REARRANGEMENT INEQUALITIES

by

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ABSTRACT

If T is a partially ordered set and the components of $\underset{\sim}{a} = (a_1, \ldots, a_n) \in T^n$ are totally ordered, then $\underset{\sim}{a}$ is called a chain and we let $\underset{\sim}{a}^* = (a_1^*, \ldots, a_n^*)$ $[\underset{\sim}{a}' = (a_1', \ldots, a_n')]$ be the vector obtained by rearranging the components of $\underset{\sim}{a}$ in decreasing [increasing] order. If $\varphi : T_1 \times T_2 \quad G$ where T_1, T_2 are partially ordered sets and G is a partially ordered abelian group, then a necessary and sufficient condition on φ is given so that

$$\sum_{j=1}^{n} \varphi(a_j^*, b_j^*) \leq \sum_{j=1}^{n} \varphi(a_j, b_j) \leq \sum_{j=1}^{n} \varphi(a_j^*, b_j')$$

for all chains $\underset{\sim}{a} \in T_1{}^n$, $\underset{\sim}{b} \in T_2{}^n$. Also a necessary and sufficient condition on φ is given so that equality holds on the right [left] iff $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are oppositely [similarly] ordered. A sufficient condition is given so that

$$\varphi(\underset{\sim}{a}^*, \underset{\sim}{b}^*) << \varphi(\underset{\sim}{a}, \underset{\sim}{b}) << \varphi(\underset{\sim}{a}^*, \underset{\sim}{b}'),$$

where $<<$ denotes a preorder relation of Hardy, Littlewood and

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Polya. Similar results to these are given when φ is a function of m variables. All these results are generalized for real valued essentially bounded measurable functions on a finite measure space. For specific choices of φ the inequalities are shown to hold for even larger classes of functions. The concept of "similarly ordered" is generalized for measurable functions to give a necessary and sufficient condition for equality.

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1. Introduction.

In recent years a number of inequalities have appeared which involve rearrangements of vectors in $\mathbb{R}^n$ and measurable functions on a finite measure space. The most famous one was proved by Hardy, Littlewood and Polya [5, Theorem 368]. If $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ let $a^* = (a_1^*, \ldots, a_n^*)$ [$a' = (a'_1, \ldots, a'_n)$] denote the vector obtained from a by rearranging the components of a in decreasing [increasing] order. Then the inequality they proved is

$$\sum_{j=1}^{n} a_j^* b_j' \le \sum_{j=1}^{n} a_j b_j \le \sum_{j=1}^{n} a_j^* b_j^* \tag{1.1}$$

for all $a, b \in \mathbb{R}^n$. Equality obtains on the right [left] iff a and b are similarly [oppositely] ordered.

Recently, Mine [11] has given the following product version: If $a_i + b_i > 0$ $(i = 1, \ldots, n)$ then

$$\prod_{i=1}^{n} (a_i^* + b_i^*) \le \prod_{i=1}^{n} (a_i + b_i) \le \prod_{i=1}^{n} (a_i^* + b_i'). \tag{1.2}$$

Motivated by these two inequalities, London [7] has proved a result which can be stated as follows:

(1.3) Let $\underset{\sim}{a}$ and $\underset{\sim}{b}$ have non-negative components. Then:

$$\text{(i)} \quad \sum_{i=1}^{n} f(1+a_i^* b_i') \le \sum_{i=1}^{n} f(1+a_i b_i) \le \sum_{i=1}^{n} f(1+a_i^* b_i^*)$$

whenever $f(e^x)$ is convex for $x \ge 0$ and $f(1) \le f(x)$ for $x \ge 1$.

$$\text{(ii)} \quad \sum_{i=1}^{n} f(a_i^* b_i') \le \sum_{i=1}^{n} f(a_i b_i) \le \sum_{i=1}^{n} f(a_i^* b_i^*)$$

whenever f is convex for $x \ge 0$ and $f(0) \le f(x)$ for $x \ge 0$. If f is strictly convex, then in either case we have equality on the left iff $(\underset{\sim}{a}\underset{\sim}{b})^* = (\underset{\sim}{a}^*\underset{\sim}{b}')^*$; we have equality on the right iff $(\underset{\sim}{a}\underset{\sim}{b})^* = \underset{\sim}{a}^*\underset{\sim}{b}^*$.

The choice $f(x) = x$ in (ii) gives (1.1). To get (1.2) when $\underset{\sim}{b}$ has positive components, use $f(x) = \log x$ in (i) and $1/\underset{\sim}{b}' = (1/\underset{\sim}{b})^*$.

A rearrangement inequality which is not a special case of London's theorem is the following one of Ruderman [13], which generalizes the left-hand inequality of (1.2):

$$\prod_{s=1}^{p} \sum_{k=1}^{n} a_{k,s} \ge \prod_{s=1}^{p} \sum_{k=1}^{n} a_{k,s}^* \qquad (1.4)$$

whenever $a_{k,s} > 0$ and $\underset{\sim}{a}_k = (a_{k,1}, \ldots, a_{k,p})$.

This inequality motivated G. G. Lorentz [8] to prove a general rearrangement theorem for non-negative bounded measurable functions on $]0,1[$. To state his theorem, we need the concept of decreasing rearrangement, which can be defined rather generally as follows.

Let (X,A,μ) be a finite measure space (m.s), and let $M = M(X,\mu)$ denote the extended real-valued measurable functions on X. If $f \in M$ then the <u>decreasing rearrangement</u> δ_f of f is defined by

$$\delta_f(t) = \inf\{s : \mu(\{x : f(x) > s\}) \le t\} \qquad 0 \le t \le a = \mu(X).$$

Also, if $E \in A$ we let 1_E denote the characteristic function of E.

Let $\varphi(x,u_1,\dots,u_n)$ be a continuous function on $]0,1[\times[0,\infty[^n$. Following Lorentz, in any inequality involving φ we will omit those variables which are the same for all occurrences of φ in the inequality. Thus

$$\varphi(x,u_1,\dots,s_k,\dots,u_n) \ge \varphi(x,u_1,\dots,r_k,\dots,u_n)$$

would simply be written $\varphi(s_k) \ge \varphi(r_k)$.

(1.5) THEOREM (Lorentz) . In order that we have

$$\int_0^1 \varphi(x, f_1(x), \ldots, f_n(x))\,dx \le \int_0^1 \varphi(x, f_1^*(x), \ldots, f_n^*(x))\,dx$$

for all non-negative bounded measurable functions $f_1, \ldots, f_n$ on $]0, 1[$ it is necessary and sufficient that the following two conditions be satisfied;

(1) $\varphi(u_i+h, u_j+h) - \varphi(u_i+h, u_j) - \varphi(u_i, u_j+h) + \varphi(u_i, u_j) \ge 0$

(2) $\int_0^\delta \Big[\varphi(x-t, u_i+h) + \varphi(x+t, u_i) - \varphi(x+t, u_i+h) - \varphi(x-t, u_i) \Big]\,dt \ge 0$

for all $0 < x < 1$, $u_k \ge 0$ $(k = 1, \ldots, n)$, $h > 0$, $0 < \delta < x$, $\delta < 1 - x$, and $i \ne j$. In addition, if φ has continuous partial derivatives, then (1) and (2) are equivalent, respectively, to

(1) $$\frac{\partial^2 \varphi}{\partial u_i \partial u_j} \ge 0$$

(2) $$\frac{\partial^2 \varphi}{\partial x \partial u_i} \le 0.$$

Ruderman's inequality (1.4) follows by taking $\varphi(u_1, \ldots, u_n) = -\log(u_1 + \ldots + u_n)$ and $f_k = \sum_{s=1}^{p} a_{ks}\, I_{](s-1)/p,\, s/p[}$, $k = 1, \ldots, n$.

Actually, Ruderman's inequality can be deduced without (1.5), using some theorems introduced by Hardy, Littlewood and Polya to handle inequalities involving rearrangements and convex functions. We may define them rather generally as follows.

Let (X_1, Λ_1, μ_1) also be a finite m.s. such that $\alpha = \mu_1(X_1) = \mu(X)$. If $f^+, g^+ \in L^1(X,\mu) \cup L^1(X_1,\mu_2)$ then $g \ll f$ means $\int_0^t \delta_g \leq \int_0^t \delta_f$ for all $0 \leq t \leq \alpha$, while $g < f$ means $g \ll f$ and $\int_0^\alpha \delta_g = \int_0^\alpha \delta_f$. In the discrete case these definitions become: $\underset{\sim}{b} \ll \underset{\sim}{a}$ iff $\Sigma_{i=1}^k b_i^* \leq \Sigma_{i=1}^k a_i^*$ for all $k = 1,\ldots,n$; $b < a$ iff in addition we have equality when $k = n$. If we write $f \sim g$ to mean $\delta_f = \delta_g$ then: $f \ll g$ and $g \ll f$ iff $f < g$ and $g < f$ iff $f \sim g$.

For example,

$$\underset{\sim}{a} + \underset{\sim}{b} < \underset{\sim}{a}^* + \underset{\sim}{b}^* \tag{1.6}$$

follows easily from (1.1), since there is a permutation σ of $\{1,\ldots,n\}$ such that $(\underset{\sim}{a}+\underset{\sim}{b})_i^* = (\underset{\sim}{a}+\underset{\sim}{b})_{\sigma(i)}$ and hence for $k = 1,\ldots,n$,

$$\sum_{i=1}^k (\underset{\sim}{a}+\underset{\sim}{b})_i^* = \sum_{i=1}^k a_{\sigma(i)} + \sum_{i=1}^k b_{\sigma(i)} \leq \sum_{i=1}^k a_i^* + \sum_{i=1}^k b_i^* = \sum_{i=1}^k (\underset{\sim}{a}^*+\underset{\sim}{b}^*)_i^* ,$$

with equality when $k = n$.

The preorder relation $<$ was first introduced in [4] where it was shown that

$$\sum_{i=1}^{n} H(b_i) \le \sum_{i=1}^{n} H(a_i) \qquad (1.7)$$

for every convex function H on an interval I with $a_i, b_i \in I$ iff $b < a$. Later it was shown that (1.7) holds for all increasing convex H iff $\tilde{b} \ll \tilde{a}$ [12, p. 164].

Using (1.6) and induction we easily deduce $\sum_{k=1}^{n} a_k < \sum_{k=1}^{n} a_k^*$ where $a_k = (a_{k1}, \ldots, a_{kp})$. If every sum is greater than 0, we may use $H(x) = -\log x$ in (1.7) and get Ruderman's inequality.

In the following section we will give a general theorem for discrete rearrangements which includes all of the inequalities above except (1.5), and which gives a necessary and sufficient condition for equality in each of these cases. In the final section these results are extended for real valued measurable functions on a finite measure space so that (1.5) is included.

2. The Discrete Case.

Let T be a partially ordered set. If $\tilde{a} = (a_1,\ldots,a_n) \in T^n$, then $\tilde{a}$ will be called a chain if $\{a_1,\ldots,a_n\}$ is totally ordered. If $\tilde{a}$ is a chain, then $\tilde{a}^*$ and $\tilde{a}'$ can be defined as in Section 1. If $\tilde{a}$ and $\tilde{b}$ are chains in a partially ordered abelian group G (written additively) then $\tilde{a} \ll \tilde{b}$ and $\tilde{a} < \tilde{b}$ can also be defined as in Section 1. It will be notationally simpler and should cause no confusion to denote every partial order under consideration by $\leq$. A partial order is understood to be anti-symmetric, and $x < y$ is used to mean $x \leq y$ and $x \neq y$.

We use the following notation in addition to that established in Section 1. Let $T_1,\ldots,T_m$ be partially ordered sets, let $\tilde{a}_k = (a_{k,1},\ldots,a_{k,n}) \in T_k^n$ $(k = 1,\ldots,m)$, let $\varphi : T_1 \times \ldots \times T_m \to G$, and let I and J be disjoint subsets of $K = \{1,\ldots,m\}$ with $L = K - (I \cup J)$. If $1 \leq i,j \leq n$, then $\varphi(a_{I,i},a_{J,j})$ is the function defined by $\varphi(a_{I,i},a_{J,j})(u_k : k \in L) = \varphi(c_1,\ldots,c_m)$ where $c_k = a_{k,i}$ for $k \in I$, $c_k = a_{k,j}$ for $k \in J$, and $c_k = u_k$ for $k \in L$. If $\tilde{b}_k = (b_{k,1},\ldots,b_{k,n}) \in T_k^n$ and $\{I,J\}$ is a partition of K, then $(\tilde{a}_I,\tilde{b}_J)$ is the sequence of vectors defined by $(\tilde{a}_I,\tilde{b}_J)_j = (c_1,\ldots,c_m)$ where $c_k = a_{k,j}$ for $k \in I$ and $c_k = b_{k,j}$ for $k \in J$. We simplify the notation slightly when

I or J is empty or a singleton, writing for example, $\underset{\sim}{a}_K$ or $(\underset{\sim}{a}_1,\ldots,\underset{\sim}{a}_m)$ in place of $(\underset{\sim}{a}_K,\underset{\sim}{b}_\emptyset)$.

We say two sequences $\underset{\sim}{s} = (s_1,\ldots,s_n) \in T_1{}^n$ and $\underset{\sim}{t} = (t_1,\ldots,t_n) \in T_2{}^n$ are <u>similarly ordered</u> if for every $1 \leq i,j \leq n$, $s_i < s_j$ implies $t_i \leq t_j$, and $t_i < t_j$ implies $s_i \leq s_j$. We say $\underset{\sim}{s}$ and $\underset{\sim}{t}$ are <u>oppositely ordered</u> if $s_i < s_j$ implies $t_j \leq t_i$, and $t_i < t_j$ implies $s_j \leq s_i$. In either case, if $\underset{\sim}{s}$ is a chain, it is equivalent to have only the first implication.

Let $\{I,J\}$ be a partition of $K = \{1,\ldots,m\}$. We consider the following two conditions on $\varphi: T_1 \times \ldots \times T_m \to G$.

(A) [(A*)] If $x_i, y_i \in T_i$ with $x_i < y_i$, and $k \neq i$, then $\varphi(y_i) - \varphi(x_i)$ is [strictly] increasing in u_k when k and i are in the same set I or J, and [strictly] decreasing in u_k when k and i are in different sets I and J, for all $1 \leq i,k \leq m$.

If $G = \mathbb{R}$, if each T_k is an open interval of $\mathbb{R}$, if the first partials of φ are continuous on $T_1 \times \ldots \times T_m$, and if the second partials of φ exist on $T_1 \times \ldots \times T_m$, then $y_i > x_i$ implies the difference in (A) is increasing [decreasing] in u_j $(j \neq i)$ iff $y_i > x_i$ implies $\frac{\partial\varphi}{\partial u_j}(y_i) - \frac{\partial\varphi}{\partial u_j}(x_i)$ is non-negative [non-positive] iff $\frac{\partial\varphi}{\partial u_j}$ is increasing [decreasing] in u_i. Hence condition (A) is equivalent to:

(A)' $\frac{\partial^2 \varphi}{\partial u_i \partial u_j}$ ≥ 0 when i and j are in same set I or J

≤ 0 when i and j are in different sets I and J

on $T_1 \times \ldots \times T_m$ for all $1 \leq i \neq j \leq m$.

(A*)' denotes the above condition with strict inequality. Clearly (A*)' implies (A*).

(2.1) THEOREM. <u>Let</u> $\varphi : T_1 \times \ldots \times T_m \to G$, <u>where each</u> T_k $(k = 1,\ldots,m)$ <u>is a partially ordered set, and</u> G <u>is a partially ordered abelian group</u>. <u>Let</u> $\{I,J\}$ <u>be a partition of</u> $K = \{1,\ldots,m\}$.

(i) <u>Condition</u> (A) <u>is necessary and sufficient that</u>

$$(1) \qquad \sum_{j=1}^{n} \varphi(\underset{\sim}{a}_1,\ldots,\underset{\sim}{a}_m)_j \leq \sum_{j=1}^{n} \varphi(\underset{\sim}{a}_I^*,\underset{\sim}{a}_J')_j$$

<u>for all chains</u> $\underset{\sim}{a}_k \in T_k^{\,n}$ $(k = 1,\ldots,m)$.

(ii) <u>Condition</u> (A*) <u>is necessary and sufficient for the following to be equivalent for all chains</u> $\underset{\sim}{a}_k \in T_k^{\,n}$.

(a) <u>Equality occurs in</u> (1).

(b) <u>a_p and a_q are similarly ordered whenever p and q are in the same set I or J, and oppositely ordered when p and q are in different sets I and J, for all $1 \leq p,q \leq m$.</u>

(c) $\varphi(a_1,\ldots,a_m) \sim \varphi(a_I^*,a_J')$.

(iii) <u>Suppose the range of φ is totally ordered. If φ satisfies condition (A) and is increasing [respectively decreasing] in u_k for $k\in I$ and decreasing [respectively increasing] in u_k for $k\in J$ then</u>

$$\varphi(a_1,\ldots,a_m) \ll \varphi(a_I^*,a_J'). \tag{2}$$

<u>for all chains $a_k \in T_k^n$</u> $(k = 1,\ldots,m)$.

<u>Proof</u>. To prove necessity of (A) for (1), let $1 \leq k,i \leq m$, let $x_i,y_i \in T_i$ with $x_i < y_i$, let $a_i = (x_i,y_i,\ldots,y_i)$, let $u_k,v_k \in T_k$ with $u_k < v_k$, and for $j \neq i,k$ let $u_j \in T_j$ and $a_j = (u_j,\ldots,u_j)$. Case 1: k,i are in the same set I or J. Let $a_k = (v_k,u_k,\ldots,u_k)$. After cancelling terms in (1) we obtain $\varphi(x_i,v_k) + \varphi(y_i,u_k) \leq \varphi(y_i,v_k) + \varphi(x_i,u_k)$, so $\varphi(y_i,u_k) - \varphi(x_i,u_k) \leq \varphi(y_i,v_k) - \varphi(x_i,v_k)$, and hence (A) is true in this case. Case 2: k,i are in different sets I and J.

Let $a_K = (u_K, v_K, \ldots, v_K)$. The proof is similar to Case 1. This completes the proof of necessity.

Before continuing we introduce some notation. For chains $a, b \in T_n$ write $b = S_{ij} a$ if $1 \leq i < j \leq n$ are such that for $P = \{k \in I : a_{k,i} < a_{k,j}\}$, $Q = \{k \in J : a_{k,i} > a_{k,j}\}$ and $L = P \cup Q$ we have: b_k for $k \in L$ is the sequence obtained from a_k by interchanging $a_{k,i}$ and $a_{k,j}$ while $b_k = a_k$ for other k.

Assume $b = S_{ij} a$ with P and Q as above, and let $\psi = \varphi(a_{P,i}, a_{Q,i}) - \varphi(a_{P,j}, a_{Q,j})$. Also for $0 \leq k \leq m$ let $P_k = P \cap \{0, \ldots, k\}$ and $Q_k = Q \cap \{0, \ldots, k\}$. Then

$$\psi = \sum_{k=0}^{m-1} \left[\varphi(a_{P,i}, a_{Q-Q_k,i}, a_{Q_k,j}) - \varphi(a_{P,i}, a_{Q-Q_{k+1},i}, a_{Q_{k+1},j}) \right]$$

$$+ \sum_{k=0}^{m-1} \left[\varphi(a_{P-P_k,i}, a_{P_k,j}, a_{Q,j}) - \varphi(a_{P-P_{k+1},i}, a_{P_{k+1},j}, a_{Q,j}) \right]$$

is a sum of differences like that in (A), so

$$(3) \qquad \psi(a_{I-P,i}, a_{J-Q,i}) \leq \psi(a_{I-P,j}, a_{J-Q,j}).$$

On writing it out, this is the same as

$$(4) \qquad \varphi(a_K)_i + \varphi(a_K)_j \leq \varphi(b_K)_i + \varphi(b_K)_j,$$ so

(5) $$\sum_{r=1}^{n} \varphi(a_R)_r \le \sum_{r=1}^{n} \varphi(b_K)_r .$$

If (A*) holds, inequality (3) and hence (5) will be strict unless P U Q = 0 or a_v . = a . . for all k e (I-P) U (J-Q) .

There are b(1), . . . , b(q) such that b(l) = a_R, b(q) = $(a_I^*, a_J^!)$ and for each $1 \le k \le m-1$ there are i and j such that $b(k+l) = S_{i,j} b(k)$. Hence

$$\Sigma_{j=1}^{n} \varphi(b(1))_j \le \cdots \le \Sigma_j \varphi(b(q))_j ,$$

which proves (1).

In (ii) it is clear that (b) => (c) => (a) always. We begin by assuming (A*) holds and show that (a) => (b) . Suppose (b) does not hold. Then an examination of cases shows there are $1 \le i < j \le n$ such that for P and Q as above we have P U Q / ^ and there is a k e (I-P) U (J-Q) such that a.. . ^ a.. .. Hence letting b__ = S. .a__ we have

$$\sum_{r=1}^{n} \varphi(a_K)_r < \sum_{r=1}^{n} \varphi(b_K)_r \le \sum_{r=1}^{n} \varphi(b^*, b')_r = \sum_{r=1}^{n} \varphi(a^*, a')_r ,$$

since $b_k^* = a_k^*$, k = l,...,m. Conversely if (a) => (b) then the arguments used in proving necessity of (A) for (1) show that (A*) holds.

We turn now to the proof of (iii). Since $\varphi(\underset{\sim}{a}_I^*, \underset{\sim}{a}_J^!) \sim \varphi(\underset{\sim}{a}_I^!, \underset{\sim}{a}_J^*)$, it suffices to prove (2) assuming φ is increasing in the I-variables and decreasing in the J-variables. In this case let $\underset{\sim}{b}_K = S_{i,j}\underset{\sim}{a}_K$. Then

$$(6) \qquad \varphi(\underset{\sim}{b}_K)_j \leq \varphi(\underset{\sim}{a}_K)_i, \quad \varphi(\underset{\sim}{a}_K)_j \leq \varphi(\underset{\sim}{b}_K)_i \; .$$

We call $\varphi(\underset{\sim}{a}_K)_i$ and $\varphi(\underset{\sim}{a}_K)_j$ the "old terms", and $\varphi(\underset{\sim}{b}_K)_i$ and $\varphi(\underset{\sim}{b}_K)_j$ the "new terms". These are the only terms where $\varphi(\underset{\sim}{a}_K)$ and $\varphi(\underset{\sim}{b}_K)$ differ

Let $1 \leq k \leq n$, define sequences

$$\underset{\sim}{\alpha} = (\varphi(\underset{\sim}{a}_K)_r^* : 1 \leq r \leq k), \qquad \underset{\sim}{\beta} = (\varphi(\underset{\sim}{b}_K)_r^* : 1 \leq r \leq k),$$

let $\Sigma\underset{\sim}{\alpha} = \sum_{r=1}^{k} \varphi(\underset{\sim}{a}_K)_r^*$ and define $\Sigma\beta$ similarly. We show that $\Sigma\underset{\sim}{\alpha} \leq \Sigma\underset{\sim}{\beta}$.

If exactly one of the old terms occurs in $\underset{\sim}{\alpha}$, then (6) implies that the only new term in $\underset{\sim}{\beta}$ is $\varphi(\underset{\sim}{b}_K)_i$. For if $\varphi(\underset{\sim}{b}_K)_j$ is in $\underset{\sim}{\beta}$, then (6) implies that $\underset{\sim}{\beta}$ contains both new terms, so there are n-k terms of $\varphi(\underset{\sim}{a}_K)$ which are $\leq$ $\varphi(\underset{\sim}{b}_K)_j$, in which case (6) implies that both old terms occur in $\underset{\sim}{\alpha}$. Hence $\underset{\sim}{\beta}$ is obtained from $\underset{\sim}{\alpha}$ by replacing an old term by the larger term cp(b) i . Thus Ta <. E£ .

If both old terms occur in a* then (4) implies their sum is < the sum of the new terms, which is <^ the sum of cp(b--) . and any term ;> cp(b--) . , in case cp(b--) _. is not in /3 . Hence Ea < E£ •

If none of the old terms occur in a , then either a = £ , or)3 is obtained from a by replacing one term of a by the larger term cp(b K) i . . Thus la ^ E3 . The proof of (iii) is finished as in (i). This completes the proof of the theorem.

When cp is a function of two variables, conditions (A) and (A*) simplify, and the arguments proving (2.1) have a symmetry which shows how small the sums can get.

(2.2) COROLLARY. Let cp : T, X T_o —*G.

(i) <u>A necessary and sufficient condition that</u>

(1) $$\sum_{j=1}^{n} cp(a^*_j, b^!_j) \leq \sum_{j=1}^{n} cp(a_j, b_j) \leq \sum_{j=1}^{n} cp(a_j^f, b_j^f)$$

<u>for</u> all chains aGT_n^n, beT_o^n is that the differenc<u>e</u>

(2) cp(d,y) - cp(c,y)

be increasing in yeT_o whenever d > *c*, d,ceT,. A <u>necessary</u>

and sufficient condition that for all chains $\underset{\sim}{a}$ *and* $\underset{\sim}{b}$ *the inequality* (1) *is strict on the left [right] unless* $\underset{\sim}{a}$ *and* $\underset{\sim}{b}$ *are oppositely [similarly] ordered is that the difference* (2) *is strictly increasing*.

(ii) *Suppose the range of* φ *is totally ordered. If in addition to* (i) φ *is increasing (or decreasing) in both variables, then*

$$\varphi(\underset{\sim}{a}^*,\underset{\sim}{b}') << \varphi(\underset{\sim}{a},\underset{\sim}{b}) << \varphi(\underset{\sim}{a}^*,\underset{\sim}{b}^*).$$

If (2) *holds with* φ *replaced by* $-\varphi$, *and* φ *is increasing in one variable and decreasing in the other, then*

$$\varphi(\underset{\sim}{a}^*,\underset{\sim}{b}^*) << \varphi(\underset{\sim}{a},\underset{\sim}{b}) << \varphi(\underset{\sim}{a}^*,\underset{\sim}{b}').$$

(2.3) REMARKS. (i) The condition (2.2.i.2) is equivalent to having $\varphi(x,d) - \varphi(x,c)$ increasing in $x \in T_1$ whenever $d > c$ and $d,c \in T_2$.

(ii) Since $\varphi(\underset{\sim}{a}^*,\underset{\sim}{b}^*) \sim \varphi(\underset{\sim}{a}',\underset{\sim}{b}')$ and $\varphi(\underset{\sim}{a}^*,\underset{\sim}{b}') \sim \varphi(\underset{\sim}{a}',\underset{\sim}{b}^*)$, the relations (2.2.ii) may be rewritten:

$$\varphi(\underset{\sim}{a}',\underset{\sim}{b}^*) << \varphi(\underset{\sim}{a},\underset{\sim}{b}) << \varphi(\underset{\sim}{a}',\underset{\sim}{b}'),$$

and similarly for the other.

(2.4) EXAMPLES. Let $G = \mathbb{R}$.

(i) $T_1 = T_2 = \mathbb{R}$ and $\varphi(x,y) = x + y$:

$$\underset{\sim}{a}^* + \underset{\sim}{b}' < \underset{\sim}{a} + \underset{\sim}{b} < \underset{\sim}{a}^* + \underset{\sim}{b}^*.$$

(ii) $T_1 = T_2 = \mathbb{R}$ and $\varphi(x,y) = x-y$:

$$\underset{\sim}{a}^* - \underset{\sim}{b}^* < \underset{\sim}{a} - \underset{\sim}{b} < \underset{\sim}{a}^* - \underset{\sim}{b}'.$$

(iii) $\varphi(x,y) = xy$: For $T_1 = T_2 = \mathbb{R}$

we obtain (1.1) with the indicated condition for equality.

For $T_1 = T_2 = [0,\infty[$ or $T_1 = T_2 =]-\infty,0]$ we obtain $\underset{\sim}{a}^*\underset{\sim}{b}' \ll \underset{\sim}{a}\underset{\sim}{b} \ll \underset{\sim}{a}^*\underset{\sim}{b}^*$ whenever $\underset{\sim}{a},\underset{\sim}{b} \in [0,\infty[^n$ or $a,b \in]-\infty,0]^n$. When $T_k = [0,\infty[$ $(k = 1,\ldots,m)$, $I = \{1,\ldots,m\}$ and $J = \emptyset$ then $\varphi(u_1,\ldots,u_m) = u_1 \ldots u_m$ satisfies (A*) and we obtain a companion to (1.4), also proved by Ruderman:

$$\sum_{j=1}^{n} \prod_{i=1}^{m} a_{i,j} \leq \sum_{j=1}^{n} \prod_{i=1}^{m} a^*_{i,j} .$$

The inequality is strict unless all of the sequences $\underset{\sim}{a}_k = (a_{k,1},\ldots,a_{k,n})$ are similarly ordered.

(iv) $\varphi(x,y) = \log(1+xy)$ satisfies (A*) with $I = \{1,2\}$, $J = \emptyset$ whenever $T_1 \times T_2 \subset \{(x,y): xy > -1\}$. Thus

$$\prod_{i=1}^{n}(1+a_i^*b_i') \le \prod_{i=1}^{n}(1+a_ib_i) \le \prod_{i=1}^{n}(1+a_i^*b_i^*)$$

whenever $a_i^*b_i' > -1$ for $i = 1$ and $i = n$. The inequality is strict except as indicated in (2.2.i). The choice $T_1 = T_2 = [0,\infty[$ or $]-\infty,0]$ gives:

$$\log(1+\underset{\sim}{a}^*\underset{\sim}{b}') \ll \log(1+\underset{\sim}{a}\underset{\sim}{b}) \ll \log(1+\underset{\sim}{a}^*\underset{\sim}{b}^*)$$

whenever $\underset{\sim}{a},\underset{\sim}{b} \in [0,\infty[^n$ or $]-\infty,0]^n$.

(v) $\varphi(x,y) = -\log(x+y)$, $T_1 \times T_2 \subset \{(x,y): x+y > 0\}$:

$$-\log(\underset{\sim}{a}^*+\underset{\sim}{b}') \ll -\log(\underset{\sim}{a}+\underset{\sim}{b}) \ll -\log(\underset{\sim}{a}^*+\underset{\sim}{b}^*)$$

whenever $a_n^* + b_n^* > 0$, and in particular we get (1.2) with the inequality strict except as indicated by (2.2.i). The example $\underset{\sim}{a} = (6,5,2,1)$ $\underset{\sim}{b} = (-3,-4,-2,1)$ shows that (1.2) may fail under the condition $a_i + b_i \ge 0$ for all i.

(vi) Suppose φ satisfies the hypotheses of (2.1.iii) and H is increasing and convex on an interval containing the range of φ. Then $\varphi_1 = H \circ \varphi$ satisfies condition (A). In this way (1.7) and (2.1.i) may be used to prove (2.1.iii). If in addition, φ satisfies (A*) and H is strictly convex, then φ_1 satisfies (A*).

We prove the first statement. Since H is convex, if $r_1 < s_1$, $r_2 < s_2$, $r_1 \neq r_2$, $s_1 \neq s_2$, then

$$\frac{H(r_2)-H(r_1)}{r_2 - r_1} \leq \frac{H(s_2)-H(s_1)}{s_2 - s_1}. \tag{1}$$

Assume φ is increasing in its I-variables and decreasing in its J-variables. Let $i,k \in I$, $y_i > x_i$, $v_k > u_k$.

We wish to prove

$$H(\varphi(y_i,u_k)) - H(\varphi(x_i,u_k)) \leq H(\varphi(y_i v_k)) - H(\varphi(x_i,v_k)). \tag{2}$$

Let $r_1 = \varphi(x_i,u_k)$, $r_2 = \varphi(y_i,u_k)$, $s_1 = \varphi(x_i,v_k)$, $s_2 = \varphi(y_i,v_k)$.

Now $r_2 - r_1 \leq s_2 - s_1$ and H is increasing, so (2) is obviously true if $r_2 = r_1$ or $s_2 = s_1$. Otherwise, we have $0 < r_2 - r_1 < s_2 - s_1$ and both terms in (1) are ≥ 0; multiplying, we obtain (2). The other cases are similar. When H is strictly convex, the same proof works except that all inequalities are strict.

(vii) London's Theorem follows from (2.2.i) and the previous example as follows. The conditions on f in (1.3) are equivalent to saying that in (ii) f and in (i) $f(e^x)$ is convex and increasing on $[0, \infty[$. We now use the previous example with $H = f$ and $\varphi(x,y) = xy$ for (ii) and $H(x) = f(e^x)$ and $\varphi(x,y) = \log(1+xy)$ for (i). If H is strictly convex, we also obtain the conditions for equality.

(viii) Ruderman [13] has observed that (1.4) generalizes the inequality between the arithmetic and geometric means. Using (2.1) we may obtain the following inequality for certain quasi-arithmetic symmetric means. Let U be an open interval of $\mathbb{R}$, let $f,g : U \to \mathbb{R}$ be strictly monotone and let $f \circ g^{-1}$ be convex on $g[U]$. If f is increasing then

$$g^{-1}\left(\frac{g(r_1)+\cdots+g(r_n)}{n}\right) \le f^{-1}\left(\frac{f(r_1)+\cdots+f(r_n)}{n}\right)$$

for all $r_1,\ldots,r_n \in U$, while if f is decreasing, the inequality reverses. If $f \circ g^{-1}$ is strictly convex, the inequality is strict unless $r_1 = \ldots = r_n$. To prove this, in (2.1.i.1) let $\underset{\sim}{a}_1 = (r_1, r_2, \ldots, r_{n-1}, r_n)$, $\underset{\sim}{a}_2 = (r_2, r_3, \ldots, r_n, r_1), \ldots,$ $\underset{\sim}{a}_n = (r_n, r_1, \ldots, r_{n-2}, r_{n-1})$ and note that

$$\varphi(u_1, \ldots, u_n) = f \circ g^{-1}\left(\frac{g(u_1)+\cdots+g(u_n)}{n}\right)$$

satisfies (A) with $I = \{1,\dots,n\}$. If $f \circ g^{-1}$ is strictly convex, then φ satisfies (A*), and the inequality is strict unless all the $\underset{\sim}{a}_k$ are similarly ordered, in which case $r_1 = \dots = r_n$.

3. The Continuous Case.

In this section we show how to generalize Theorems (2.1) and (2.2) for L^{∞} functions on a finite measure space when φ is jointly continuous. Let (X,Λ,μ) be a finite measure space with $\alpha = \mu(X) < \infty$, let R_f denote the essential range of $f \in L^{\infty}$ and let $I_f = [\text{ess. inf } f, \text{ ess. sup } f] = [\delta_f(\alpha-), \delta_f(0)]$. If $f_1,\dots,f_m \in L^{\infty}$ and $\varphi : R_{f_1} \times \dots \times R_{f_m} \longrightarrow \mathbb{R}$ is bounded, then the function $\varphi(f_1,\dots,f_m)$ defined by $x \mapsto \varphi(f_1(x),\dots,f_m(x))$ is in L^{∞}. If $f \in M(X,\mu)$ then ι_f denotes the <u>increasing rearrangement</u> of f which is defined by $\iota_f(t) = \delta_f((\alpha-t)-)$ for all $0 \leq t \leq \alpha$. If $\{I,J\}$ is a partition of $\{1,\dots,m\}$ then $(\delta_{\underset{\sim}{f}_I}, \iota_{\underset{\sim}{f}_J})$ denotes $(g_1,\dots,g_m)$ where $g_i = \delta_{f_i}$ for $i \in I$ and $g_i = \iota_{f_i}$ for $i \in J$.

We will say that $f,g \in M$ are <u>similarly</u> [<u>oppositely</u>] <u>ordered</u> if ess. sup $f|A <$ ess. inf $f|B$ implies ess. sup $g|A \leq$ ess. inf $g|B$ [ess. sup $g|B \leq$ ess. inf $g|A$] whenever A,B are disjoint measurable sets, each of positive measure. In particular, if f and g are similarly ordered, then for all $t \in \mathbb{R}$,

$$\text{ess. sup } g|\{f<t\} = \lim_{n\to\infty} \text{ ess. sup } g|\{f<t-\tfrac{1}{n}\} \le \text{ ess. inf } g|\{f\ge t\}.$$

Thus if f and g are similarly [oppositely] ordered, then

$$\text{ess. sup } g|A \le \text{ ess. inf } g|B \quad [\text{ess. sup } g|B \le \text{ ess. inf } g|A]$$

whenever $A \subset \{f<t\}$ and $B \subset \{f\ge t\}$. The numbers involved in these inequalities may be extended real numbers.

(3.1) THEOREM. Let $\varphi : T_1 \times \cdots \times T_m \to \mathbb{R}$ be continuous, where $T_1,\ldots,T_m$ are intervals of $\mathbb{R}$, and let $\{I,J\}$ be a partition of $\{1,\ldots,m\}$.

(i) If φ satisifes condition (A) then

$$\int \varphi(f_1,\ldots,f_m)\,d\mu \le \int_0^\alpha \varphi(\delta_{\underset{\sim}{f}_I}, \iota_{\underset{\sim}{f}_J}) \tag{1}$$

for all $f_i \in L^\infty$ such that $I_{f_i} \subset T_i$, $i = 1,\ldots,n$. If (X,Λ,μ) is non-atomic, then (A) is necessary for (1).

(ii) If φ satisfies (A*) then the following are equivalent:

(a) Equality holds in (1).

(b) f_i and f_j are similarly ordered whenever i and j are in the same set I or J, and oppositely ordered whenever i and j are in different sets I and J for all 1 ≤ ij j £ m.

(c) cp(f,, . . . 3£_) ~ cp(6^ , t -) .

(iii) IjE cp satisfies (A) and i^ increasing [respectively decreasing] in u_i for iel and decreasing [respectively increasing] for ieJ^ then for all f_i as in (i) we have

$$V(f_r...,fj \ll cp(6_{f_{\sim I}}, t_{f_{\sim J}}).$$

(3.2) COROLLARY. Let cp : T, x T_0 —» IR be continuous, where T_1 and T_0 are intervals of *M*, and let $f,g \in L^{QD}$ with $R_f \Leftarrow T_1$ and R_g c T_2.

(i) EP (1): cp(d^y) - cp(c^y) is increasing in $y \in T_2$ whenever d > c and d^ceT,$_1$, then

(2) $$J_0^\alpha cp(6_f, t_g) \lesssim J\ cp(f,g)djii\ 1\ J_0^\alpha cp(6_f, \delta_g).$$

If the monotonicity in (1) JLS strict, then the inequality (2) is strict on the left [right] unless f and g are oppositely [similarly] ordered.

(ii) <u>If in addition to</u> (i) φ <u>is increasing in both variables or decreasing in both variables, then</u>

$$\varphi(\delta_f, \iota_g) \ll \varphi(f,g) \ll \varphi(\delta_f, \delta_g).$$

(3.3) REMARK. If (*) $\varphi(d,y) - \varphi(c,y)$ is decreasing in $y \in T_2$ whenever $d > c$ and $d, c \in T_1$, or in addition φ is increasing in one variable and decreasing in the other, then apply (3.2) to $\varphi_1(x,y) = \varphi(x, r+s-y)$, f and $g_1 = r + s - g$ where $I_g = [r,s]$. The result is that the inequalities reverse:

(i) $$\int_0^{\alpha} \varphi(\delta_f, \delta_g) \leq \int \varphi(f,g)\,d\mu \leq \int_0^{\alpha} \varphi(\delta_f, \iota_g)$$

(ii) $$\varphi(\delta_f, \delta_g) \ll \varphi(f,g) \ll \varphi(\delta_f, \iota_g)$$

If the monotonicity in (*) is strict, the inequality on the left [right] is strict iff f and g are similarly [oppositely] ordered.

We begin by showing that it sufficies to prove (3.1) and (3.2) for non-atomic measure spaces by embedding (X, Λ, μ) in a non-atomic measure space $(X^\#, \Lambda^\#, \mu^\#)$, which we define as follows. Now $X = X_o \cup \bigcup_{n \in P} A_n$, where X_o is non-atomic, each A_n is an atom, $\mu(A_i \cap A_j) = 0$ when $i \neq j$, and $P = \{1, \ldots, p\}$ or $\{1,2,3,\ldots\}$. Let $I[a_n, b_n]$ $n \in P$ be disjoint intervals of $\mathbb{R}$

with end points a_n and b_n such that $b_n - a_n = \mu(A_n)$, and define $(X^{\#}, \Lambda^{\#}, \mu^{\#})$ to be the direct sum of $(X_o, \Lambda \cap X_o, \mu)$ and $(I[a_n,b_n], \lambda)$, $n \in P$, where λ is Lebesgue measure. If $f \in M(X,\mu)$ then f is constant μ- a.e. on each atom, and we define $f^{\#} = f1_{X_o} + \sum_{n\in P} (f|A_n) 1_{I[a_n,b_n]}$. Then $f^{\#} \sim f$ so $\delta_{f^{\#}} = \delta_f$ and $\iota_{f^{\#}} = \iota_f$.

Let $\varphi : T_1 \times \dots \times T_m \to \mathbb{R}$ and let $f_i \in M(X,\mu)$ with $R_{f_i} \subset T_i$ $(i = 1,\dots,m)$. Then

$$\varphi(f_1^{\#},\dots,f_n^{\#}) = \varphi(f_1,\dots,f_n)1_{X_o} + \sum_{n\in P} [\varphi(f_1,\dots,f_n)|A_n] 1_{I[a_n,b_n]}$$

$$= \varphi(f_1,\dots,f_n)^{\#} \sim \varphi(f_1,\dots,f_n).$$

In addition it is not hard to see that f and g are similarly [oppositely] ordered iff $f^{\#}$ and $g^{\#}$ are similarly [oppositely] ordered. Thus if (3.1) and (3.2) are true when (X,Λ,μ) is non-atomic, then they are true for any finite m.s.

Before proceeding with the proof when (X,Λ,μ) is non-atomic, we require some lemmas.

(3.4) LEMMA. <u>The following three statements are equivalent</u>.

(i) (X,Λ,μ) <u>is non-atomic</u>.

(ii) <u>There is a measure preserving map</u> $\sigma : X \to [0,\mu(X)]$.

(iii) <u>There is a map</u> $\phi : [0,\mu(X)] \to \Lambda$ <u>such that</u> $\mu(\phi(t)) = t$ <u>and</u> $t \leq u$ <u>implies</u> $\phi(t) \subset \phi(u)$.

Essentially, if σ is given, then σ is not constant on any set of positive measure. The maps σ and ϕ are related by $\phi(t) = \sigma^{-1}[0,t[$ and $\sigma(x) = \inf \{t : x \in \phi(t)\}$. For a construction of σ see [1, (3.1)].

(3.5) LEMMA. <u>Let</u> (X,Λ,μ) <u>be non-atomic. Suppose</u> $\{D_k\}_{k=1}^N$ <u>is a partition of</u> X <u>by measurable sets. If</u> $\epsilon > 0$, <u>then there is a partition</u> $\{E_i\}_{i=1}^n$ <u>of</u> X <u>by measurable sets such that</u> $\mu(E_i) = \mu(X)/n$ $(i = 1,\dots,n)$ <u>and</u> $\mu(\cup\{E_i : E_i$ intersects more than one $D_k\}) < \epsilon$.

<u>Proof</u>. Let $\alpha = \mu(X)$. If $\alpha = 0$, the lemma is trivially true. Otherwise, rename the sets D_k so that $\mu(D_k) = 0$ for $1 \leq k < p$ and $\mu(D_k) > 0$ for $p \leq k \leq N$. There is a $\phi : [0,\alpha]$ Λ such that $\mu(\phi(t)) = t$, $t \leq u$ implies $\phi(t) \subset \phi(u)$, $\phi(0) = \bigcup_{1 \leq k < p} D_k$, and $\phi(\sum_{1 \leq k \leq q} \mu(D_k)) = \bigcup_{1 \leq k \leq q} D_k$ for $q = p,\dots,N$. For any n such that $\alpha/n \leq \min\{\mu(D_k) : p \leq k \leq N\}$ and for $E_i = \phi(\alpha i/n) - \phi(\alpha(i-1)/n)$ $(i = 1,\dots,n)$ we have that

each E_i intersects at most two sets D_k of positive measure, and at most $N-1$ of these E_i intersect more than one D_k. To finish the proof, choose n so that also $a(N-1)/n < \epsilon$.

(3.6) LEMMA. *Suppose* (X, A, μ) *is non-atomic. Let* $\{s(k)_i\}_{i=1}^{\infty}$ $(k = 1, \ldots, m)$ *be* m *sequences of simple functions. Then there are* m *sequences* $\{t(k)_i\}_{i=1}^{\infty}$, $(k = 1, \ldots, m)$ *of simple functions such that*

(i) *For each* i, $t(1)_i, \ldots, t(m)_i$ *have the same sets of constancy, and these sets have equal measure;*

(ii) *For each* $k = 1, \ldots, m$, $s(k)_i - t(k)_i \to 0$ μ-a.e. *as* $i \to \infty$;

(iii) *For each* $k = 1, \ldots, m$ *and* $i \geq 1$, $|t(k)_i| \leq |s(k)_i|$.

Proof. For clarity of exposition, we prove the lemma in the case $m = 2$. The proof for larger m will be readily apparent. Before considering sequences, let $s(1) = \sum_{i=1}^{n} a_i 1_{A_i}$ and $s(2) = \sum_{j=1}^{p} b_j 1_{B_j}$ where $\{A_i\}$ and $\{B_j\}$ partition X, and let $\{I_k\}_{k=1}^{N} = \{A_i \cap B_j : 1 \leq i \leq n, \ 1 \leq j \leq p\}$. Let $\epsilon > 0$. Then there is a measurable partition $\{E_q\}_{q=1}^{r}$ as in Lemma (3.5).

For each $q = 1,\ldots,r$, if E_q intersects only $A_i \cap B_j$ then $E_q \subset A_i \cap B_j$, and for $k = 1,2$ we define $t(k)|E_q = s(k)|A_i \cap B_j$; we define $t(k) = 0$ elsewhere. Then $|t(k)| \leq |s(k)|$ and $\mu(\{s(k) \neq t(k)\}) < \epsilon$. Hence given $\{s(k)_i\}_{i=1}^{\infty}$ there are sequences $\{t(k)_i\}_{i=1}^{\infty}$ satisfying (i) and (iii) such that $\mu(\{s(k)_i \neq t(k)_i\}) < 2^{-i}$. Then

$$\mu(\{s(k)_i - t(k)_i \not\to 0\}) = \mu\Big(\bigcup_{q=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{i=N}^{\infty} \{|s(k)_i - t(k)_i| > 1/q\}\Big)$$

$$\leq \lim_{q\to\infty} \lim_{N\to\infty} \sum_{i=N}^{\infty} 2^{-i} = 0,$$

and the proof is finished.

(3.7) PROPOSITION. Suppose (X,Λ,μ) is non-atomic, let $\{I,J\}$ be a partition of $\{1,\ldots,m\}$ and let $f_1,\ldots,f_m \in M(X,\mu)$. Then the following two conditions are equivalent.

(i) f_i and f_j are similarly ordered if i and j are in the same set I or J, and oppositely ordered if i and j are in different sets I and J.

(ii) There is a measure preserving $\sigma : X \to [0,\alpha]$ such that $\delta_{f_i} \circ \sigma = f_i$ μ-a.e. if $i \in I$, and $\iota_{f_j} \circ \sigma = f_j$ μ-a.e. if $j \in J$.

<u>Proof</u>, (ii) $\Rightarrow$ (i): Let i and j be in the same set I or J and let i and k be in different sets I and J. Suppose A and B are disjoint sets of positive measure such that ess. sup $f_i|A <$ ess. inf $f_i|B$. There are $A_i \subset A$ and $B_i \subset B$ such that $\mu(A \setminus A_i) = 0 = \mu(B \setminus B_i)$, ess. sup $f_i|A =$ sup $f_i|A_i$ and ess. inf $f_i|B =$ inf $f_i|B_i$. Then te<y$[A_i]$ and U£g$[B_i]$ implies $\delta_{f_i}(t) < \delta_{f_i}(u)$ so $u < t$ and hence $\delta_{f_j}(t) \leq \delta_{f_j}(u)$. Thus

$$\text{ess. sup } f_j|A = \text{ess. sup } f_j|A_i \leq \sup f_j|A_i \leq \inf f_j|B_i$$
$$\leq \text{ess. inf } f_j|B_i = \text{ess. inf } f_j|B.$$

Similarly, we get ess. sup $f_k|B \leq$ ess. inf $f_k|A$, which proves the result.

(i) $\Rightarrow$ (ii): We introduce some notation for this part of the proof. If $g, h \in M(X, \mu)$, let

$$|g-h|_\mu = \inf \{s + \mu(\{|g-h| > s\}) : s > 0\},$$

the metric of convergence in μ-measure [2, p. 102]. We let $|\cdot|_\lambda$ denote the metric of convergence in Lebesgue measure on $[0,\alpha]$.

For each $n = 1,2,3,\ldots$ let $U_n = \{U_{n,i}\}_{i}$ be an enumeration of the finite collection of intervals

$\{[n, +\infty[, \;]-\infty, -n[\} \cup \{[(k-1)2^{-n}, k2^{-n}[\; : -n2^n+1 \le k \le n2^n\}$,

and for each i let $u_{n,i} \in U_{n,i} \cap \mathbb{R}$. For each $h \in M(X,\mu)$ let $h_n = \sum_i u_{n,i} 1_{h^{-1}(U_{n,i})}$. Then $h_n \to h$ pointwise μ-a.e., so $\delta_{h_n} \to \delta_h$ pointwise a.e. [14, p. 508, (18.21)]. The intervals of U_n (being disjoint) are ordered in the obvious way: for any two of them, the larger is the one whose points are all greater than those of the other. Then $(U^*_{n,i})_{i \ge 1}$ and $(U^\dagger_{n,i})_{i \ge 1}$ denote as usual these intervals in decreasing and increasing order, respectively. Clearly $u^*_{n,i} \in U^*_{n,i}$ and $u^\dagger_{n,i} \in U^\dagger_{n,i}$.

Finally, if A and B are disjoint sets of positive measure and $s = \text{ess. sup } h|A \le \text{ess. inf } h|B = t$, then $\text{ess. sup } h_n|A \le \text{ess. inf } h_n|B$ for all $n \ge 1$. Indeed, we have $s \in U^\dagger_{n,i}$ and $t \in U^\dagger_{n,j}$ for some $i \le j$, so $\text{ess. sup } h_n|A = u^\dagger_{n,i} \le u^\dagger_{n,j} = \text{ess. inf } h_n|B$.

We will illustrate the proof of (i) $\Rightarrow$ (ii) in the case $m = 3$, $I = \{1,2\}$ and $J = \{3\}$. The general method of proof will then be evident. For convenience let $f = f_1$, $g = f_2$, $h = f_3$, and let sequences (f_n), (g_n), (h_n) be defined as above.

Let $A_{ni} = f^{-1}(U^\dagger_{n,i})$, $B_{ni} = g^{-1}(U^\dagger_{n,i})$, and $C_{ni} = h^{-1}(U^\dagger_{n,i})$.

In addition, let $\mathcal{D}_n = \{D_{n,i}\}_{i=1}^{\ell_n}$ be a sequence of partitions of X such that for each n, $\mathcal{D}_{n+1}$ is a refinement of $\mathcal{D}_n$, and $\mu(D_{n,i}) \leq \alpha/n$ for all $1 \leq i \leq \ell_n$. Finally, let

$$a(i,j,k,p)_n = \sum_{1\leq q<i} \mu(A_{n,q}) + \sum_{1\leq q<j} \mu(A_{n,i} \cap B_{n,q}) +$$

$$+ \sum_{1\leq q<k} \mu(A_{n,i} \cap B_{n,j} \cap C_{n,q}) +$$

$$+ \sum_{1\leq q<p} \mu(A_{n,i} \cap B_{n,j} \cap C_{n,k} \cap D_{n,q});$$

let $V_n(i,j,k,p) = [a(i,j,k,p)_n,\ a(i,j,k,p+1)_n[$; for all i,j,k,p such that $V_n(i,j,k,p)_n \neq \emptyset$ let $\sigma_n : A_{n,i} \cap B_{n,j} \cap C_{n,k} \cap D_{n,p} \to V_n(i,j,k,p)$ be measure preserving, and let $\mathcal{V}_n = \{V_n(i,j,k,p) : i,j,k,p \geq 1\}$. Since

$$\sigma_n : A_{n,i} \to [\sum_{1\leq q<j} \mu(A_{n,q}), \sum_{1\leq q\leq j} \mu(A_{n,q})[$$

whenever $\mu(A_{n,i}) \neq 0$ we have $f_n = \delta_{f_n} \circ \sigma_n$ μ-a.e.

In view of the remarks preceding (3.1) we have:

ess. sup $g|A_{n,i+1} \leq$ ess. inf $g|A_{n,i}$ and

ess. sup $h|A_{n,i} \cap B_{n,j} \leq$ ess. inf $h|A_{n,i} \cap B_{n,j+1}$

whenever these sets have non-zero measure, so the same inequalities hold with g and h replaced by g_n and h_n respectively. The first inequality implies that $g_n = \Sigma_j u^*_{n,j} 1_{A_{n,i_j} \cap B_{n,j}}$ for some increasing sequence (i_j) having $i_1 = 1$. Hence as with f_n, $\delta_{g_n} \circ \sigma_n = g_n$ μ-a.e. Similarly, $\iota_{h_n} \circ \sigma_n = h_n$ μ-a.e.

Now for all $q \geq 0$ we have by construction that $\mathcal{V}_{n+q}$ refines $\mathcal{V}_n$, and $\sigma_n(x)$ and $\sigma_{n+q}(x)$ lie in the same member of $\mathcal{V}_n$. Hence $|\sigma_n(x) - \sigma_{n+q}(x)| \leq \frac{\alpha}{n}$ μ-a.e., and thus there is a $\sigma : X \rightarrow [0,\alpha]$ such that $\sigma_n \rightarrow \sigma$ μ-a.e. Then $\delta_{\sigma_n} \rightarrow \delta_\sigma$ a.e. so σ is measure preserving.

Let G be the set of points of discontinuity of δ_f on $[0,\alpha]$. Then G is at most countable so $\mu(\sigma^{-1}(G)) = 0$, and hence $\delta_f \circ \sigma_n \rightarrow \delta_f \circ \sigma$ pointwise μ-a.e. Now for finite measure spaces, pointwise convergence implies convergence in measure, and $|\delta_{f_n} \circ \sigma_n - \delta_f \circ \sigma_n|_\mu = |\delta_{f_n} - \delta_f|_\lambda$, so

$$|\delta_{f_n} \circ \sigma_n - \delta_f \circ \sigma|_\mu \leq |\delta_{f_n} - \delta_f|_\lambda + |\delta_f \circ \sigma_n - \delta_f \circ \sigma|_\mu \rightarrow 0$$

and hence $f_n = \delta_{f_n} \circ \sigma_n \rightarrow \delta_f \circ \sigma$ in measure. Then a subsequence of f_n converges to $\delta_f \circ \sigma$ pointwise a.e. so $f = \delta_f \circ \sigma$. Similarly, $g = \delta_g \circ \sigma$ and $h = \iota_h \circ \sigma$ μ-a.e., completing the proof.

We can now prove (3.1) and (3.2). For clarity of exposition we will only present a proof of (3.2). The proof of (3.1) will then be clear. With regard to (3.1.ii) we remark that (3.7) shows that (b) $\Longrightarrow$ (c) $\Longrightarrow$ (a) always. The proof of (3.2) will illustrate the proof of (a) $\Longrightarrow$ (b) when $m = 2$.

<u>PROOF OF (3.2)</u>. Recall that $\alpha = \mu(X)$. Let $v = \sum_{j=1}^{n} a_j 1_{E_j}$ and $w = \sum_{j=1}^{n} b_j 1_{E_j}$ where $R_v \subset T_1$, $R_w \subset T_2$ and $\mu(E_j) = \alpha/n$ $(j = 1,\ldots,n)$. Then $\delta_v = \sum_{j=1}^{n} a_j^* 1_{[(j-1)\alpha/n,\ j\alpha/n[}$ and δ_w, ι_v, ι_w have similar expressions. When $t = k\alpha/n$, $0 \leq k \leq n$ we have

$$\int_0^t \delta_{\varphi(\delta_v,\iota_w)} = \int_0^t \sum_{j=1}^{n} \varphi(\underset{\sim}{a}^*,\underset{\sim}{b}')_j^* 1_{[(j-1)\alpha/n,\ j\alpha/n[}$$

$$= \frac{\alpha}{n} \sum_{j=1}^{k} \varphi(\underset{\sim}{a}^*,\underset{\sim}{b}')_j^*$$

and similar expressions for $\int_0^t \delta_{\varphi(v,w)}$ and $\int_0^t \delta_{\varphi(\delta_v,\delta_w)}$. In particular $t = \alpha$ gives expressions of this type for $\int_0^\alpha \varphi(\delta_v,\iota_w)$, $\int \varphi(v,w)\,d\mu$ and $\int_0^\alpha \varphi(\delta_v,\delta_w)$. In case (i), (2.2.i) gives

$$(*) \qquad \int_0^\alpha \varphi(\delta_v,\iota_w) \leq \int \varphi(v,w)\,d\mu \leq \int_0^\alpha \varphi(\delta_v,\delta_w)$$

while in case (ii), (2.2.ii) gives for $t = k\alpha/p$

$$(**) \qquad \int_0^t \delta_{\varphi(\delta_v, \iota_w)} \leq \int_0^t \delta_{\varphi(v,w)} \leq \int_0^t \delta_{\varphi(\delta_v, \delta_w)} .$$

Now in (**) each of the integrands is constant on each of the intervals $[(j-1)\alpha/n, j\alpha/n[$, so the integrals are linear functions of t on these intervals, and hence (**) holds for all $0 \leq t \leq \alpha$. When $|v| \leq |f|$ and $|w| \leq |g|$, then each of the integrands in (*) and (**) is bounded by a constant which depends only on f and g, because φ is bounded on $I_f \times I_g$ and $R_v \times R_w \subset I_f \times I_g$. Using now (3.6) there are sequences v_i and w_i of simple functions like v and w above such that $v_i \to f$, $w_i \to g$, $|v_i| \leq |f|$ and $|w_i| \leq |g|$, so $\delta_{v_i} \to \delta_f$ and $\delta_{w_i} \to \delta_g$ a.e. Taking limits and using the dominated convergence theorem, we have that (*) or (**) holds with v and w replaced by f and g respectively.

We now show the condition for equality on the right in (3.2.i). Assume φ satisfies (A*), suppose f and g are not similarly ordered, and we will show that the inequality on the right is strict. There are disjoint sets A and B of positive measure such that ess. sup $f|A <$ ess. inf $f|B$ and $t =$ ess. sup $g|A >$ ess. inf $g|B = r$. Let $r < s_1 < s_2 < t$ and let $D \subset \{x \epsilon A : g(x) \geq s_2\}$ and $E \subset \{x \epsilon B : g(x) \leq s_1\}$ with

$0 < \mu(D) = \mu(E) = \beta$. Then let $\sigma_D : D \to [0,\beta[$ and $\sigma_E : E \to [0,\beta[$ be measure preserving and define

$$f' = \begin{cases} \delta_{f|D} \circ \sigma_D & \text{on } D \\ \delta_{f|E} \circ \sigma_E & \text{on } E, \\ f & \text{elsewhere} \end{cases} \qquad g' = \begin{cases} \delta_{g|E} \circ \sigma_D & \text{on } D \\ \delta_{g|D} \circ \sigma_E & \text{on } E. \\ g & \text{elsewhere} \end{cases}$$

Then $f' \sim f$, $g' \sim g$, ess. sup $f|D <$ ess. inf $f|E$, ess. sup $g|E <$ ess. inf $g|D$, and for all $0 \leq u \leq \beta$ we have $\delta_{f|D}(u) < \delta_{f|E}(u)$ and $\delta_{g|E}(u) < \delta_{g|D}(u)$. Hence

$$\int_D \varphi(f,g)d\mu + \int_E \varphi(f,g)d\mu \leq \int_0^\beta \varphi(\delta_{f|D},\delta_{g|D}) + \varphi(\delta_{f|E},\delta_{g|E}) <$$

$$< \int_0^\beta \varphi(\delta_{f|D},\delta_{g|E}) + \varphi(\delta_{f|E},\delta_{g|D}) =$$

$$= \int_D \varphi(f',g')d\mu + \int_E \varphi(f',g')d\mu.$$

Adding $\int_{X-(D\cup E)} \varphi(f,g)d\mu = \int_{X-(D\cup E)} \varphi(f',g')d\mu$ we obtain

$$\int \varphi(f,g)d\mu < \int \varphi(f',g')d\mu \leq \int_0^\alpha \varphi(\delta_{f'},\delta_{g'}) = \int_0^\alpha \varphi(\delta_f,\delta_g) ,$$

and the proof is finished.

REMARK. Depending on the choice of cp and the intervals T_i, Theorems (3.1) and (3.2) may hold for a larger set of functions than L^{∞}. Indeed, in the proof of (3.2), to get inequalities (1) or (2) we only needed to be able to interchange limit and integral in equations (*) and (**) respectively. To get the condition for equality, we only needed to know that if equation (3.2.1) holds for f and g then it also holds for f|A and g|A whenever A *e* A-

For example, suppose $f_1, \ldots, f_m e h^1$ implies $cp(f_1, \ldots, f_m) GL^1$«. Now it follows from [10, p. 93] that $|v| \leq |f|$ implies $|6_v| \leq |6_f|$ and $|t_v| \leq |t_f|$, so we may use [3] and the dominated convergence theorem to conclude that (3.1.1) and (3.2.1) hold for all L^1 functions. Finally since $f_1, \ldots, f_m eL^1$ implies $f_1|A, \ldots, f_m|AeL^1$ the condition for equality also holds for all L^1 functions. Other illustrations appear in the following examples.

(3.8) EXAMPLES.

(i) $6_f + t_g < f + g < 6_f + 6_g$ for all f,geL^1.

(ii) $6_f - 6_g < f - g < 6_f - i_g$ for all f,geL^1.

The (i) and (ii) are easily seen to be equivalent using [10,p. 93]. While $6_{f+g} < 6_f + 6_g$ is well-known (see [10, p. 108]), the fact that $6_f - 6_g$ •< f - g is new. In [10,p. 107] it is proved that: g < feL ^{1} implies |g| << |f|. Hence I6$_f$ - 6$_g$I « If - $g\backslash$, generalizing [9, Prop. 1, p. 34]. It then follows that \\f_R - f||,-% 0 implies ||6- - 6j|, -* 0,

p -L r^ r x

where {f$^{\beta}$} is a net. Using [10, (9.1), p. 103], the inequality 6$_f$ - 6^g < f - g can be written equivalently:

$$\int_E {}^6f + J_E^r \gg_g(«-t)dtij_0^{m(E)} 6_{f+g}$$

for all Lebesgue measurable E c [0,a], where m denotes Lebesgue measure. This is an interesting generalization of [10,(10.1)].

(iii) f_0^{α} 6.1 < ffg du < $\backslash_0^{\alpha}$ 6^6 holds for all £,g∈L^{∞}.

If 0 <. f,geM, we may approximate f and g by non-negative simple functions and use monotone convergence to show that it holds for these f and g also. By decomposing f,geM into their positive and negative parts, this inequality can, as in [10, p. 102], be shown to hold for all f,geM such that 6i$_f$|6i|g| e L^1 [0,a]. The inequalities are strict except as indicated in (3.2). Similarly, 6$_{fg}$ « fg « 6$_f$6$_g$ for all 0 <. f,geM or 0 > f,geM such that 6$_f$6$_g$ € L^1[0,a].

(iv) (1) J_0^{α} log(l + 6$_f$l$_g$) _< J log(l + fg)d$_M$ < J_0^{α} log(l + 6$_f$6$_g$)

holds for all f^eL^{00} satisfying

(2) both 6$_f$(0)t$_g$(o) > -1 and 6$_f$(a-)t$_g$(a-) > -1,

because (2) is equivalent to: $I_f \times I_g \subset \{(x,y) : xy > -1\}$. In addition, using monotone convergence, (1) can be shown to hold if $0 \leq f, g \in M$ or $0 \geq f, g \in M$. Then (1) can be shown to hold for all $f, g \in M$ satisfying (2) using the following observations. First, $\log(1+fg) = \log(1+f^+g^+) + \log(1-f^+g^-) + \log(1-f^-g^+) + \log(1+f^-g^-)$. Next, when (2) holds for the pair f,g it also holds for each of the pairs: f^+, g^+; $f^+, -g^-$; $-f^-, g^+$; $-f^-, -g^-$. Finally, when (2) holds, then: f unbounded above implies $g \geq 0$; f unbounded below implies $g \leq 0$; and the same is true when f and g are interchanged. Clearly if $f, g \in M$ satisfy (2) so do $f|A$ and $g|A$ for any $A \in \Lambda$. Hence the inequalities are strict except as indicated in (3.2).

Similarly, $\log(1+\delta_f \iota_g) \ll \log(1+fg) \ll \log(1+\delta_f \delta_g)$ for all $0 \leq f, g \in M$ or $0 \geq f, g \in M$ such that $\log(1+\delta_f \delta_g) \in L^1[0,\alpha]$.

(v) (1) $\int_0^\alpha \log(\delta_f + \delta_g) \leq \int \log(f+g)\,d\mu \leq \int_0^\alpha \log(\delta_f + \iota_g)$

for all $f, g \in L^\infty$ such that

(2) $\delta_f(\alpha-) + \delta_g(\alpha-) > 0$,

since (2) is equivalent to $I_f \times I_g \subset \{(x,y) : x+y > 0\}$. Actually, (1) holds for all $f, g \in M$ satisfying (2) since f and g are then bounded below, so we may approximate them by increasing sequences of bounded functions satisfying (2) and use the B. Levi Monotone Convergence theorem [6, p. 172]. The inequalities are strict except as directed in (3.3). Similarly, if $f, g \in M$ satisfy (2) and $\log(\delta_f + \iota_g) \in L^1[0,\alpha]$ then $-\log(\delta_f + \iota_g) \ll -\log(f+g) \ll -\log(\delta_f + \delta_g)$.

(vi) We have the following continuous version of London's Theorem. Suppose $0 \leq f, g \in M$ or $0 \geq f, g \in M$.

(1) If H is convex, increasing and continuous on $[0,\infty[$ then

$$\int_0^\alpha H(\delta_f \iota_g) \leq \int H(fg)\,d\mu \leq \int_0^\alpha H(\delta_f \delta_g).$$

(2) If $H(e^x)$ is convex, increasing and continuous on $[0,\infty[$ then

$$\int_0^\alpha H(1+\delta_f \iota_g) \leq \int H(1+fg)\,d\mu \leq \int_0^\alpha H(1+\delta_f \delta_g).$$

In either case, if H is strictly convex, then we have equality on the left [right] iff f and g are oppositely [similarly] ordered iff $\delta_f \iota_g \sim fg$ $[\delta_f \delta_g \sim fg]$.

(vii) Theorem (3.1) gives a necessary and sufficient condition for the inequality in Lorentz' Theorem (1.5) which simplifies his condition (1.5.2). In (3.1) take $X =]0,1[$ with

Lebesgue measure, $T_1 = [0,1]$, $T_i = [0,\infty[$ $(i = 2,\dots,m)$, $J = \{1\}$, $I = \{2,\dots,m\}$ and $f_1(x) = x$.

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