between (29) and (31) gives $b = b_o$ where b_o is as in (23). Eliminating λ between (29) and (30) and substituting $b = b_o$ gives $\delta = \delta_o$ where δ_o is as in (22). Now $\lambda = \lambda_o$ where λ_o is as in (24) follows from (22), (23) and (31).

It is contained in Lemma 3 that p_o and u_o satisfy the differential equation (1) and conditions (2)-(5) for $b = b_o$. This was enabled by the relations (17) and (18) of Lemma 2 which render (15) an equality for $p = p_o$, $u = u_o$ and $b = b_o$. Further it will be seen in Section 3 how the inequality (15) can be used to solve the remaining part of Problem 1a. In fact, we show rigorously that $p = p_o$, $u = u_o$, $b = b_o$ is the unique solution set solving Problem 1a.

3. Comparison Relations and the Main Proof

     In this section we restate Problem 2a relaxing certain

conditions and show that $p = p_o$, $u = u_o$ of Lemma 3 provide

the unique solution of the new problem, called Problem 3,   It

is then shown that the solution of Problem 3 is also the unique

solution of Problem 2a and finally that $p = p_o$, $u = u_o$, $b = b_o$

is the unique solution set having the desired property of

satisfying (1)-(5) and maximizing & given by (6).

     PROBLEM 3* Find

(32)   H = sup inf E(p,u)
                            P   **u**

subject to the following conditions; $p \geq 0$ and has ja finite

support which is an interval with 0 for the left-hand end

point, p is continuous in the closure of its support; the

region of integration is the support of p, and the integral

of p^ρ over the support is not to exceed a given positive

constant K^ρ; the function u JLS continuous, has piecewise

continuous first derivative and takes value 1 at. x = 0.

The quantity q > 0 occuring in the expression for E and

$P \geq 1$ are given constants.

LEMMA 4. Let p be an arbitrary function satisfying the conditions of the p-functions of Problem 3. Let p_o and u_o be the functions given by (20)-(23). Then

$$(33) \qquad E(p_o, u_o) \geq E(p, u_o).$$

Proof: By Lemma 3, for $b = b_o$, the functions p_o and u_o satisfy (17) and (18) of Lemma 2 and (15) becomes an equality for $p = p_o$, $u = u_o$ and $b = b_o$. Hence

$$(34) \qquad E(p_o, u_o) = K\left(\int_0^{b_o} |u_o'|^\alpha dx\right)^{\frac{2}{\alpha}} + \int_0^{b_o} qu_o^2 dx.$$

Now let b be the right-hand end point of the support of p. It is convenient to consider the two cases $b_o < b$ and $b_o \geq b$ separately:

Case (i) $b_o < b$: Since $u_o(b_o) = 0$ by defining u_o as zero for $x > b_o$ we have a continuous extension of u_o and thus the extended function has the admissibility properties of the u-functions of Problem 3. So,

$$(35) \quad E(p, u_o) = \int_0^b [pu_o'^2 + qu_o^2] dx \leq K\left(\int_0^b |u_o'|^\alpha dx\right)^{\frac{2}{\alpha}} + \int_0^b qu_o^2 dx.$$

Here we have applied the Hölder's inequality

$$\int_0^b pu_o'^2 dx \leq \left(\int_0^b p^\rho dx\right)^{\frac{1}{\rho}}\left(\int_0^b |u_o'|^\alpha dx\right)^{\frac{2}{\alpha}}$$

and the property of p that $I_0^b p^\wedge dx \leq^\wedge K^\rho$. Now since $u_o = 0$

in $x > b_o$ the right side of the inequality in (35) becomes

$E(p_o, u_o)$ on using (34), thus giving (33).

Case (ii) $b_o \geq b$: Since the restriction of u_o to $0 \leq X \leq b$

satisfies the admissibility properties of the u-functions of

Problem 3, we have also in this case

$$(36) \quad E(p, u_o) = J_0^b [pu^{\wedge 2} + qu_o^2] dx \leq K(J_0^b |u^\wedge|^a dx)^{\frac{2}{a}} + J_0^b qu_o^2 dx$$

on applying Hölder's inequality as in Case (i) and on using

$(J_0^b p^{\bar{p}} dx) \leq K^\rho$. Now, in the right side of the inequality of (36)

$$\int_0^b |u_o'|^\alpha \, dx \leq \int_0^{b_o} |u_o'|^a \, dx$$

and

$$\int_0^b u_o^2 \, dx \leq \int_0^{b_o} u_o^2 \, dx$$

since $b_o \geq b$. Using these and (34) in (36) gives (33). This

completes the proof of the lemma.

THEOREM JL. The pair of functions $[p_o, u_o]$ given by (20)-

(23) is a solution of Problem 3.

Proof; Let p be an arbitrary function satisfying the

admissibility conditions of the p-functions of Problem 3 and u

be an arbitrary function satisfying the admissibility conditions

of the u-functions of Problem 3. Then, since both the restriction

of u_o to any $0 \leq x \leq b$ with $b \leq b_o$ and extension of u_o

as zero beyond $x = b_o$ are admissible,

$$\inf_{u} E(p,u) \leq E(p,u_o) \leq E(p_o,u_o) \qquad \text{by (33)}$$

and therefore

$$(37) \qquad \sup_{p} \inf_{u} E(p,u) \leq E(p_o,u_o).$$

Also,

$$(38) \qquad \sup_{p} \inf_{u} E(p,u) \geq \inf_{u} E(p_o,u) = E(p_o,u_o).$$

The last equality in (38) is true because u_o satisfies the Euler

differential equation, (1) with $p = p_o$, of the positive definite

quadratic functional $E(p_o,u)$ and the boundary conditions (2)

and (3). The relations (37) and (38) now imply the desired

result that

$$(39) \qquad E(p_o,u_o) = \sup_{p} \inf_{u} E(p,u).$$

The relation (39) is the same as (32) of Problem 3 and

shows that H of (32) equals $E(p_o,u_o)$.

THEOREM 2. The pair of functions $[p_o,u_o]$ given by (20)-

(23) is the unique solution of Problem 3.

Proof: A proof of uniqueness can be given in essentially the same manner as the uniqueness proof given by R. J. Duffin and D. K. McLain in Theorem 2 of reference [2]. We delay giving this proof until Section 5.

THEOREM 3. Problem 2a and Problem 3 are equivalent and the pair of functions $[p_o, u_o]$ given by (20)-(23) is the only solution of Problem 2a.

Proof: By Theorem 1, the sup inf of (32) of Problem 3 is actually assumed for functions $p = p_o$ and $u = u_o$ and hence the solution $[p_o, u_o]$ of Problem 3 is also a solution of Problem 2a. Here we have also used the fact that the functionals being optimized in the two problems are the same. This shows the equivalence of two problems and that the minimax values of the two problems are equal. Now the solution of Problem 2a is unique. For, otherwise it would mean that Problem 3 has another solution besides $[p_o, u_o]$. But this contradicts Theorem 2 and the proof is complete.

THEOREM 4. Problem 1a i^ equivalent to Problem 2a and Problem 3 and the pair of functions $[p_o, u_o]$ given by (20)-(23) is the unique solution of Problem 1. Summarizing: Problem 1a has the unique solution $p = p_o$, $u = u_o$, $b = b_o$ given by

$$(20) \qquad u_o = (1 - \frac{x}{b_o})^{\rho}$$

$$(21) \qquad p_o = \delta_o (1 - \frac{x}{b_o})^2$$

where

$$(22) \qquad \delta_o = \frac{q b_o^2}{\rho(\rho+1)}$$

$$(23) \qquad b_o = \left[\frac{K\rho(\rho+1)(2\rho+1)^{1/\rho}}{q} \right]^{\rho/(1+2\rho)}$$

and the maximum value H of the functional

$$(6) \qquad \mathcal{H}(u) = q\int_o^b u(x)\,dx$$

is

$$(40) \qquad H = \mathcal{H}(u_o) = E(p_o, u_o) = \frac{q b_o}{(\rho+1)}$$

where b_o is given by (23) above.

Proof: Since p_o is continuous in $0 \le x \le b_o$ and u_o, u_o' and $(p_o u_o')'$ are also continuous in $0 \le x \le b_o$, these conditions are sufficient to justify the integration by parts in (9) for $b = b_o$, $p = p_o$ and $u = u_o$. Thus the relation (13) with $b = b_o$, $p = p_o$ and $u = u_o$ shows that the maximum value of Problem 1a cannot exceed the minimax value H of Problem 2a. Here the result of Lemma 3, that the pair of functions

$[p_o, u_o]$ satisfies the differential equation (1) and the side conditions (2)-(5), with $b = b_o$, of Problem 1a, has also been used. By (13) with $b = b_o$, $p = p_o$ and $u = u_o$ we have

$$(41) \qquad\qquad H = \mathcal{H}(u_o) = E(p_o, u_o).$$

These considerations show that $[p_o, u_o]$ is a solution of Problem 1a and the maximum value of Problem 1a equals the minimax value of Problem 2a. The solution of Problem 1a is unique. For, otherwise Problem 2a would have another solution apart from $[p_o, u_o]$. But this contradicts Theorem 3. To complete the proof of the theorem, it is now enough to see that by (11) with $p = p_o$, $u = u_o$, $b = b_o$ relation (41) becomes

$$H = \mathcal{H}(u_o) = E(p_o, u_o) = -[p_o u_o']_{x=0}.$$

Substituting (20)-(23) in this gives (40).

4. A Dual Variational Approach to Problem 1 and the Duality

    Inecruality

    In this section we give a different approach to the solution
of Problem 1 and derive a pair of sharp upper and lower estimates
for the maximum value H of Problem 1a.   The proof of the
following theorem does not depend on the results of the previous
sections.

    THEOREM 5. Let u(x) and y(x) be continuous functions
in $0 \pounds x \leqslant\pounds b$ with piecewise continuous derivatives there.
Let u and y satisfy the boundary conditions

(42) $u(0) = 1$

and

(43) $y(_0) \doteq 1, \quad y(b) = 0.$

Let

(44) $\|u'\|_{\alpha} = (J_0^b |u\ll|^a dx)^{\frac{1}{a}}$ and $\|y\|_{\beta} = (J_0^b |y|^{\beta} dx)^{\frac{1}{\beta}}.$

Then define dual norms as

$$\|u\|_{2,\alpha} = [K\|u\ll\|^{\wedge} + J_0^b qu^2 dx]^{\frac{1}{2}}$$

and

$$\|y\|_{2,\beta} = [K^{-1}\|y\|_{\beta}^2 + \int_0^b \frac{y'^2}{q} dx]^{\frac{1}{2}}$$

where a and £ are positive constants satisfying

(45)
$$\alpha^{-1} + \beta^{-1} = 1$$

and K is also a positive constant and $q(x) > 0$ is continuous

in $0 \leq x \leq b$. Then the following inequality holds

(46)
$$\|u\|_{2,\alpha} \geq H^{*\frac{1}{2}} \geq \|y\|_{2,\beta}^{-1}$$

where H^* is a constant independent of u and y.

   Proof: Substituting the boundary conditions (42) and (43)

in the identity

$$[(uy)]_0^b = \int_0^b (u'y + uy')\,dx$$

gives

$$-1 = \int_0^b (u'y + uy')\,dx$$

and therefore

$$1 \leq |\int_0^b u'y\,dx| + |\int_0^b uy'\,dx|.$$

Here applying Hölder's inequality to the first term of the

right side and a weighted Schwarz's inequality to the second

term gives

$$1 \leq \|u'\|_\alpha \|y\|_\beta + (\int_0^b qu^2\,dx)^{\frac{1}{2}}(\int_0^b \frac{y'^2}{q}\,dx)^{\frac{1}{2}}.$$

Applying Cauchy's inequality to this gives

$$1 \leq \|u\|_{2,\alpha} \|y\|_{2,\beta}.$$

The factors on the right side cannot vanish and therefore division gives

$$\|u\|_{2,\alpha} \geq \|y\|_{2,\beta}^{-1}.$$

Since the left side is independent of y and the right side is independent of u relation (46) follows for some constant H^* independent of u and y. This completes the proof of the theorem.

The relation (46) furnishes upper and lower bounds for H^* and so we term (46) a "<u>duality inequality</u>". The quantity H^* is conceivably non-unique. However, it would seem that under the constraint

$$\int p^{\rho} dx \leq K^{\rho}, \qquad \frac{1}{\rho} + \frac{2}{\alpha} = 1, \qquad \rho \geq 1,$$

relation (46) determines a unique H^* equal to the maximum value of Problem 1 (with q(x) not a constant). We do not pursue, in this study, either this question or the solution of Problem 1.

However, we show in this section that for α defined by $\frac{1}{\rho} + \frac{2}{\alpha} = 1$ and q(x) = constant = q and the usual meaning of ρ, K and q -- as in Problem 1a -- and for b = ∞ (in fact,

any $b \geq b_o$, b_o as in Problem 1a, suffices) (46) determines a

unique H^* and that $H^* = H$ the maximum value of Problem 1a.

In such a case when H^* is unique we say that there is no gap

in the duality inequality.

THEOREM 5a. Let $u(x)$ and $y(x)$ be continuous with piecewise

continuous first derivatives in $x \geq 0$ so that the integrals in

(44a) below are convergent. Let u and y satisfy the boundary

conditions

(42a) $u(0) = 1$, u bounded as $x \longrightarrow \infty$

and

(43a) $y(0) = 1$, $y(x) \longrightarrow 0$ as $x \longrightarrow \infty$.

Let

(44a) $\|u'\|_\alpha = (\int_o^\infty |u'|^\alpha dx)^{\frac{1}{\alpha}}$, $\|y\|_\beta = (\int_o^\infty |y|^\beta dx)^{\frac{1}{\beta}}$.

Then define dual norms as

$$\|u\|_{2,\alpha} = [K\|u'\|_\alpha^2 + q\int_o^\infty u^2 dx]^{\frac{1}{2}}$$

$$\|y\|_{2,\beta} = [K^{-1}\|y\|_\beta^2 + \frac{1}{q}\int_o^\infty y'^2 dx]^{\frac{1}{2}}$$

where α and β are positive constants satisfying

(45a) $\alpha^{-1} + \beta^{-1} = 1$, $\rho^{-1} + 2\alpha^{-1} = 1$

and where $\rho \geq 1$, $K > 0$ and $q > 0$ are the constants appearing in Problem 1a (or Problem 3). Then the following duality inequality holds for a constant H^* independent of u and y:

$$(46a) \qquad \|u\|_{2,\alpha} \geq H^{*\frac{1}{2}} \geq \|y\|_{2,\beta}^{-1}.$$

Proof: This is an immediate consequence of Lemma 5 on letting $b = \infty$.

THEOREM 6. Let u and y satisfy the hypothesis of Theorem 5a and H be the maximum value of Problem 1a. Let $\tilde{u}_0$ and $\tilde{y}_0$ be defined by

$$(47) \qquad \tilde{u}_0(x) = \begin{cases} u_0(x) & \underline{in} \ \ 0 \leq x \leq b_0 \\ 0 & \underline{in} \ \ x > b_0 \end{cases}$$

and

$$(48) \qquad \tilde{y}_0(x) = \begin{cases} -\left(\dfrac{p_0(x) u_0'(x)}{H}\right) & \underline{in} \ \ 0 \leq x \leq b_0 \\ 0 & \underline{in} \ \ x > b_0. \end{cases}$$

Then $\tilde{u}_0$ and $\tilde{y}_0$ satisfy the admissibility conditions of Theorem 5a and

$$(49) \qquad \|\tilde{u}_0\|_{2,\alpha}^2 = H = \|\tilde{y}_0\|_{2,\beta}^{-2}$$

and hence H^* of (46a) is unique and equals H.

Proof: Let us define y_o by

(50) $$y_Q(x) = \frac{p_o(x)u^\wedge(x)}{H} \quad \text{in } 0 \pounds x \pounds b_Q.$$

Let us recall for convenience the properties of p and u.
By Lemma 2 and Lemma 3 and (41) of Theorem 4, we have

(15a) $$H = E(p_o, U_Q) = K(J^{b_o}_o |u^\wedge|^a dx)^{\frac{2}{a}} + J^{b_o}_o qu^\wedge dx;$$

by Lemma 3 and (17) we have

(17a) $$[^{b_o}_o p^p dx = K^p;$$

by Lemma 3 and (18) we have

(18a) $$u'_o = -A p^{\frac{\varrho}{a}}_o \quad \text{in } 0 \leqslant x \leqslant b_o;$$

because u_o satisfies (1) in $0 \pounds x \pounds b_o$ and y_o satisfies (50) we have

(51) $$y^\wedge = - \frac{qu}{\wedge} \cdot \text{in } 0 \wedge. x \leqslant b_o$$

and lastly by (50) and (18a)

(52) $$y_Q = s^{\frac{\lambda}{2}} P_Q \quad \text{in } 0 \leqslant x \leqslant b_o.$$

Taking norms in (51) and (52) we have

follows:

Substituting $u = \tilde{u}_0$ and $y = \tilde{y}_0$ in (46a) gives

$$H = \|\tilde{u}_0\|_2^2{}^{\wedge}{}_a \wedge H^* \wedge \|\tilde{y}_0\|_{2fj}^{-2}8 = H$$

and hence $H^* = H$. This completes the proof of the Theorem.

THEOREM 2- (DUAL EXTREMUM PRINCIPLES.) Let u and y satisfy the hypothesis of Theorem 5a. Then the following two "dual extremum principles" characterizing the maximum value H of Problem 1a hold:

$$H = \min_{u}[K(\int_0^{\infty} |u'|^a dx)^a + \int_0^{ro} qu^2 dx]$$

and

$$H^{-1} = \min_{y}[K^{-1}(\int_0^{\infty} y^p dx)^{\frac{B}{p}} + \int_0^{\infty} \frac{v'^2}{*-} dx].$$

Proof; This is a direct consequence of Theorem 6 and Theorem 5a, in particular (46a) and (49).

COROLLARY 1. Theorem 7 shows that (46a) with H^* replaced by H, can be used to give sharp estimates of H the maximum value of Problem 1a.

5. Uniqueness Proof

In this section we prove a statement made in Section 3.

THEOREM 2. The pair of functions $[p_o, u_o]$ given by (20)-(23) is the unique solution of Problem 3.

Proof: Assume that $P(x)$ is a function satisfying the admissibility conditions of the p-functions of Problem 3 and that $U(x)$ satisfies the admissibility conditions of the u-functions such that $E(P,u)$ is minimized by U and

$$(56) \qquad E(P,U) = E(p_o, u_o).$$

Then

$$(57) \qquad E(P,u_o) \geq E(P,U).$$

But by Lemma 4

$$(58) \qquad E(p_o, u_o) \geq E(P, u_o).$$

Now (56), (57) and (58) imply that (57) is actually an equality. Thus

$$(59) \qquad E(P, u_o) = E(P,U) = E(p_o, u_o).$$

Considering $E(P,u)$ as a quadric functional in u, by the parallelogram law for quadratic functionals we have

$$(60) \qquad 0 \leq E(P, U-u_o) = 2E(P,U) + 2E(P, u_o) - 4E(P,Z)$$

where $Z = (U + u_o)/2$. Because Z satisfies the boundary condition $Z = 1$ at $u = 0$, $E(P,Z) \geq E(P,U)$. Now this and (59) and (60) imply that $E(P,U-u_o) = 0$ and hence

(61) $U \equiv u_o$ in $0 \leq x \leq b$

where b is the right-hand end point of Support of P. This also shows that

(62) $E(P,u_o) = \min_u E(P,u)$

since the left side equals $E(P,U)$ and $E(P,U) = \min_u E(P,u)$.
Now, it remains to show that $P \equiv p_{*o}$ and $b = b_o$.

Since P is continuous and u^o is continuous with piecewise continuous derivative in $0 \leq x \leq b$, and u^o has the minimizing property (62) in the class of continuous functions with piecewise continuous first derivatives in $0 \leq x \leq b$ and satisfying the boundary condition $u(0) = 1$, we may apply the "Second Lemma" of the Calculus of Variations [12]. The details will be omitted but it results that P satisfies

(63) $Pu'_o = \text{constant} + \int_o^x qu_o \, dx$ in $0 \leq x \leq b$,

and the natural boundary condition

(64) $Pu^1_o = 0$ at $x = b$.

Integrating $E(P,u_0)$ by parts and using (63) and (64) gives

$$(65) \qquad\qquad E(P,u_0) = - [Pu_0']_{x=0}.$$

Similarly,

$$(66) \qquad\qquad E(p_0,u_0) = - [p_0 u_0']_{x=0}.$$

It follows from (59), (65) and (66) that

$$(67) \qquad\qquad P = p_0 = \delta_0 \quad \text{at} \quad x = 0.$$

Here we have used (20) and (21) and δ_0 is given by (22). Now, on using the expression for u_0 given by (20) and differentiating, (63) gives a first order linear equation of the Cauchy-Euler type in D, namely

$$(68) \qquad (1 - \frac{x}{b_0})P' - \frac{(\rho-1)}{b_0} P = - \frac{qb_0}{\rho}(1 - \frac{x}{b_0})^2$$

where

$$(69) \qquad D = (\text{Support of } P) \cap (\text{Support of } p_0).$$

Let B be the right-hand end point of the Support of D. Then $B = \min[b,b_0]$. Now as p_0 satisfies (68) and P and p_0 satisfy the same initial condition (67) it follows that

$$(70) \qquad\qquad P \equiv p_0 \quad \text{in} \quad D.$$

So it is enough to show $b = b_0$. Suppose that $b < b_0$. Then

(71)
$$E(P, U_0) = \int_0^b [Pu'^2 + qu_0^2]dx$$

$$= \int_0^b [p_0 u'^2_0 + qu_0^2]dx$$

$$< \int_0^{b_0} [p_0 u'^2_0 + qu_0^2]dx = E(p_0, u_0).$$

Here (70) and the fact $p \wedge 0 \wedge u$ in $b_0 \leq x \leq b_0$ are used.
The inequality (71) contradicts (59) and hence $b > b_0$ is the
only other possibility if not $b = b_0$. So, if $b > b_0$,

(72)
$$\int_0^b P^P dx = (\int_0^{b_0} P^P dx) + (\int_{b_0}^b P^P dx) > K^P.$$

Here $\int_0^{b_0} p_0^P dx = K^P$, (70) and also the fact $P \neq 0$ in $b_0 \leq x \leq b$
are used. But (72) is absurd and thus $b > b_0$ is untenable and
$b = b_0$. Thus the proof of the theorem is complete.

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