DECAY OF SOLUTIONS OF THE TWO-DIMENSIONAL WAVE EQUATION IN EXTERIOR DOMAINS

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Abstract

This report concerns the behavior, at large times, of solutions of two-dimensional wave equations in exterior regions. This behavior is related to low frequency calculations for the reduced wave equation. Both Dirichlet and Neumann boundary data are considered, and it is shown that there are differences in the two cases. The results also differ greatly from the corresponding ones in three dimensions.

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1. Introduction

This paper concerns the large time behavior of solutions of the two-dimensional wave equation in a region exterior to a cylinder. The corresponding problem in three dimensions has been extensively studied, [1], [2], and [4]. The basic result, in three dimensions, is that if a solution of the wave equation vanishes on a star-shaped obstacle, and has initial data of compact support, then it decays exponentially in time.

It is known, by examples, that exponential decay does not hold in two dimensions: Even the free space solutions decay only like t^{-1}. The difference between two and three dimensions reflects differences in solutions of the reduced wave equation,

$$Au + k^2 u = 0, \tag{1.1}$$

in exterior domains. In three dimensions, solutions of (1.1) are meromorphic functions of k while in two dimensions they have branch points at $k = 0$, as observed in [1].

A detailed study of the behavior, near $k = 0$, of two-dimensional solutions of (1.1) was provided in [5]. The purpose of this paper is to indicate, in a partially heuristic manner, the effects of this behavior on solutions of the wave equation. In two dimensions, there is a significant difference between Dirichlet and Neumann problems; this difference is not present in three dimensions since everything decays so rapidly.

In order to illustrate our results, while minimizing technical details, we confine ourselves to two simple, but representative, situations:

(1) Initial data of compact support and zero boundary data;

(2) Zero initial data and time-independent boundary data.

With Dirichlet boundary data we obtain the following conclusions:

(i) In case (1) the scattered wave "cancels" the free space solution so that the total solution decays more rapidly than it would in free space.

(ii) In case (2) the solution tends to a steady state.

With Neumann boundary data the corresponding conclusions are:

(i) The scattered wave decays more rapidly than the free space solution so that the latter is left as the dominant term.

(ii) The solution need not tend to steady state and may, in fact, become infinite as t tends to infinity.

Our results require a use of the fine structure of solutions of (1.1). The most effective tool for this seems to be the Laplace transform and this is what we use. This yields very sharp results but creates a technical difficulty which we have not entirely overcome. We require the validity of certain high frequency asymptotic series for solutions of (1.1), as indicated in [6]. For the Dirichlet problems we provide a partial justification by using results of Ludwig and Morawetz [3]. For the Neumann problems, no such justification has been obtained (see, however, Remark 5.4). We observe that our work requires the validity of the high frequency theory only in the special case of boundary data which are independent of k.

In Section 2, we state our results precisely. In Section 3 we provide the necessary results for Laplace transforms and in Section 4 we use these to prove the statements of Section 2. Section 5 contains our partial justification of the high frequency theory.

2. Statement of Results

Let γ denote a C^{∞} closed, convex curve in the plane and Ω denote its exterior. We assume γ has non-vanishing curvature. We consider the two problems:

$$
\text{(D)}\qquad
\begin{aligned}
&D_{tt} = \Delta D && \text{in } \Omega,\\
&D(x,0) = f(x), \quad D_t(x,0) = g(x) && \text{in } \Omega\\
&D(x,t) = F(x,t) && \text{on } \gamma;
\end{aligned}
$$

$$
\text{(N)}\qquad
\begin{aligned}
&N_{tt} = \Delta N && \text{in } \Omega,\\
&N(x,0) = f(x), \quad N_t(x,0) = g(x) && \text{in } \Omega,\\
&N_\nu(x,t) = F(x,t) && \text{on } \gamma.
\end{aligned}
$$

We restrict ourselves to two special cases:

<u>Case</u> (1)*: $f \equiv 0, \quad g \in C_0^\infty(\Omega), \quad F \equiv 0.$

<u>Case</u> (2): $f \equiv g \equiv 0, \quad F(x,t) = F_0(x).$

We denote by $U_0(x,t;g)$ the free space solution

$$
U_0(x,t;g) = (2\pi)^{-1}\int_\Omega \frac{g(y)}{\sqrt{t^2-|x-y|^2}}\,dy. \tag{2.1}
$$

We will need to consider also the reduced problems:

$$
(\mathfrak{D})\qquad
\begin{aligned}
&\Delta \mathfrak{D} = s^2 \mathfrak{D} && \text{in } \Omega,\\
&\mathfrak{D} = \varphi(x) && \text{on } \gamma;
\end{aligned}
$$

$$
(\mathfrak{n})\qquad
\begin{aligned}
&\Delta \mathfrak{n} = s^2 \mathfrak{n} && \text{in } \Omega,\\
&\mathfrak{n}_\nu = \varphi && \text{on } \gamma.
\end{aligned}
$$

---

*The case $f \in C_0^\infty(\Omega)$, $g \equiv 0$ can be obtained from this one by differentiation with respect to t.

Here s is to be in $\operatorname{Re} s \geq 0$ and the solutions are subject to a radiation condition. We denote the (unique) solutions by $\mathcal{D}(x;\varphi;s)$ and $\mathcal{N}(x;\varphi;s)$. Note that φ is independent of s.

Finally we need the functions $\mathcal{D}^o(x;\varphi)$ and $\mathcal{N}^o(x;\varphi)$ which are the respective (unique) solutions of the problems:

$$(\mathcal{D}^o)\qquad \begin{aligned} &\Delta w = 0 && \text{in } \Omega, \\ &w = \varphi && \text{on } \gamma, \\ &w = O(1) && \text{as } |x| \to \infty; \end{aligned}$$

$$(\mathcal{N}^o)\qquad \begin{aligned} &\Delta w = 0 && \text{in } \Omega, \\ &w_\nu = \varphi && \text{on } \gamma, \\ &w - (2\pi)^{-1}\left(\int_\gamma \varphi\right)\log|x| = o(1) && \text{as } |x| \to \infty. \end{aligned}$$

The functions $\mathcal{D}$ and $\mathcal{N}$ possess formal asymptotic expansions of the form:

$$\text{(I)}\qquad \mathcal{D} \sim e^{-s\tau(x)} \sum_{n=0}^{\infty} \frac{a_n(x)}{s^n}\,,$$

$$\text{(II)}\qquad \mathcal{N} \sim e^{-s\tau(x)} \sum_{n=0}^{\infty} \frac{b_n(x)}{s^n}\,.$$

The construction of these series is indicated in Section 5. We say such series are valid if the error after N terms is

$O(s^N)$ uniformly for x in compact subsets of Ω and $s \longrightarrow \infty$ in $\operatorname{Re} s \geq 0$. We will use the following hypotheses:

H_I (H_{II}): The series I and II and those obtained by termwise differentiations with respect to x are valid.

H'_I (H'_{II}): The series obtained from (I) and (II) by termwise differentiations with respect to s are valid.

THEOREM 1. H_I and H_{II} imply that the solutions of (D) and (N) exist for cases (1) and (2) and that the following estimates hold:

Case (1): $D \longrightarrow 0$ and $N \longrightarrow 0$ as $t \longrightarrow \infty$;

Case (2): $D - \mathscr{D}^o(x;F_o) \longrightarrow 0$ as $t \longrightarrow \infty$,

$$N - \alpha \log t - \mu\alpha - \hbar^o(x;F_o) \longrightarrow 0 \quad \text{as} \quad t \longrightarrow \infty,$$

where μ is a constant and,

$$\alpha = (2\pi)^{-1} \int_\gamma F_o ds.$$

THEOREM 2. H'_I and H'_{II} yield the following sharpened estimates:

$\underline{\text{Case}}$ (1): $D = O(t^{-1}(\log t)^{-2})$ $\underline{\text{as}}$ $t \to \infty$,

$N = U^{o}(x,t;g) + O(t^{-2}\log t)$ $\underline{\text{as}}$ $t \to \infty$;

$\underline{\text{Case}}$ (2): $D - \mathscr{D}^{o}(x;F_o) = O((\log t)^{-2})$ $\underline{\text{as}}$ $t \to \infty$;

$N - \alpha \log t - \mu\alpha - \mathfrak{n}^{o}(x;F_o) = O(t^{-1})$ $\underline{\text{as}}$ $t \to \infty$.

REMARK 2.1. In Section 5 we prove that H_I holds thus validating part of the conclusions of Theorem 1.

REMARK 2.2. The solutions in case (2) will be piecewise continuous. It is shown in the next sections that jumps occur only on the surface $t = \tau(x)$ and that their magnitudes can be computed from the coefficients in series (I) and (II).

REMARK 2.3. It will become clear that the results can be extended to the case in which $F(x,t)$ merely tends to the value $F_o(x)$ as t tends to infinity. By linearity, the general case of problems (D) and (N) can then be treated by combinations of the two special cases.

REMARK 2.4. Another special case which can be studied is that in which $F(x,t) = F_o(x)e^{i\omega t}$. The methods here show that for both Dirichlet and Neumann problems, the solutions tend to limit periodic values (the limiting amplitude principle) but at different rates.

3. Laplace Transforms

We introduce the class of functions with which we will have to work in the next section. Let R_δ denote the region (Re s > 0) U (0 < I si < 6,| arg s| < *ir)* . The class C, will consist of functions x which are analytic in R_δ, have continuous derivatives of all orders on Re s ^ 0, s ^ 0 and satisfy

$$\text{(A)}\quad sx(s) \sim e^{-sT} \sum_{n=0}^{\infty} a_n s^{-n}, \quad T > 0, \quad \text{as} \quad 6 \to \infty \quad \text{in} \quad \text{Re } s \geq 0.$$

(A) is to be a valid asymptotic approximation in the sense of Section *2.*

For $X \in C_{\delta,T}$ we define a function U by

$$(3.1)\qquad U(t;x) = (2\pi i)^{-1} \int_{\beta-i\infty}^{\beta+i\infty} e^{st} x(s)\,ds.$$

LEMMA 3.1, U is independent of j3 *in* $ > 0 and defines a piecewise smooth function of t with discontinuities only at t = T. Moreover,

$$\frac{\partial^k U(0;x)}{\partial t^k} = 0 \qquad k = 0\ 1\ 2$$

Proof; Define functions $J_k(t,r)$, k = 0,1,2,..., by

(3.2) $$J_k(t,\tau) = \begin{cases} 0 & \text{for } t < \tau \\ \dfrac{(t-\tau)^k}{k!} & \text{for } t > \tau . \end{cases}$$

The transforms of these functions are $e^{-s\tau}s^{-k-1}$. Thus (A) yields

(3.3) $$U(t;x) = \sum_{n=0}^{N} a_n J_n(t,\tau) + (2\pi i)^{-1}\int_{\beta-i\infty}^{\beta+i\infty} e^{st} R_N(s)\,ds,$$

where $R_N = O(s^{-N-1})$. The last term can be differentiated (N-1) times with respect to t and these derivatives are all zero at $t = 0$.

We want to relate the behavior of $U(t;x)$ for large t to that of $X(s)$ near $s = 0$. We assume that $X \in C_{0,\tau}$ satisfies the condition

(B) $$x - \alpha s^{-1}\log s - \beta s^{-1} = \hat{M}(s)\,, \quad \hat{M}(s) = o(s^{-1}) \text{ as } s \to 0 \text{ in } R_{c_0}.$$

Define the function $L(t)$ by,

(3.4) $$L(t) = \begin{cases} 0 & \text{for } 0 < t < 1, \\ -\log t & \text{for } t > 1, \end{cases}$$

and let $\mathcal{L}(s)$ denote the transform of L. It is not difficult to verify that $\mathcal{L}$ is analytic in $R_{-\delta}$ for any δ, and that, (3.5) $\mathcal{L}(s) = e^{-s}s^{-2}(1 + o(1))$ as $s \to \infty$ in R_δ,

(3.6) $\mathcal{L}(s) = s^{-1}\log s + r(s)$ as $s \to 0$ in R_δ,

where $r(s)$ is regular and single-valued in $|s| < \delta$. Thus, we can replace (B) by,

(B') $x(s) - a\mathcal{L}(s) - \beta s^{-1} = M(s)$, $M(s) = o(s^{-1})$ as $s \to 0$ in R_δ.

It follows that (3.1) can be written,

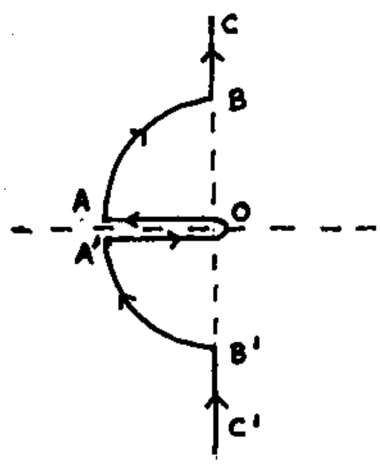
(3.7) $$U(t;x) = aL(t) + \beta + V(t;x),$$

where

(3.8) $$V(t;x) = (2\pi i)^{-1}\int_{\beta-i\infty}^{\beta+i\infty} e^{st}M(s)\,ds.$$

Conditions (A), (3.5), and (B') show that we can deform the contour in (3.8) into Γ indicated below. Then we have

$V = I + II + III$



where I and II are integrals over C'B'A' and ABC respectively and

(3.9) $$III = \int_0^{\varepsilon} e^{-\eta t}\psi(\eta)\,d\eta,$$

where,

(3.10) $$\psi(\eta) = (M(\eta e^{-\pi i}) - M(\eta e^{\pi i}))(2\pi i)^{-1}.$$

It follows from (A) and (3.5) that,

(3.11) $$V = a_o s^{-1} e^{-s\tau} - \beta s^{-1} + O(s^{-2}) \text{ as } s \longrightarrow \infty \text{ in } R_\delta.$$

This estimate and the Riemann-Lebesgue Lemma imply that the integrals I and II both tend to zero as t tends to infinity. Thus, we have proved:

<u>LEMMA 3.2.</u> <u>If</u> $\chi \in C_{\delta,\tau}$ <u>satisfies</u> B <u>then</u>

$$U + \alpha \log t - \beta - III \longrightarrow 0 \text{ as } t \longrightarrow \infty,$$

<u>where</u> III <u>is given by</u> (3.9).

Lemma 3.2 can be refined if the series (A) can be differentiated k times with respect to t. For then we can integrate by parts k times in I and II and deduce that those terms are order t^{-k} as t tends to infinity. Thus we have:

LEMMA 3.3. *If the series* A *can be differentiated* k *times, then*

$$U + \alpha \log t - \beta - III = O(t^{-k}) \text{ as } t \to \infty.$$

4. Proofs of Theorems 1 and 2

We proceed by Laplace transforms. The object is to obtain the solutions by applying the operator U of Section 3 to solutions of problems ($\mathfrak{D}$) and ($\mathfrak{N}$). The estimates necessary to verify estimate (B) all derive from the following result in [5].

LEMMA 4.1. *Suppose* $\varphi = \sum_{n=0}^{\infty} \varphi_n(x) s^{2n}$, *the series converging for* $|s|$ *sufficiently small. Then there is an* s_o *such that for* $0 < |s| < s_o$*, $\mathfrak{D}(x;\varphi;s)$ *and* $\mathfrak{N}(x;\varphi;s)$ *both exist and can be expressed in the following forms:*

$$(4.1) \qquad \mathfrak{D}(x;\varphi;s) = \{ \sum_{j=0}^{\infty} \sum_{i=j}^{\infty} A_{ij} s^{i-1} (s \log s)^{j-1} \}$$

$$\cdot \{ \sum_{j=0}^{\infty} \sum_{i=j}^{\infty} B_{ij} s^{i-1} (s \log s)^{j-1} \}^{-1},$$

with $A_{00} = A_{10} = B_{00} = B_{10} = 0$ *and*,

$$\frac{A_{11}}{B_{11}} = \mathfrak{D}^{o}(x;\varphi_o);$$

---

*On the Riemann surface for log s.

$$(4.2) \qquad s\mathfrak{n}(x;\varphi;s) = \sum_{j=1}^{\infty} \sum_{i=j-1}^{\infty} C_{ij} s^i (s \log s)^j + \sum_{i=1}^{\infty} D_i s^i,$$

with,

$$C_{01} = (2\pi)^{-1} \int_{\partial\Omega} \varphi_o ds, \qquad D_1 = \mu C_{01} + \mathfrak{n}(x;\varphi_o), \quad \mu \text{ a constant.}$$

Case 2: If we transform problems (D) and (N) we obtain, formally, problems ($\mathfrak{d}$) and ($\mathfrak{n}$) with $\varphi = s^{-1}F_o(x)$. The functions $\mathfrak{d}(x;s^{-1}F_o;s)$ and $\mathfrak{n}(x;s^{-1}F_o;s)$ are analytic in $\text{Re } s > 0$ and differentiable of all orders in $\text{Re } s \geq 0$, $s \neq 0$. This follows from results of [9] (see also the argument given in [6]). Lemma 4.1 and hypotheses H_I and H_{II} imply, then, that both functions belong to $C_{s_o, \tau(x)}$. It follows from Lemma 3.1 that the functions,

$$(4.3) \qquad D(x,t) = U(t;\mathfrak{d}(x,s^{-1}F_o;s), \qquad N(x,t) = U(t;\mathfrak{n}(x,s^{-1}F_o;s)$$

are defined and piecewise differentiable with respect to t with discontinuities only on $t = \tau(x,y)$. Further they satisfy the initial and boundary conditions. It follows from the differentiability of the series I and II with respect to x that ΔD and ΔN exist and the equations for $\mathfrak{d}$ and $\mathfrak{n}$ imply that D and N satisfy (E).

We turn to the estimates for large t. Lemma 4.1 yields the estimate,

$$(4.4)\qquad \mathscr{S}(x;F_o(x)s^{-1};s) = s^{-1}\mathscr{S}^o(x;F_o)[1 + \gamma_1(\log s)^{-1} + \gamma_2(\log s)^{-2}] + O(s^{-1}(\log s)^{-3}).$$

Thus $\mathscr{S}$ satisfies (B) with $\alpha = 0$, $\beta = \mathscr{S}^o(x;F_o)$. For the quantity ψ of (3.9) and (3.10) we find, from (4.4),

$$(4.5)\qquad \psi(\eta) = -\gamma_1\mathscr{S}^o(x;F_o)\eta^{-1}[(\log \eta)^2 + \pi^2]^{-1} + O(\eta^{-1}(\log \eta)^{-3}).$$

It follows from (4.5) and (3.9) that,

$$(4.6)\qquad III = O((\log t)^{-2}) \text{ as } t \longrightarrow \infty.$$

Equation (4.6) and Lemma 3.2 yield the estimate for D in Theorem 1. The stronger hypotheses H_I', together with Lemma 3.3, yields the estimate in Theorem 2.

Lemma 4.1 also yields the estimate,

$$(4.7)\qquad h(x;F_o(x)s^{-1};s) = \alpha s^{-1}\log s + (\mu\alpha + h^o(x;\varphi_o))s^{-1} + R(x)\log s + O(s \log s),$$

where,

$$\alpha = (2\pi)^{-1}\int_\gamma F_o ds.$$

This also has the form (B), with $\beta = (\mu\alpha + h^o(x;\varphi_o))$. The function ψ of (3.9) has the form,

$$\psi(\eta) = -R(x) + O(\eta \log \eta)$$

and then (3.9) yields,

$$\text{(4.8)} \qquad III = O(t^{-1}) \text{ as } t \longrightarrow \infty.$$

Thus we obtain, as before, the estimates for N in Theorems 1 and 2.

<u>Case</u> 1: We subtract the free space solution U_o from D and N thus obtaining a problem with zero initial conditions but non-zero boundary conditions. We continue to denote the solutions by D and N, remembering that at the end we must add U^o to both. The transforms of problems (D) and (N) will then yield ($\mathfrak{D}$) and ($\mathfrak{n}$) once more but with,

$$\text{(4.9)} \qquad \begin{aligned} \varphi &= \varphi^{\mathfrak{D}}(x,s) = -\int_o^\infty e^{-st}[U^o(x,t;g]_\gamma dt, \\ \varphi &= \varphi^{\mathfrak{n}}(x,s) = -\int_o^\infty e^{-st}[\frac{\partial}{\partial\nu} U^o(x,t;g)]_\gamma dt, \end{aligned}$$

respectively. We cannot proceed directly here since $\varphi^{\mathfrak{D}}$ and $\varphi^{\mathfrak{n}}$ are not independent of s. We overcome this difficulty with a kind of Duhamel principle. We illustrate with $\mathfrak{D}$.

We seek to express $\mathfrak{D}$ in the form,

$$\text{(4.10)} \qquad \mathfrak{D}(x;\varphi^{\mathfrak{D}};s) = \int_o^\infty e^{-st} u(x;[U^o(\cdot,t)]_\gamma;s)dt.$$

The function $U(x; [U^{o}(\cdot,t)]_{\gamma};s)$ is a solution of ($) with boundary data which are independent of s; t appears as a parameter. Thus the series I applies, that is,

$$(4.11) \qquad U(x;[U^{o}(\cdot,t)]_{\gamma};s) \sim e^{-s\tau(x)} \sum_{n=0}^{\infty} s^{-n} a_n(x,t).$$

For each fixed t, $U(x; [U^{o}(\cdot,t)]_{\gamma};s)$ belongs to $C_{s_0,\tau}$. (In Lemma 4.1, the constant s^{o} is independent of φ if φ does not depend on s.) On the other hand, formula (2.1) for U^{o} shows that,

$$(4.12) \qquad [U^{o}(x,t)]_{\gamma} \sim \sum_{n=1}^{\infty} t^{-n} \varphi_n(x) \qquad \text{as} \quad t \to \infty.$$

For any fixed s it follows from linearity that the estimate (4.12) carries over to $U(x; [U^{o}(\cdot,t)]_{\gamma};s)$ and its derivatives with respect to s. From this, one can see that, for fixed s, the integral in (4.10) exists and defines a function $\Phi(x;\varphi;s)$ which satisfies all the conditions for class $C^{\sim}_{\delta,\tau(x)}$ save possibly for condition (A).

We show now that $\Phi(x;\varphi;s)$ also satisfies (A). We indicate in the next section (Remarks 5.1 and 5.2) that the estimate (4.12) also carries over to the series (4.11) (including the error term after truncation). This means that, for large s, we can substitute (4.11) with (4.10), obtaining,

---

The formal substitution is made rigorous by truncating (4.11) and including an error term.

$$(4.13)\qquad \mathfrak{s}(x;\varphi^{\mathfrak{s}};s) \sim \sum_{n=0}^{\infty} s^{-n}\int_0^{\infty} e^{-st}\alpha_n(x,t)\,dt = \sum_{n=0}^{\infty} s^{-n}\beta_n(x,s),$$

where the β_n's are the transforms of the α_n's. Next we observe that (2.1) yields the relations,

$$(4.14)\qquad \frac{\partial^k}{\partial t^k}[U^o(x,0)]_\gamma = 0 \qquad\qquad k = 0,1,2,\ldots$$

Once again these relations carry over to the α_n's (see Remark 5.1). But then it follows, on repeated integrations by parts, that the transforms β_n satisfy,

$$(4.15)\qquad \beta_n(x;s) = O(s^{-M}) \text{ for arbitrary } M \text{ as } s \longrightarrow \infty.$$

Thus (4.13) implies that $\mathfrak{s}(x;\varphi^{\mathfrak{s}};s) = O(s^{-M})$ for arbitrary M, that is $\mathfrak{s}$ satisfies (A) with all the coefficients a_n equal to zero. Hence $\mathfrak{s}(x;\varphi^{\mathfrak{s}};s) \in C_{s_o,\tau(x)}$ and the existence portion of Theorem 1 follows as in Case 2.

We turn now to the behavior near $s = 0$. The function $\varphi^{\mathfrak{s}}$ which is the transform of $-[U_o(x;t)]_\gamma$ can be determined explicitly. It is,

$$(4.16)\qquad \varphi^{\mathfrak{s}}(x;s) = -(2\pi)^{-1}\int_\Omega g(y)\,[K_o(s|x-y|)]_{x\in\gamma}\,dy,$$

where $K_o(z)$ is the singular Bessel function with imaginary argument. It has the expansion,

$$(4.17) \qquad K_o(z) = \sum_{n=0}^{\infty} \alpha_n z^{2n} \log z + \sum_{n=0}^{\infty} \beta_n z^{2n}, \qquad \alpha_o = -1.$$

Thus we can write,

$$(4.18) \qquad \varphi^{\mathfrak{s}}(x;s) = \varphi^1(x,s)\log s + \varphi^2(x,s),$$

where

$$(4.19) \quad \varphi^1(x;s) = \sum_{n=0}^{\infty} J_n(x)s^{2n} \qquad J_o(x) = (2\pi)^{-1}\int_\Omega g(y)\,dy,$$

$$(4.20) \quad \varphi^2(x,s) = \sum_{n=0}^{\infty} L_n(x)s^{2n}, \qquad L_o(x) = -\beta_o(2\pi)^{-1}\int g(y)\,dy + (2\pi)^{-1}\int_\Omega g(y)\,[\log|x-y|]_{x\in\gamma}dy.$$

We write, in obvious notation,

$$(4.21) \qquad \mathfrak{s}(x,\varphi^{\mathfrak{s}};s) = \log s\ \mathfrak{s}(x;\varphi_1;s) + \mathfrak{s}(x;\varphi_2;s).$$

We can apply Lemma 4.1 to each term in (4.21). The result has the form

$$(4.22) \quad \begin{aligned} \mathfrak{s}(x,\varphi_1;s) &= \mathfrak{s}^o(x;J_o) + P_1(x)(\log s)^{-1} + P_2(x)(\log s)^{-2} + O((\log s)^{-3}) \\ \mathfrak{s}(x,\varphi_2;s) &= \mathfrak{s}^o(x;L_o) + Q_1(x)(\log s)^{-1} + Q_2(x)(\log s)^{-2} + O((\log s)^{-3}). \end{aligned}$$

It follows that $\mathfrak{s}(x,\varphi^{\mathfrak{s}};s)$ satisfies (B) with $\alpha = \beta = 0$. Equation

(4.22) shows that the function J) of (3.10) has the form,

(4.23) j>(T7) = -&°(x;J$_o$) + (P$_2$(x)+Q$_1$(x)) [(log r\)^2 + ir^2]'^{1}(l + o(1))

as 77—^0.

It follows then from (3.9) that,

(4.24) III + JS°(x;j$_o$)t~^{1} = O(t"^{1}(log t)~^{2}) as t — * oo.

Since, by (4.19), J$_o$ is a constant we have,

(4.25) ©°(x;J$_o$) = J$_Q$ = (27T)"^{1}! g(y)dy.
Ω

It follows from Lemma 3.2 that,

(4.26) D - III = D + J8°(x;J)t~^{1} = D + (2TT)~^{X} f g(y)dy t"^{1}-→ O.
Ω

Under the stronger hypothesis H!L$_I$, Lemma 3.3 and (4.25) sharpen (4.26) to,

(4.27) D + (2TT)"^{1}r g(y)dy t"^{1} = O(t"^{1}(log t)"^{2}).
Ω

We recall, however, that to obtain the true D we must add U$_o$.
From (2,1) we see that,

U°(x,t) = (STr)^^{1}! 9(Y)dY t"^{1} + 0(t"^{2}),
Ω

Thus the estimates of Theorems 1 and 2 for D follow from (4.26) and (4.27).

The calculations for the Neumann problem proceed in the same way. The only difference occurs in the behavior for small s. From (4.9), (4.16) and (4.17) we have, in this case,

$$\varphi^{\partial}(x,s) = -(2\pi)^{-1}\int_{\partial\Omega} g(y)\,\frac{\partial}{\partial\nu}\,[k_0(s|x-y|)]_{x\in\gamma}\,dy \tag{4.28}$$

$$= \varphi^1(x,s)\log s + \varphi^2(x,s),$$

where,

$$\varphi^1(x,s) = \sum_{n=1}^{\infty} M_n(x)s^{2n},$$

$$\varphi^2(x,s) = \sum_{n=0}^{\infty} N_n(x)s^{2n}, \quad N_0(x) = (2\pi)^{-1}\int_{\Omega} g(y)\,[\tfrac{\partial}{\partial\nu}\log|x-y|]_{x\in\gamma}\,dy. \tag{4.29}$$

Then $h(x,\varphi;s) = \log s\; h(x;\varphi_1;s) + h(x;\varphi_2;s)$. Lemma 4.1 yields,

$$h(x;\varphi_1;s) = O(s^2\log s)$$

$$h(x;\varphi_2;s) = (2\pi)^{-1}\left(\int_{\gamma} N_0\,ds\right)\log s + D_1(x) + O(s\log s). \tag{4.30}$$

It follows from (4.30) that h satisfies (B) with α and $\beta = 0$. It is not difficult to verify that the integral of N_0 over γ is zero. Hence, the function Φ of (3.9) is $O(\eta\log\eta)$ as $\eta\to 0$. It follows that $III = O(t^{-2}\log t)$ as $t\to\infty$ and the estimates in Theorems 1 and 2 follow from Lemmas 3.2 and 3.3.

5. High Frequency Asymptotics for the Reduced Equation

In this section we establish that hypothesis H_I is satisfied. The techniques are those of [3] and [7], with only minor variations.

We introduce a co-ordinate system in which μ is arc-length along γ and τ is distance, along the normals, from γ. This is possible since γ is convex. This co-ordinate system is orthogonal with the fundamental form $Q^2 d\mu^2 + d\tau^2$, where,

$$(5.1)\qquad Q^2 = 1 + 2\tau k(\mu) + \tau^2 (X''(\mu)^2 + Y''(\mu)^2).$$

Here X and Y are the parametric representation of γ and k is the curvature. We have,

$$(5.2)\qquad \nabla = \underset{\sim}{e}_\mu Q^{-1} \frac{\partial}{\partial \mu} + \underset{\sim}{e}_\tau \frac{\partial}{\partial \tau}$$

$$(5.3)\qquad \Delta = Q^{-1}\{\frac{\partial}{\partial \mu}(Q^{-1} \frac{\partial}{\partial \mu}) + \frac{\partial}{\partial \tau}(Q \frac{\partial}{\partial \tau})\}.$$

The formal series (I) for $\mathfrak{s}(x;\varphi;s)$ has the form,

$$(5.4)\qquad \mathfrak{s} \sim e^{-s\tau} \sum_{n=0}^{\infty} a_n(x) s^{-n},$$

where,

$$(5.5)_0\qquad \Delta\tau\, a_0 + 2\nabla\tau\cdot\nabla a_0 = 0, \quad \tau > 0; \qquad a_0\big|_{\tau=0} = \varphi.$$

$$(5.5)_m \qquad \Delta\tau\, a_m + 2\nabla\tau\cdot\nabla a_m = \Delta a_{m-1}, \quad \tau > 0; \quad a_m\big|_{\tau=0} = 0 \qquad m \geq 1.$$

From (5.2) and (5.3) one finds that $\nabla\tau = \underset{\sim}{e}_\tau$ and $\Delta\tau = a^{-1}Q_\tau$. It follows that equation $(5.5)_m$ can be integrated in the form,

$$(5.6)_0 \qquad a_0(\mu,\tau) = Q(\mu,\tau)^{-1/2}\varphi(X(\mu),Y(\mu))$$

$$(5.6)_m \qquad a_m(\mu,\tau) = Q(\mu,\tau)^{-1/2}\int_0^\tau (Q(\mu,\xi))^{1/2}\Delta a_{m-1}(\mu,\xi)\,d\xi \qquad m \geq 1.$$

<u>REMARK 5.1.</u> It follows from these explicit formulas that any dependence of φ on parameters carries over to the a_m's. This confirms the statements made in conjunction with equations (4.12) and (4.15).

<u>LEMMA 5.1.</u> <u>For</u> $N \geq 2$ <u>write</u>,

$$(5.7) \qquad U(x;\varphi;s) - e^{-s\tau}\sum_0^N a_n(x)\, s^{-n} = R_N(x,s).$$

<u>Then</u>,

$$(5.8) \qquad \Delta R_N - s^2 R_N = e^{-s\tau} s^{-N}\Delta a_N(x).$$

<u>Moreover there exist constants</u> s_0 <u>and</u> k <u>such that for all</u> s <u>in</u> $\mathrm{Re}\, s \geq 0$, $|s| \geq s_0$

$$(5.9) \qquad |R_N| \leq k|s|^{-N}.$$

Proof: Relation (5.8) follows immediately from (5.5). In addition, we note that (5.5) yields,

(5.10) $$R_N \equiv 0 \quad \text{on} \quad \gamma.$$

We want to use the ideas of [3] to estimate R_N from (5.8) and (5.10). To this end we first estimate the right side of (5.8) for large $|x|$. A fairly straightforward induction argument based on (5.1), (5.3) and (5.6) yields,*

(5.11) $$\Delta a_m(\mu,\tau) = O(\tau^{-5/2}) \quad \text{as} \quad \tau \longrightarrow \infty,$$

and the fact that all derivatives of Δa_m with respect to μ satisfy the same estimate.

Since $\tau \longrightarrow |x|$ for large $|x|$ the estimate (5.11) together with (5.8) shows that,

(5.12) $$|x|\,|LR_N| \le C|s|^{-N}|x|^{-3/2} \quad \text{as} \quad x \longrightarrow \infty,$$

where $LR_N = \Delta R_N - s^2 R_N$. The work of [3] would yield an estimate for R_N from (5.8), (5.10) and (5.12) provided that $s = ik$. Our only task is to extend the work of [3] to complex s in $\text{Re } s \ge 0$. The only non-trivial step in the extension is that the identity (1.2) of [3] must be extended to the complex form,

*This estimate uses the fact that the curvature is non-zero.

$$(5.13)\quad 2\mathrm{Re}\, r\, \overline{\mathcal{D}u}\, Lu = \mathrm{div}\{2\mathrm{Re}\, r\, \overline{\mathcal{D}u}\, \nabla u - |\nabla u|^2 x + (\eta^2-\sigma^2)|u|^2 x\}$$
$$- 2r\sigma|\nabla u|^2 - (|\nabla u|^2 - |u_r|^2) - |\mathcal{D}u - \frac{1}{2}\frac{u}{r}|^2 - 2i\sigma\eta r(\overline{u}_r u - u_r\overline{u})$$
$$+ |u|^2(-2\sigma|s|^2 r + 2\sigma^2).$$

In this formula, $r = |x|$, $s = \sigma + i\eta$, $Lu = \Delta u - s^2 u$ and $\mathcal{D}u = u_r + su + \frac{1}{2}\frac{u}{r}$. This formula is proved in the same way as (1.2) of [3] and reduces to it when $\sigma = 0$ and $\eta = -\lambda$.

We have,

$$|2i\sigma\eta_r(\overline{u}_r u - u_r\overline{u})| \leq \sigma r|\nabla u|^2 + \sigma r\eta^2|u|^2.$$

Hence (5.13) yields,

$$(5.14)\quad 2\mathrm{Re}\, r\, \overline{\mathcal{D}u}\, Lu = \mathrm{div}\, Z - r\sigma|\nabla u|^2 - (|\nabla u|^2 - u_r^2)$$
$$-|\mathcal{D}u - \frac{1}{2}\frac{u}{r}|^2 + |u|^2[(-2|s|^2 + \eta^2)r\sigma + 2\sigma^2],$$

where Z is the bracketed quantity in (5.13). We can assume that $r \geq \underline{r} > 0$ in Ω. It follows that the last term in (5.14) is positive for all s in $|s| \geq M$ provided M is sufficiently large. Hence we obtain the inequality,

$$(5.15)\quad 2\mathrm{Re}\, r\, \overline{\mathcal{D}u}\, Lu \leq \mathrm{div}\, Z - r\sigma|\nabla u|^2 - (|\nabla u|^2 - u_r^2) - |\mathcal{D}u - \frac{1}{2}\frac{u}{r}|^2.$$

The remaining portions of the argument in [3] can be carried through intact, using the inequality (5.15) instead of the identity (1.2) of [3] when $\sigma \geq 0$.

REMARK 5,2. It follows from (5.8) and (5.10) and Remark 5.1 that dependence of cp on a parameter carries over to R_N thus confirming the statements in Section 4. It also follows from (5.8) and (5.10), and a little further analysis, that the derivatives with respect to x of R_N satisfy estimates similar to (5.9). Note in particular that (5.8) and (5.9) yield,

$$\Delta R_N = 0(s^{-N+2}). \tag{5.16}$$

REMARK 5.3. We have not been able to verify that the derivatives of R_N with respect to s satisfy the kinds of estimates needed for hypothesis H_1. If we differentiate (5.8) with respect to s we find that $T = T_N$ is a solution of

$$\Delta T_N - s^2 T_N = 2sR_N + Te^{-s}V^N \Delta a_N(x) - Ne^{-s}V^{N-1}\Delta a_N(x). \tag{5.17}$$

We still have $T_N = 0$ on y« The difficulty is that we cannot obtain a sufficiently good estimate of the right side for large |x| in order to be able to apply the results of [3]. We believe this is a technical difficulty which can be overcome by a more careful analysis of R_N, but we have not succeeded in doing it.

REMARK 5.4. Some results for the Neumann problem are contained in [8]. If those could be completed we could validate hypothesis H_{II}.

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