REGULARITY OF GROUP VALUED

BAIRE AND BOREL MEASURES

by

K. Sundaresan[1] and Peter W. Day[2]

Research Report 71-50

October 1971

[1]The research work of the author was in part supported by the Scaife Faculty Grant 1971 administered by Carnegie-Mellon University.

[2]Research in part supported by NSF Grant GU-2056.

/nlc - 10/29/71

REGULARITY OF GROUP VALUED BAIRE AND BOREL MEASURES

by

K. Sundaresan and Peter W. Day

ABSTRACT. It is known that a real valued measure (1) on the σ-ring of Baire sets of a locally compact Hausdorff space, or (2) on the Borel sets of a complete separable metric space is regular. Recently Dinculeanu and Kluvánek used regularity of non-negative Baire measures to prove that any Baire measure with values in a locally convex Hausdorff topological vector space (TVS) is regular. Subsequently a direct proof of the same result was offered by Dinculeanu and Lewis. Here we show just as directly that any measure defined as in (1) or (2) is regular, even when it takes values in a Hausdorff topological group. In particular, when the group is a Hausdorff TVS, our result improves the Dinculeanu-Kluvánek-Lewis theorem.

REGULARITY OF GROUP VALUED BAIRE AND BOREL MEASURES

K. Sundaresan[1] and Peter W. Day[2]

1. *Introduction*. Of the several proofs of regularity of Baire measures in the literature, the simplest seems to be the one offered in a recent paper of Dinculeanu and Lewis [2]. By bringing in the *p*-quasi variation, however, these authors make heavy use of the fact that the measures they consider take values in a Hausdorff locally convex space. We will show that in fact their result holds even when the range space is only a Hausdorff topological group. With minimal additional work we will also show that a group valued Borel measure on a complete separable metric space is also regular.

We briefly recall some notations and basic definitions required in the subsequent discussion.

In what follows, G is always a Hausdorff topological group written multiplicatively, with identity element 1, and

---

AMS 1971 subject classifications. Primary 2210, 2850; Secondary 2635, 2810.

Key Words and Phrases. Regular measure, group valued measure, Baire measure, Borel measure.

[1]The research work of the author was in part supported by the Scaife Faculty Grant 1971 administered by Carnegie-Mellon University.

[2]Research in part supported by NSF Grant GU-2056.

ft denotes the filter of all neighborhoods of 1. Let ft be a ring of sets. A function m : ft—>G is said to be a measure if $m(\bigcup_{i=1}^{oo} A_i) = \lim_{n \to oo} m(A_1)^{###} m(A_n)$ exists in G whenever $[A^{\wedge}]_{i \geq 1}$ is a sequence of pairwise disjoint sets in ft such that $\bigcup_{i \geq 1} A_i$ eft. It is verified that (1) when m is a measure and $\{A_i\}_{i \geq 1}$ is as above, then $\lim_{n \to ^{\wedge} oo} m(A_n) = 1$; and (2) m is a measure iff $\lim_{1 \to oo} m(B_i) = 1$ whenever $B_i \wedge 0$, B_i eft.

Let X be a Hausdorff topological space, and let ft be a ring of subsets of X. A measure m : ft —> G is called regular at Aeft if for every Ueft there is a compact set KG ft and an open set Geft such that K c A c G, and m(S)eU whenever S c: G\K, S€ft. The pair (K,G) is then said to work for (A,U) . The measure m is called regular on ft if it is regular at each member of ft.

2. The Lemmas. Before proceeding to the main theorem we state two useful results. The first one is due to Gould [4] for Banach space valued measures. The second one is a principle which seems to be known. We sketch a proof here for the sake of completeness. Recall that a ring is called a 6-ring if it is closed for countable intersections.

LEMMA 1. Let R be a 6-ring and let m : R—• G be a measure. If A_n * 0, $A_n$eft, then for each Ueft there is a natural number N such that if s ^ A_N and SGR, then m(S)GU.

PROOF. Suppose that the lemma is false. Then there is a sequence A_n ^ *0,* A_neR, a Ueft, and a sequence $\{S_n\}_{n\geq}$^ in R such that S_n c A_n and m(S_n) ^U for all n ^ 1. Let V,WGH be such that V*W (= u, and let n_1 =1. If L_k is defined, then, since m(S_k 0 A_i) —• 1 as i —> oo, there is a n_{k+1} > n_k such that m(S_{n_k} HA$_{n_{k+1}}$) GW. Let B_k = S_{n_k} \ (S_{n_k} ∩ $A_{n_{k+1}}$). Then B_k = S_{n_k} n (A$_{n_k}$ \A$_{n_{k+1}}$) c A_1 and B_k U S$_{\wedge}$ PI A$_{n_{k+1}}$ = S_{n_k}. Hence m(B_k)-m($S_{\wedge}$ Pi A$_{n_{k+1}}$) = m(S_{n_k} *) f£\J,* so m(B_k)^V. Now

$\bigcup_{k\geq 1}$ B_keR because $\bigcup_{k\,1}$ B_k c A and R is a 6-ring. But the sets $\{B_k\}_{n\geq}$ are disjoint, so m(B_k) —^ 1 as k~—too. This contradicts m(B_k)^V for all k ^≥ 1 and completes the proof of the lemma.

LEMMA 2. :rf R is a a-ring [5-ring] of sets generated by a^ class *&,* and J^ m : R—-# G is a^ measure which is regular at each member of $5 then m is regular on R.

PROOF. Let G = {AGR : m is regular at A}. We show first that G is a ring. Let A_1,A_2GQ, let UG^, and let V_1,V_2Gh be such that V_1#V_2 CU* Let (K*^G_r)$_1$ work for (A_i,V_i) i = 1,2. Then as in \2, page 93, (b)] it is verified that $(K_1 \cup K_2, G_1 \cup G_2)$ works for $(A_1 \cup A_2, U)$, and $(K_1 \cup G_2, G_1 \cup K_2)$ works for $(A_1 \backslash A_2, U)$. Thus G is a ring.

Next let ft be a 6-ring. We show that G is a S-ring. Suppose A_n ^ A, $A_n \in G$, and $U \in ft$. Let V_o €^ be such that $V_o * V_o \subset U$. By induction there is a sequence $\{V_i\}_{i \geq 1} \subset$ fa such that $V_{i+1} - V_{i+1} \subset V_i$, $i \geq 0$. Then $V_1 + \cdots + V_n \subset V_0$ for all $n \geq 1$. Let $(C_i, \&_i)$ work for (A_i, V_i), let $K_n = \bigcap_{i=1}^{n} C_i$, $G_n = \bigcup_{i=1}^{n}$ ^ , and $K = \bigcap_{n \geq 1} K_n$. By Lemma 1 there is an N such that if $S \subset K_N \backslash K$ and $S \in ft$ then $m(S) \in V_o$. it is then verified that (K_N, G_N) works for (A_N, V_o) so (K, G_N) works for (A,U) as in [2, p. 93, fc)]. Thus G is a 6-ring, so G = 54.

If ft is a a-ring, then a similar argument shows that A^n *f* A, $A^n \in G$ implies $A \in G$, so G is a a-ring, and again G = ft.

3. The Theorem. We are now ready to prove our theorem.

THEOREM. *Let X be a locally compact Hausdorff space [complete separable metric space], and let B be the a-ring of Baire [Borel] sets in X. Then every G-valued measure on B is regular.*

PROOF. (1) Suppose X is a locally compact Hausdorff space, and let *&* be the collection of all compact G_δ subsets of X. Let *Ke&.* It follows from [3, p. 296, Prop. 14] that there is a sequence $\{G_n\}_{n\geq 1}$ of open Baire sets such that G_n 4/ K. Let UeH. By Lemma 1 there is a natural number N such that S c $G_N\backslash K$, SeB implies m(S)eU, i.e., (K,G_N) works for (K,U) . Since IB is the a-ring generated by *&,* it follows from Lemma 2 that m is regular.

(2) Now let (X,d) be a complete, separable metric space, let *&* be the set of all closed subsets of X, let AeS, let Ueft, and let Veft be such that W c u. Since A is closed, it is a G_δ, so as in (1) there is an open set G ^ A such that if S c G\A and Se& then m(S)eV.

We now construct a suitable compact set. Let $\{a_i\}_{i\geq 1}$ be a countable, dense subset of X. For each natural number n ^ 1 let $B_{1/n}(a_i) = \{y€A : d(y,a_i) £ \frac{1}{n}\}$ and let $F_K(n) = \bigcup_{i=1}^{k} B_{1/n}(a_i)$. From the denseness of $\{a_i\}_{i\geq 1}$ it follows that for each n, $F_k(n)$ t_k A. Let $W_o$eft be such that W_o-W_o civ, and define fw.}.^.. c: n by induction so that W_{i+1}-W_{i+1} c W_i for all i > 0. Then W_1-»-W_n c W_o for all n ^> 1. By Lemma 1 there is a k_1 such that if S c: A\F. (1) and Se(B then m(S)eW_1. By induction there is a sequence $\{k_i\}_{i\geq 1}$ such that if S c $\bigcap_{i=1}^{n} F_{k_i}(i)\backslash \bigcap_{i=1}^{n+1} F_{k_i}(i)$ and SGB then m(S)€W_{n+1}. Indeed, if $k_1,...,k_n$ are defined, then

$\bigcap_{i=1}^{n} F_{k_i}(i) \setminus \left(\bigcap_{i=1}^{n} F_{k_i}(i) \cap F_k(n+1) \right) \downarrow_k \emptyset$, so the existence of k_{n+1} follows from Lemma 1. Let $K_n = \bigcap_{i=1}^{n} F_{k_i}(i)$ and $K = \bigcap_{n \geq 1} K_n$. It is verified that each sequence in K has a convergent subsequence. Since K is closed, K is in fact compact. Since $K_n \downarrow K$ there is an N such that if $S \subset K_N \setminus K$ and $S \in \mathcal{B}$ then $m(S) \in W_o$. Thus if $S \subset A \setminus K$ and $S \in \mathcal{B}$, then $S = S_1 \cup \cdots \cup S_{N+1}$ where $S_i \in \mathcal{B}$ for $1 \leq i \leq N+1$, $S_1 \subset A \setminus K_1$, $S_i \subset K_{i-1} \setminus K_i$ for $2 \leq i \leq N$, and $S_{N+1} \subset K_N \setminus K$, so $m(S) = m(S_1) \cdots m(S_{N+1}) \in W_1 \cdots W_N W_o \subset V$. It follows that (K,G) works for (A,U). Thus m is regular at every closed set, so by Lemma 2, m is regular at every Borel set, and the theorem is proved.

REFERENCES

[1] N. Dinculeanu and I. Kluvánek, "On vector measures", Proc. Lond. Math. Soc. (3)17(1967), 505-512. MR35#5571.

[2] N. Dinculeanu and P. W. Lewis, "Regularity of Baire measures", Proc. Amer. Math. Soc. 26(1970), 92-94. MR41#5588.

[3] N. Dinculeanu, Vector Measures, Pergamon Press, 1967.

[4] G. G. Gould, "Integration over vector -valued measures", Proc. Lond. Math. Soc. (3)15(1965), 193-225. MR30#4894.

DEPARTMENT OF MATHEMATICS
CARNEGIE-MELLON UNIVERSITY
PITTSBURGH, PENNSYLVANIA 15213 U.S.A.

/nlc
10/29/71