NETWORK MODELS FOR
MAXIMIZATION OF HEAT TRANSFER
UNDER WEIGHT CONSTRAINTS*

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Abstract

Of concern is a network in which the conductance of certain branches are variable. The problem posed is the maximization of the joint conductance subject to a bound on the ℓ_p norm of the variable conductances. It is shown that at an optimum state the conductance of a variable branch is proportional to the $2/(p+1)$ power of the current through the branch. This relation together with a dual variational principle leads to a "duality inequality" giving sharp upper and lower estimates of the maximum joint conductance. Such a network serves as a discrete model for a cooling fin subject to a weight limitation. Thus the model shows what analogous properties should hold for the cooling fin.

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1. Introduction.

A common problem of heat transfer is the design of machinery so that the structure can dissipate excess heat. For example, cooling fins are used on the cylinders of air cooled engines. Suppose in this example that the fin is not permitted to exceed a given weight. Then an optimum design problem is to find how the thickness of the fin should taper so that the rate of heat dissipation is a maximum.

The cooling fin design problem was solved for circular cylinders by R. J. Duffin in [3] and for convex cylinders by R. J. Duffin and D. K. McLain in [4]. They employed the calculus of variations to recast the question into a max-max problem in [3] and a saddle point problem in [4]. Such variational principles led to the following key lemma -- <u>for an optimum fin the magnitude of the temperature gradient is constant</u>. Using this lemma it is then easy to obtain explicit expressions for the thickness function of the optimum fin.

In this paper an electrical network model for such cooling devices is formulated and studied. Thus consider a lumped network having a finite number of conducting branches. Certain branches, termed set B, are allowed to vary their conductance but the total conductance is limited by the following ℓ_p norm type constraint

(I) $$(\Sigma_B g_s^{\rho})^{1/\rho} \le K.$$

Here g_s is the conductance of branch s and K and ρ are positive constants. Then the design problem is to maximize the joint conductance of the network between two specified input points. Thus it is desired to find Γ, the maximum conductance subject to constraint I.

By a variational argument we establish the following key lemma -- <u>for an optimum network the branch voltages v_s satisfy</u>

(II) $$|v_s| = \lambda (g_s)^{\frac{\rho-1}{2}}, \quad s \in B.$$

Here λ is a constant and $g_s \neq 0$.

Our network question may be characterized as a maximizing problem of mathematical programming. This suggests that there is a dual minimizing problem. Pursuing this idea leads to the following duality inequality if $\rho > 1$,

(III) $$\|v\|_{2,\alpha} \ge \Gamma^{1/2} \ge 1/\|y\|_{2,\beta}.$$

Here $\| \ \|_{2,\alpha}$ and $\| \ \|_{2,\beta}$ are certain dual norms. The vector v is an arbitrary normalized voltage distribution satisfying Kirchhoff's voltage law. The vector y is an arbitrary normalized current distribution satisfying Kirchhoff's current law. There is no "duality gap", in other words the duality inequality could be used to give a sharp estimate of Γ.

In a previous paper [1] R. J. Duffin empolyed the same network model but confined attention to the linear constraint expressed by I with $\rho = 1$. In that paper relations were obtained corresponding to II and III when ρ is given the value 1.

Presumably there are analogous results for more general non-linear constraints. However we feel that the I_p constraint is sufficiently important to be singled out for special treatment.

In Section 6 we return to the cooling fin problem and introduce a L^p constraint analogous to I. Then reasoning by analogy suggests the form of the key lemma II and the duality inequality III for the cooling fin.

2. Some network definitions and properties.

An electrical network may be depicted as a graph diagram with m nodes and n arcs, each arc connecting two distinct nodes. An 'arc' corresponds to a 'branch' of the network. Let the nodes be designated by 1,2,...,m and let the arcs be designated by 1,2,...,n. A direction is assigned to each arc. Suppose arc s connects nodes i and j and that the positive direction is from i to j. If u_i and u_j are the potentials of node i and node j then the branch voltage v_s of branch s is defined as the potential difference

(1) $$v_s = u_i - u_j .$$

Thus, assigning potentials to the nodes automatically assigns voltages to the branches.

Let currents $y_1,\ldots,y_n$ be assigned to the branches. Then the current source w_i at node i is defined as

(2) $$w_i = \Sigma_1 y_s - \Sigma_2 y_s .$$

Here Σ_1 denotes summation over the arcs starting at node i, and Σ_2 denotes summation over the arcs terminating at node i. Thus, assigning currents to the branches uniquely assigns current sources to the nodes.

In the remaining part of this section we state a number of lemmas whose proofs are either simple or can be found in reference [1].

Lemma 1. Let $\{u_i\}$ be an arbitrary assignment of node potentials and let $\{y_i\}$ be an arbitrary assignment of branch currents. Then

(3) $$\Sigma_1^m w_i u_i = \Sigma_1^n y_s v_s.$$

The common value is termed power.

PROOF: A direct consequence of equations (1) and (2).

Lemma 2. Let the voltages $(v_1, v_2, \ldots, v_n)$ of the branches arise from arbitrary assignment of the node potentials $(u_1, \ldots, u_m)$. Let the current sources $(w_1, \ldots, w_m)$ at the nodes arise from an arbitrary assignment of currents $(y_1, \ldots, y_n)$ through the branches. Then

(4) $$(\Sigma_1^m w_i u_i)^2 \leq \Sigma_1^n g_s v_s^2 \quad \Sigma_1^n g_s^{-1} y_s^2$$

where $(g_1, \ldots, g_n)$ is a set of positive constants. This is an equality if and only if y_s and $g_s v_s$ are proportional. Moreover some of the g_s can be allowed to vanish in (4) provided corresponding y_s also vanish.

PROOF: See Lemma 2 of reference [1] or it is enough to observe that an application of Cauchy inequality to (3) yields (4).

The constants $\{g_s\}$ are termed branch conductances. Ohm's law is satisfied if

(5) $$y_s = g_s v_s.$$

If this relation holds for all branches we shall say that there is an equilibrium state. In an equilibrium state it is seen that relation (4) is an equality rather than an inequality. Moreover, it is a corollary of Lemma 1 that

(6) $$\Sigma_1^m w_i u_i = \Sigma_1^n g_s v_s^2 = \Sigma_1^n g_s^{-1} y_s^2$$

at equilibrium.

A situation of central interest in this work arises when the potential of nodes 1 and m are given by $u_1 = 1$, $u_m = 0$ and all other nodes are <u>insulated</u>. A node i is said to be insulated if the current source w_i there vanishes. Then the <u>input conductance</u> between nodes 1 and m is defined as $\gamma = w_1$. It then follows from (6) that

(7) $$\gamma = \Sigma_1^n g_s v_s^2.$$

The solution of the input conductance problem can be obtained from a minimum principle of Maxwell which is stated here as a lemma.

<u>Lemma</u> 3. <u>Suppose some of the nodes of a network have prescribed potentials and the others are insulated. Then there is an equilibrium state in which the potentials of the insulated nodes take on values to minimize the power function</u>

(8) $$E = \Sigma_1^n g_s v_s^2.$$

<u>Moreover, the branch power</u> $g_s v_s^2$ <u>is uniquely determined for each branch</u>.

<u>PROOF</u>: See Lemma 3 of reference [1], for example.

<u>Lemma</u> 4. <u>The input conductance</u> $\gamma(g)$ <u>as a function of the branch conductance</u> g_j <u>is continuous, concave, non-decreasing and homogeneous of degree</u> 1.

<u>PROOF</u>: This is a consequence of Maxwell's principle and is proved in Lemma 4 of reference [1].

Two nodes of a network are said to be *positively connected* if it is possible to traverse from one to the other along a chain of branches with positive conductance. A network is said to be positively connected if any two nodes are positively connected.

Lemma 5. J*Tf a. network is positively connected, then*

(9) $g_{S}^{*\%\&\bullet} \; . \; T^{2}_{S}$

~~where~~ v J~~LS_ an equilibrium voltage corresponding to unit input voltage~~.

PROOF; This is yet another consequence of Maxwell's principle. See Lemma 5 of reference [1].

3. <u>The concept of a duality inequality</u>.

The developments in this treatment evolve from a scalar product inequality stated in the following lemma.

<u>Lemma</u> 6. <u>Let</u> v_s <u>and</u> y_s <u>for</u> $s = 1,2,\ldots,n$ <u>be real numbers. Let the terms of the scalar product sum be partitioned into two sets</u> A <u>and</u> B. <u>Thus</u>

$$(10) \qquad \Sigma_1^n v_s y_s = \Sigma_A v_s y_s + \Sigma_B v_s y_s.$$

<u>Let</u> α <u>and</u> β <u>be positive constants such that</u> $\alpha^{-1} + \beta^{-1} = 1$ <u>and let</u>

$$(11) \qquad \|v\|_\alpha = (\Sigma_B |v_s|^\alpha)^{1/\alpha}, \quad \|y\|_\beta = (\Sigma_B |y_s|^\beta)^{1/\beta}.$$

<u>Then a scalar product inequality is</u>

$$(12) \qquad |\Sigma_1^n v_s y_s| \leq \|v\|_{2,\alpha} \, \|y\|_{2,\beta}.$$

<u>Here,</u> $\| \ \|_{2,\alpha}$ <u>and</u> $\| \ \|_{2,\beta}$ <u>are mixed norms defined in terms of positive numbers</u> g_s <u>and</u> K <u>as</u>

$$(13) \qquad \|v\|_{2,\alpha} = (\Sigma_A g_s v_s^2 + K\|v\|_\alpha^2)^{1/2}$$

$$(14) \qquad \|y\|_{2,\beta} = (\Sigma_A g_s^{-1} y_s^2 + K^{-1}\|y\|_\beta^2)^{1/2}.$$

<u>PROOF</u>: In relation (10) apply Cauchy's inequality to Σ_A and Hölder's inequality to Σ_B to obtain

$$|\Sigma_1^n v_s y_s| \leq (\Sigma_A v_s^2)^{1/2} \, (\Sigma_B y_s^2)^{1/2} + \|v\|_\alpha \|y\|_\beta.$$

Then apply Cauchy inequality here to obtain

$$|\Sigma_1^n v_s y_s| \leq (\Sigma_A v_s^2 + \|v\|_\alpha^2)^{1/2} \, (\Sigma_A y_s^2 + \|y\|_\beta^2)^{1/2}.$$

To obtain more generality let g_s for $s = 1,2,\ldots,n$ be positive constants such that $g_s \equiv K$ in the set B. Then replacing v_s and y_s by $g_s^{1/2} v_s$ and $y_Q / g_Q^{1/2}$ gives the desired inequality. Hie mixed norms appearing in this inequality seem somewhat artificial; however it will now be shown that they have a natural application to a class of optimization problems.

We apply the inequality (12) to the power balance equation

(3) $$\Sigma \wedge w_i u_i = \Sigma_1^n y_s v_s$$

of Section 2, and this gives

(15) $$(\Sigma_1^m w_i u_i)^2 \leq (\Sigma_A g_s v_s^2 + K\|v\|_\alpha^2)\ (\Sigma_A g_s^{-1} y_s^2 + K^{-1}\|y\|_\beta^2)\,.$$

Here the constant g_s is interpreted as the conductance of branch s. The constant K has the dimension of conductance and will be given a definite interpretation later, A duality inequality results from (15) by specializing boundary conditions, This main result is stated as a theorem.

Theorem 1. Let v_s b<5 a. set of branch voltages subject to the boundary condition $u_1 = 1$ and $u = 0$. Let y be a set of branch currents subject to the boundary condition $w_i = 0$ except $w_1 = 1$ and $w_m = -1$. Then

(16) $$\Sigma_A g_s v_s^2 + K\|\,fa * r * \wedge\ (\Sigma_A g_s^{-1} y_s^2 + K^{-1}\|y\|_\beta^2)^{-1}.$$

The constant r^* is independent of v_s and y_s.

PROOF; With this choice of boundary conditions we see that $\wedge\Sigma^n w_i u_i = w_1 u_1 = 1$- 1<hus tlle two norms on the right side of (15) do not vanish and division gives

$$\|v\|_{2,\alpha}^2 \geq \|y\|_{2,\beta}^{-2}.$$

Since the left side does not depend on y and the right side does not depend on v it follows that there exists a constant Γ^* satisfying

$$\|v\|_{2,\alpha}^2 \geq \Gamma^* \geq \|y\|_{2,\beta}^{-2}$$

and the proof of (16) is complete.

The constant Γ^* has the dimension of conductance. Relation (16) furnishes upper and lower bounds for Γ^* and so we term (16) a duality inequality. Conceivably the duality inequality gives a non-unique Γ^*. In that case we say there is a <u>duality gap</u>. It is desired to show that under the constraint

$$(17) \qquad (\Sigma_B g_s^\rho)^{1/\rho} \leq K, \quad \frac{1}{\rho} + \frac{2}{\alpha} = 1, \ \rho > 1,$$

there is no duality gap. In other words Γ^* is uniquely determined by (16).

If the set B is empty then the duality inequality (16) reduces to a classical result. In that case Γ^* is simply the input conductance γ. The limiting case, when $\alpha = \infty$, $\beta = 1$, $\rho = 1$ was treated in detail in a previous work [1]. For other ramifications of duality inequalities see references [2] and [5].

4. *Maximization of the input conductance*,

The central network question of this work may be phrased as follows.

Problem I. *Let* $y(g)$ *be the joint conductance of a network with* n *branches. Let the branch conductances be fixed for a set of the branches denoted by* A. *Let* B *be the set complementary to* A. *Find*

(18) $$T = \max y(g)$$

subject to the constraints $g_s \geq 0$ *and*

(17) $$(\Sigma_B g_s^\rho)^{1/\rho} \leq K$$

where $\rho > 0$ *and* $K > 0$ *are given constants*.

In the remainder of the treatment of this question it shall be assumed that *the network is positively connected by the* A *branches alone*. This simplifies the discussion without loss of generality because disconnected networks can be treated aposteriori by continuity argument.

Theorem 2. *Problem* I *always has a solution*.

PROOF: The constraints of the problem I restrict g to a compact set. Now the proof is immediate by Lemma 4.

Lemma 7. T of problem I is also given by

$$F = \max y(g)$$

subject to the constraints $g_s \geq 0$ and

$$\{\Sigma_B g_s^\rho\}^{1/\rho} = K.$$

PROOF: A solution of Problem I exists (Theorem 2) and let this be g'. If $\Sigma_{s\in B} g_s'^{\rho} < K^{\rho}$, let $B = \{1,\ldots,\ell\}$ and define $\tilde{g}_1 = \{K - g_2'^{\rho} \ldots - g_\ell'^{\rho}\}^{1/\rho}$ and $\tilde{g}_s = g_s'$, $s = 2,\ldots,\ell$. Then clearly $\tilde{g}_1 > g_1'$ and $(\Sigma_B \tilde{g}_s^{\rho})^{1/\rho} = K$ and by Lemma 4 (non-decreasing property of $\gamma(g)$) $\Gamma \geq \gamma(\tilde{g}) \geq \gamma(g') = \Gamma$ and the proof is complete by another application of Theorem 2.

Problem I is said to be degenerate if there is a solution with $g_s = 0$ for all s in the set B. Otherwise the problem is said to be nondegenerate. It is seen that the degenerate case is trivial.

We regard the next result as the "main lemma" in our analysis of the problem.

Theorem 3. If $\tilde{g}$ is a solution in the nondegenerate case then there exists a constant λ such that for the equilibrium voltage $\tilde{v}$

$$|\tilde{v}_s| = \lambda \tilde{g}_s^{\frac{\rho-1}{2}} \tag{19}$$

in the set B if $\tilde{g}_s > 0$. If $\rho > 1$ this also holds for $\tilde{g}_s = 0$, and moreover $\lambda > 0$.

PROOF: Let $B = \{1,\ldots,\ell\}$. Since

$$\Sigma_B \tilde{g}_s^{\rho} = K^{\rho} > 0$$

at least one $\tilde{g}_s$ in the set B is positive. Say $\tilde{g}_1$ is positive; then $\tilde{g}_1 = (K^{\rho} - \Sigma' \tilde{g}_s^{\rho})^{1/\rho} > 0$. Here Σ' is the sum over the set B in which the g_1 term is omitted. Thus by Lemma 7 the function $\tilde{\gamma}$ defined as

$$\tilde{\gamma}(g_2,\ldots,g_\ell) = \gamma((K^{\rho} - \Sigma' \tilde{g}_s^{\rho})^{1/\rho}, g_2,\ldots,g_\ell) \tag{20}$$

$g_2 \geq 0,\ldots,g_\ell \geq 0$ has a maximum at $(\tilde{g}_2,\ldots,\tilde{g}_\ell)$. Thus, if

$\tilde{g}_s > 0$ for some s in the set $B - \{1\}$, then

$$\frac{\partial\tilde{\gamma}}{\partial\tilde{g}_s} = 0 \quad \text{i.e.,} \quad -\frac{\partial\gamma}{\partial\tilde{g}_1}\tilde{g}_s^{\rho-1}(K^{\rho}-\Sigma'\tilde{g}_s^{\rho})^{\frac{1-\rho}{\rho}} + \frac{\partial\gamma}{\partial\tilde{g}_s} = 0$$

(21) i.e., $$\frac{\partial\gamma}{\partial\tilde{g}_s} = \lambda^2\tilde{g}_s^{\rho-1} \text{ where } \lambda^2 = \frac{\partial\gamma}{\partial\tilde{g}_1}\tilde{g}_1^{1-\rho} = \tilde{v}_1^2\,\tilde{g}_1^{1-\rho} \geq 0$$

i.e., $$\frac{\partial\gamma}{\partial\tilde{g}_s} = \lambda^2\tilde{g}_s^{\rho-1} \text{ for } \tilde{g}_s > 0 \text{ in the set } B.$$

On the other hand if $\tilde{g}_s = 0$ for some s in the set $B - \{1\}$ and $\rho > 1$, then, similarly $\frac{\partial\tilde{\gamma}}{\partial\tilde{g}_s} \leq 0$ leads to

$$0 \leq \tilde{v}_s^2 = \frac{\partial\gamma}{\partial\tilde{g}_s} \leq \lambda^2\tilde{g}_s^{\rho-1}.$$

But if $\rho > 1$ the right side vanishes, i.e.,

(22) $$\tilde{v}_s^2 = \lambda^2\tilde{g}_s^{\rho-1}, \text{ for } \tilde{g}_s = 0 \text{ in the set } B - \{1\}.$$

Combining (21) and (22) we have (19). Now $\lambda > 0$ if $\rho > 1$. For otherwise if $\lambda = 0$ (19) would imply $\tilde{v}_s = 0$ in the set B and $\Gamma = \Sigma_A g_s \tilde{v}_s^2 = \gamma(g')$ with $g'_s \equiv 0$ in B, contrary to our hypothesis.

We note that the relation (19) of Theorem 3 serves as a check to see whether an assumed solution is optimal. Moreover if it is not optimal an iterative procedure is suggested by (19) to redefine g_s to come closer to the optimal.

<u>Theorem 4. Let Γ be the maximum input conductance and let</u> v <u>be a voltage resulting from a potential which is arbitrary except for a unit potential difference at the input. Then</u>

W

(23) $$\Gamma \le \Sigma_A g_s v_s^2 + K(\Sigma_B |v_s|^\alpha)^{\frac{2}{\alpha}}$$

where $n > 1$, and $\frac{1}{p} + \frac{2}{a} = 1$, Moreover this becomes an equality for the equilibrium voltage $\tilde{v}$.

PROOF: Let $\tilde{g}$ be a solution of Problem I and as in Theorem 3, then, $\tilde{v}$ being equilibrium voltage,

$$\Gamma = \Sigma_A g_s \tilde{v}_s^2 + \Sigma_B \tilde{g}_s \tilde{v}_s^2$$

$$\le \Sigma_A g_s v_s^2 + \Sigma_B \tilde{g}_s v_s^2 \qquad \text{(Lemma 3)}$$

$$\le \Sigma_A g_s v_s^2 + (\Sigma_B \tilde{g}_s^\rho)^{\frac{1}{\rho}} (\Sigma_B |v_s|^\alpha)^{\frac{2}{\alpha}} \qquad \text{(Hölder)}$$

$$= W s^2 + K(\Sigma_B |v_s|^\alpha)^{\frac{2}{\alpha}}.$$

This proves (23).

Now, if $v = \tilde{v}$, by Theorem 3, the right side of (23) becomes

$$\Sigma_A g_s \tilde{v}_s^2 + K\lambda^2 (\Sigma_B \tilde{g}_s^{\frac{\alpha}{2}(\rho-1)})^{\frac{2}{\alpha}} =$$

$$\Sigma_A g/_s^2 + \wedge\wedge y^{\frac{1.2B.}{a}} =$$

$$\Sigma_A g_s \tilde{v}_s^2 + \lambda^2 K^\rho.$$

However, by the same Theorem 3

$$\Gamma = \Sigma_A g_s \tilde{v}_s^2 + \lambda^2 \Sigma_B \tilde{g}_s \cdot \tilde{g}_s^{\rho-1}$$

$$= \Sigma_A g_s \tilde{v}_s^2 + \lambda^2 K^\rho.$$

This shows that there is equality in (23) if $v = \tilde{v}$, the equilibrium voltage.

Corollary JL. If g jj^ an arbitrary set of conductances which satisfy the constraints and if $p > 1$ and $\frac{1}{p} + \frac{2}{\&} = 1$, then

$$(24)\qquad r\,i\;\; \gamma(g) - \Sigma_B g\, J_s^2 + K\{\Sigma_s \wedge^{\frac{\alpha}{2}}\}^{\frac{2}{\alpha}}.$$

This is an equality for the optimum solution $\tilde{g}$ oj| Theorem 2L

PROOF; In (23) let v be the equilibrium solution according to the choice g. Then

$$(25)\qquad \gamma = L_A g_s v_s^2 + S_B g_s v_s^2 \qquad \text{and}$$

$$(26)\qquad \frac{\partial\gamma}{\wedge_s} = v_s^j .$$

Substituting (25), (26) into (23) proves (24).

To study equality, let $g = \tilde{g}$, then r.h.s. of (24) reduces to $S_{a\,r\backslash}\, g_s\, \tilde{v}_s + K(S_{T3\,D}|\tilde{v}_s|^{a})^{\frac{2}{a}}$, and this equals r by Theorem 4, and the proof is complete.

5. Closing the duality gap.

It follows from Theorem 4 that we can take the constant Γ^* of the duality inequality equal to the maximum conductance Γ. The following theorem is the dual of Theorem 4 and shows that there is no duality gap.

Theorem 5. *Let Γ be the maximum joint conductance in Problem I. Then*

(27)
$$\Gamma^{-1} \le \Sigma_A g_s^{-1} y_s^2 + K^{-1}(\Sigma_B |y_s|^{\beta})^{\frac{2}{\beta}}$$

where $\rho > 1$, $2\beta^{-1} = 1 + \rho^{-1}$ and $\{y_s\}$ is any set of branch currents such that the current source has unit magnitude at the input nodes and vanishes at other nodes. Moreover this becomes an equality for the equilibrium state.

PROOF: Let $\tilde{v}_s$ and $\tilde{g}_s$ be optimal solutions of Theorem 3. According to lemma 7, we may assume $\|\tilde{g}\|_\rho = K$. Theorem 3 states that $\tilde{v}_s^2 = \lambda^2 \tilde{g}_s^{\rho-1}$ in B. Equilibrium currents satisfy $\tilde{y}_s = \tilde{g}_s \tilde{v}_s$. Thus

(28)
$$|\tilde{y}_s| = \lambda\, \tilde{g}_s^{\frac{\rho+1}{2}}.$$

Taking the norm here gives

(29)
$$\|\tilde{y}\|_\beta^2 = \lambda^2 (\Sigma_B \tilde{g}_s^{\rho})^{\frac{2}{\beta}} = \lambda^2 K^{\rho+1}.$$

On the other hand $\tilde{g}_s \tilde{v}_s^2 = \lambda^2 \tilde{g}_s^{\rho}$ so

(30)
$$K\|\tilde{v}\|_\alpha^2 = \Sigma_B \tilde{g}_s \tilde{v}_s^2 = \lambda^2 K^{\rho}.$$

Then (29) and (30) show that

(31) $$\Sigma_A g_s \tilde{v}_s^2 + K\|\tilde{v}\|_\alpha^2 = \Sigma_A g_s^{-1} \tilde{y}_s^2 + K^{-1}\|\tilde{y}\|_\beta^2.$$

Of course $\tilde{y}_s$ may not satisfy the boundary conditions, but $y_s^* = \tilde{y}_s/\Gamma$ does satisfy the boundary conditions and if we substitute $\tilde{v}$ and y^* into the duality inequality (16) it follows by virtue of (31) that it is actually an equality. This means the right side of (27) is minimized for $y = y^*$ and the minimum equals Γ^{-1} and the proof is complete.

Actually this theorem and the optimal conductance-current relation (28) remain true for $\rho > 0$. However we are unable to prove Theorem 4 in that case.

Corollary 2. *If g is an arbitrary set of conductances which satisfies the constraints then*

(32) $$\Gamma^{-1} \leq \gamma^{-1} - \gamma^{-2} \Sigma_B g_s \frac{\partial\gamma}{\partial g_s} + K^{-1}\gamma^{-2}\left(\Sigma_B g_s^\beta \left(\frac{\partial\gamma}{\partial g_s}\right)^{\frac{\beta}{2}}\right)^{\frac{2}{\beta}}.$$

This becomes an equality for the optimum solution $\tilde{g}$ of Theorem 3.

PROOF: In (27), let y be the equilibrium solution corresponding to the choice of g. Then,

(33) $$\gamma^{-1} = \Sigma_A g_s^{-1} y_s^2 + \Sigma_B g_s^{-1} y_s^2$$ and

the corresponding branch voltages are related by

(34) $$y_s = \frac{1}{\gamma} g_s v_s.$$

By Lemma 5 and (34) we obtain

$$(35) \qquad Y_s^2 = \frac{g_s^2}{\gamma^2} \frac{\partial \gamma}{\partial g_s} .$$

Substituting (33) and (35) in (27) yields (32). The case of equality follows as in Corollary 1 under Theorem 4 and the proof is complete.

6. Cooling Fins.

The lumped network problems just discussed are analogous to conduction problems for continuous systems. In particular this analogy will be developed here for cooling fins. Such fins are used to conduct heat away from machines to the ambient media.

The cooling fin problem is to maximize the conductance of a fin R of thickness $p(x,y)$ subject to the constraint $\iint_R p^\rho ds \le K^\rho$ where $\rho > 1$ and $K > 0$ are given. This is to be accomplished by suitably tapering the fin. The linear case of $\rho = 1$ has been treated rigorously in two previous works [3] and [4]. Here it is proposed to give a heuristic treatment based on the network model. Thus we will be led to formulate a duality inequality providing upper and lower bounds for the conductance of the optimum cooling fins.

For convenience let us adopt electrical rather than thermal terminology and treat the equivalent electrical problem.

The power input to the plate R is

$$E = \iint_R (p|\nabla u|^2 + qu^2)\, ds \tag{36}$$

where u is the electrical potential, p is the specific conductance, and q is the leakage conductance to ground. It is supposed that ground is at zero potential. The boundary conditions are that $u = 1$ on the part ∂R_1 of the boundary of R and $p \frac{\partial u}{\partial n} = 0$ on the complementary part ∂R_2 of the boundary. Then E is equal to the conductance γ of the plate, it being assumed that u is the equilibrium potential satisfying the differential equation

(37) $$\nabla.(p\nabla u) = qu.$$

Problem II. Maximize the conductance γ subject to the constraint

(38) $$\iint_R p^\rho ds \leq K^\rho, \quad \rho > 1,\ K > 0$$

being given. Without loss of generality it may be assumed that the variation in p is due to a variation in thickness of the plate. It is assumed that q may be function of position but is not subject to variation.

Reasoning by analogy from Theorem 3 the optimum plate should be tapered so that

(39) $$|\nabla u| = \lambda p^{\frac{\rho-1}{2}}$$

for some constant λ. By analogy with Theorems 4 and 5, the optimum conductance Γ will have upper and lower bounds given by

(40) $$\iint_R qu^2 ds + K\left(\iint_R |\nabla u|^\alpha ds\right)^{2/\alpha} \geq \Gamma \geq$$
$$\left[\iint_R \frac{(\nabla \circ y)^2}{q} ds + K^{-1}\left(\iint_R |y|^\beta ds\right)^{\frac{2}{\beta}}\right]^{-1}.$$

The function u is arbitrary except that $u = 1$ on ∂R_1 and the vector field y, which corresponds to a current flow, is also arbitrary except that the net flow across boundary ∂R_1 is unity and current flow across ∂R_2 vanishes at all points. A similar conjecture for the limiting case, when $\alpha = \infty$, $\beta = 1$, $\rho = 1$ was given in reference [1]. Rigorous proofs of this conjecture and of (40) are yet to be supplied.

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