NORMALITY OF POWERS IMPLIES COMPACTNESS

by

S. P. Franklin and R. C. Walker

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Abstract

In this note we reprove the theorem of N. Noble which asserts that

each power of a T_1 space being normal implies that the space is compact.

Our short but non-elementary proof seems to point the way toward an elementary proof, i.e., one that proceeds directly from the definitions without relying on any powerful theorem.

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Recently N. Noble showed that

1. <u>each power of a T_1 space being normal implies that the space is compact</u> [N_2].

This remarkable theorem has been applied by Herrlich and Strecker [H-S] and by Franklin, Lutzer and Thomas [F-L-T] to give two different categorical characterizations of the category of compact Hausdorff spaces as a subcategory of the Hausdorff spaces.

Noble's original proof derived this theorem as a corollary of a more complicated one which in turn depended on a long chain of previous results of Noble and others. Keesling has given a short and very elegant derivation of Noble's theorem from two well-known theorems, one of Stone and one of Morita [K].

Our purpose in this note is to present another short proof of Noble's theorem which we have reason to believe may point the way to an elementary proof, i.e., one which proceeds directly from the definitions without relying on any powerful theorem along the way.

Our short but non-elementary proof relys on theorems of Noble and Glicksburg and on a lemma of Glicksburg ([N_1],[G]):

2. (Noble) <u>If a product of T_1 spaces is normal, then some partial product with cocountable index set is countably compact.</u>

3. (Glicksburg) *Pseudocompactness of a product of completely regular spaces is equivalent to the pseudocompactness of every countable partial product*.

$4_{\#}$ (Glicksburg) p *distributes over pseudocompact products*.

Proof of 1.

Let X^{111} be an uncountable power of X. By 2. there exists a cocountable subset n of m such that x^{n} is countably compact. Every countable partial product is homeomorphic to a closed subspace of X^ and is therefore pseudocompact. Thus by 3. A^{m} is pseudocompact and we can apply 4. to obtain p(X™) = $(pX)^{m}$. Now suppose that p is a free z-ultrafilter on X and consider the product X^{P} formed by choosing the zero-sets in p as the indexing set. The proof is completed by showing that the assumption that p is free will provide disjoint closed sets of X^{P} which cannot be separated. Write A for the diagonal of X^{P}, i.e. the set of all constant functions. Write C for the set of choice functions, i.e., C = [x€ X^{P} |· x_{z}e z}. A and C are closed sets, and are disjoint since p is free. Since X^{P} is normal, there is a Urysohn function f : X^{P}—> I completely separating A and C. But f cannot extend to $p(X^{p})$ since the function in $(pX)^{P} = p(X^{P})$ which is constantly p belongs to the closure in $p(X^{P})$ of both A and C. Thus it must be that A fl C ^ *0,* i.e., p is fixed.

An elementary proof would result if one could show directly that Δ and C cannot be separated in X^p.

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Carnegie-Mellon University
Pittsburgh, Pennsylvania 15213

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