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Abstract

The structure that we shall describe here is a dual version of the Leray spectral sequence of a covering (Cf. [4], p. 212) but is not extensive enough to satisfy the precise definition of "spectral sequence." Yet, it does enable one to derive a functorial spectral sequence that relates the (generalized) homology sheaf of a space to the homology of the space. (Cf. [2] for an earlier attempt to obtain such a spectral sequence.)

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Let X be a space with base point $* \in X$. For some finite set $\mathcal{J}$ let $\mathfrak{X} = (X_s \mid s \subset \mathcal{J})$ be a family of subspaces of X, each containing $*$. Assume that $\mathfrak{X}$ is <u>covariant</u>, i.e. that an inclusion $s \subset s'$ implies an inclusion $X_s \subset X_{s'}$. We then have the notion of an alternating non-degenerate n-cochain ξ $(n \geq 0)$ of $\mathfrak{X}$ with coefficients in π_q $(q \geq 2)$: such a ξ is any family $\{\xi^{i_*} \mid i_* = (i_0, \ldots, i_n) \in \mathcal{J}^{n+1}\}$ such that (i) each $\xi^{i_*} \in \pi_q(X_{i_*})$, (ii) the condition $\xi^{i_*} = 0$ holds if $i_* = (i_0, \ldots, i_n)$ contains a repetition, (iii) the condition $\xi^{i'_*} = (\text{sign } \sigma)\xi^{i_*}$ holds if σ is any permutation map $\{0,1,\ldots,n\} \to \{0,1,\ldots,n\}$ and $i'_* = (i_{\sigma(0)}, \ldots, i_{\sigma(n)})$. We have also the coboundary $\delta\xi$ of ξ, which is defined as the $(n+1)$-cochain such that

$$(\delta\xi)^{i_*} = \sum_{t=0} (-1)^t \, \xi^{i_*(t)}\Big|_{X_{i_*}},$$

where $i_* \in \mathcal{J}^{n+2}$ and $i_*(t) = (i_0, i_1, \ldots, i_{t-1}, i_{t+1}, \ldots, i_{n+1})$. Denote the function group of all n-cochains as $C^n(\mathfrak{X}; \pi_q)$.

In the linear space $\bigoplus_{i \in \mathcal{J}} \mathbb{R}^{\{i\}} = \mathbb{R}^{\mathcal{J}}$ denote the standard basis somehow, say as $\{e_i \mid i \in \mathcal{J}\}$, and for each subset $s \subset \mathcal{J}$ denote the convex hull of the corresponding set (e.$_1$| i e s) as As. Regard Ac9 as a simplicial complex with vertex set (e$_i$|ie c9}. Each pair (K,L) of subcomplexes of Ac? provides a function space

3S(K,L) = {cp:K->x| <pL = *; for each As c K$_5$ (pAs c X$_s$ }•

We consider the groups *rr*$_q$I5(K,L) (c£>_l) .

<u>Proposition</u> 1. If (K,L,M) is a triple of subcomplexes of △J, then 3(K,M) is a fibre space over 3(L,M) with fibre 5J(K,L).

<u>Proof</u>. The assertion is that the transformation

6 : 3 (K,M) ⟶ 𝔉 (L,M)

of each <p€3(K,M) to its restriction *tp* is a map with the homotopy lifting property, while the kernel of 9 is 3(K,L). The latter is obvious, as is continuity of 9. Now^ since <J> is finite, 0 is the composite of the maps

θ_n: 𝔉 (K_n∪L,M) ⟶ 𝔉 (K_{n-1}∪L,M) .

Each has the homotopy lifting property, according to the following argument: Suppose Y is any space, together with a map H$_Q$: Y-*3(K$_n$UL,M) and a homotopy h$_t$: Y -* 5 (K^j^U'L^M) (O^t^l) such that h$_o$ = 9$_n$H$_o$. Regard these, respectively, as a map H^ : Y X (K$_n$UL)-*X and homotopy h^ : Y X (K$_{n-1}$UL)~>X (O^t^l) . For each n-simplex As of K not in L there are H$^?_{°|YXAS}$ and h!$_{r|YXA\Delta s}$ (Q^t^l) , which take values in at most X$_s$, so by the Homotopy Extension Property for (YXAS,YX5AS) they extend to form a homotopy

H$_t^s$: Y X AS-VX_ (O≤t≤1) .

Doing this for each n-simplex Δs in K and not in L, and extending to agree with $\{h'_t\}$, we define a homotopy

$$H'_t : Y \times (K_n \cup L) \to X \quad (0 \leq t \leq 1),$$

which is the same as the required

$$H_t : Y \to \mathfrak{F}(K_n \cup L, M) \quad (0 \leq t \leq 1).$$

So, θ_n, and therefore θ, has the homotopy lifting property. □

Corollary 1.1. For each such triple (K,L,M) there is an exact sequence of base-pointed sets

$$\cdots \pi_{q+1}\mathfrak{F}(L,M) \xrightarrow{\partial} \pi_q \mathfrak{F}(K,L) \to \pi_q \mathfrak{F}(K,M) \cdots \to \pi_0 \mathfrak{F}(L,M).$$

Corollary 1.2. There is essentially a spectral sequence $\{E^n_{r;q}; d^n_{r;q} : E^n_{r;q} \to E^{n+r}_{r;q-1}\}$ with $E^n_{r;q}$ defined for all $n \in Z$, all $r \geq 1$, all $q \geq 2$, and with $d^n_{r;q}$ defined for all $n \in Z$, all $r \geq 1$, all $q \geq 3$. The formulas are:

(i) $E^n_{r;q}$ = the homology subquotient of the half-exact sequence

$$\pi_{q+1}\mathfrak{F}(\Delta\mathcal{J}_{n-1}, \Delta\mathcal{J}_{n-r}) \xrightarrow{\partial} \pi_q \mathfrak{F}(\Delta\mathcal{J}_n, \Delta\mathcal{J}_{n-1}) \xrightarrow{\partial} \pi_{q-1}\mathfrak{F}(\Delta\mathcal{J}_{n+r-1}, \Delta\mathcal{J}_n);$$

(ii) $d^n_{r;q}$ is induced by the additive relation

$$\begin{array}{ccc}
\pi_q \mathfrak{F}(\Delta\mathcal{J}_n, \Delta\mathcal{J}_{n-1}) & \xrightarrow{\partial} & \pi_{q-1}\mathfrak{F}(\Delta\mathcal{J}_{n+r}, \Delta\mathcal{J}_n) \\
& \searrow & \uparrow \\
& & \pi_{q-1}\mathfrak{F}(\Delta\mathcal{J}_{n+r}, \Delta\mathcal{J}_{n+r-1}).
\end{array}$$

<u>Corollary</u> 1.3. The spectral sequence converges to the set of homology subquotients $E^n_{\infty;q}$ of the half-exact sequences

$$\pi_{q+1}\mathfrak{F}(\Delta\mathcal{J}_{n-1},\emptyset)\xrightarrow{\partial}\pi_q\mathfrak{F}(\Delta\mathcal{J}_n,\Delta\mathcal{J}_{n-1})\xrightarrow{\partial}\pi_{q-1}\mathfrak{F}(\Delta\mathcal{J},\Delta\mathcal{J}_n),$$

and these in turn are isomorphic to the subquotients

$$\frac{F^n\pi_q\mathfrak{F}(\Delta\mathcal{J},\emptyset)}{F^{n+1}\pi_q\mathfrak{F}(\Delta\mathcal{J},\emptyset)}$$

of $\pi_q\mathfrak{F}(\Delta\mathcal{J},\emptyset)$, where

$$F^n\pi_q\mathfrak{F}(\Delta\mathcal{J},\emptyset) = \mathrm{Im}[\pi_q\mathfrak{F}(\Delta\mathcal{J},\Delta\mathcal{J}_{n-1}) \to \pi_q\mathfrak{F}(\Delta\mathcal{J},\emptyset)].$$

The isomorphism in question is induced by the additive relation

$$\pi_q\mathfrak{F}(\Delta\mathcal{J}_n,\Delta\mathcal{J}_{n-1}) \leftarrow \pi_q\mathfrak{F}(\Delta\mathcal{J},\Delta\mathcal{J}_n) \rightarrow \pi_q\mathfrak{F}(\Delta\mathcal{J},\emptyset).$$

(The Corollaries 1.2-1.3 assume familiarity with the often-referred-to section in [3], p.333.)

<u>Proposition</u> 2. There is a natural isomorphism

$$\Phi: E^n_{1;q} \cong C^n(X)(\pi_{q+n}),\ q \geq 2, \text{ such that } \Phi d_{1;q} = \delta\Phi\ (q \geq 3).$$

<u>Proof</u>. First, $E^n_{1;q} = \pi_q\mathfrak{F}(\Delta\mathcal{J}_n,\Delta\mathcal{J}_{n-1})$, by definition. But $\mathfrak{F}(\Delta\mathcal{J}_n,\Delta\mathcal{J}_{n-1}) \cong \times_{\dim\Delta s=n}\mathfrak{F}(\Delta s,\partial\Delta s)$ by restriction of the members of $\mathfrak{F}(\Delta\mathcal{J}_n,\Delta\mathcal{J}_{n-1})$ to each n-simplex. This means $E^n_{1;q} \cong \prod_{\dim\Delta s=n}\pi_q\mathfrak{F}(\Delta s,\partial\Delta s)$ (naturally). Sticking momentarily to one n-simplex Δs, and putting its indices into a sequence $i_* = (i_0,\ldots,i_n)$, we form next an isomorphism

$\Phi^{i_*}: \pi_q\mathfrak{F}(\Delta s,\partial\Delta s) \xrightarrow{\cong} \pi_{q+n}(X_s)$: Its formula entails the set $s(t) = \{i_t, i_{t+1},\ldots,i_n\}$ $(0\leq t\leq n)$ via the diagram

$$\begin{array}{c}
\pi_q\mathfrak{F}(\Delta s,\partial\Delta s)\\
\parallel\\
\pi_q \mathrm{Top}_o(\Delta s(0),\partial\Delta s(0);X_s)\\
\partial\uparrow\\
\vdots\\
\pi_{q+k}\mathrm{Top}_o(\Delta s(k),\partial\Delta s(k);X_s)\\
\partial\uparrow\cong\\
\pi_{q+k+1}\mathrm{Top}_o(\partial\Delta s(k),\partial\Delta s(k)-\mathrm{int}\ \Delta s(k+1);\ X_s)\\
\downarrow\cong\\
\pi_{q+k+1}\mathrm{Top}_o(\Delta s(k+1),\partial\Delta s(k+1);X_s)\\
\vdots\\
\pi_{q+n}\mathrm{Top}_o(\Delta s(n),\emptyset;X_s)\\
\parallel\wr\\
\pi_{q+n}(X_s).
\end{array}$$

(Top_o stands for function space, with base point of X_s understood.) The rest of the proof is exactly as presented in [1], Lemma 2.5. □

Thus, the groups $\pi_q\mathfrak{F}(\Delta\mathcal{J},\emptyset)$ for $q\geq 2$ are related to the groups $\pi_j(X_s)$ for $j\geq 2$ and $s\subset\mathcal{J}$: we have

$$\frac{F^n\pi_q\mathfrak{F}(\Delta\mathcal{J},\emptyset)}{F^{n+1}\pi_q\mathfrak{F}(\Delta\mathcal{J},\emptyset)} \cong \text{a subquotient of } \prod_{\dim\Delta s=n}\pi_{q+n}(X_s)$$

(where this isomorphism entails both Corollary 1.3 and Proposition 2). To approximate this subquotient, one forms subquotients $E^n_{r;q}$ $(r\geq 2)$ by reference to differentials, with the exception of the case $q = 2$. (The $d^n_{r;2}$'s are missing, since the corresponding $E^{n+r}_{r;1}$'s are not available to serve as their codomains).

<u>Application to general homology</u>. Before going any further we note that there is a map $b : \bigcap_{s\subset\mathcal{J}} X_s \to \mathfrak{F}(\Delta\mathcal{J},\emptyset)$ which sends each $x \in \bigcap_{s\subset\mathcal{J}} X_s$ into the constant map which has value x. The map b is obviously natural.

Now let $h = \{h_q, \partial_q \mid q \in Z\}$ be any general homology theory (Cf. [5]). If h^* is its polar general cohomology theory, there is an S-module (syn: spectrum) $\underset{\sim}{W} = \{W(k) \mid k = 0,1,\ldots\}$ of base-pointed spaces that classifies h^*. For compact polyhedral pairs (X,A) we must then have $h_q(X,A) \cong \lim_{k\to\infty} \pi_{q+k}(W(k)\wedge X, W(k)\wedge A)$ for all $q \in Z$ (where for any base-pointed space M one defines $M \wedge X$ to be $(M\times X)/(\{*\}\times X)$).

Let X be any compact polyhedron without base point and $\{(X_i,A_i) \mid i \in \mathcal{J}\}$ a finite family of pairs of subpolyhedra of X. For each $k = 0,1,\ldots$ define $Y(k-1) = \Omega(W(k)\wedge X, W(k)\wedge X)$, $Y_s(k-1) = \Omega(W(k)\wedge X_s, W(k)\wedge A_s)$ $(s\subset\mathcal{J})$, where $(X_s,A_s) = (\bigcup_{i\in s} X_i, \bigcup_{i\in s} A_i)$. It follows that $\underset{\sim}{Y} = \{Y(k)\}$ and $\underset{\sim}{Y}_s = \{Y_s(k)\}$ are S-modules, with $\lim_{k\to\infty} \pi_{q+k}(Y_s(k))$ (henceforth denoted $\pi_q(\underset{\sim}{Y}_s)$) $= h_q(X_s,A_s)$. The function space $\mathfrak{F}(K,L)$ for $(Y(k);\mathcal{J};\{Y_s(k)\mid s\subset\mathcal{J}\})$ will be denoted $\mathfrak{F}(K,L)(k)$. The structure of S-module for $\underset{\sim}{Y}$ carries over, so that there is an S-module $\underset{\sim}{\mathfrak{F}}(K,L) = \{\mathfrak{F}(K,L)(k)\mid k = 0,1,\ldots\}$. Returning now to b, we see that we have a map

$$b(k) : \bigcap_{s\subset\mathcal{J}} Y_s(k) \longrightarrow \mathfrak{F}(\Delta\mathcal{J},\emptyset)(k)$$

which commutes with the S-module structure, so that there is an S-module transformation $\underset{\sim}{b} : \cap\underset{\sim}{Y}_s \longrightarrow \underset{\sim}{\mathfrak{F}}(\Delta\mathcal{J},\emptyset)$.

<u>Proposition</u> 3. $\pi_q(\underset{\sim}{b})$ is an isomorphism

$$h_q\Big(\bigcap_{i\in\mathcal{J}} X_i, \bigcap_{i\in\mathcal{J}} A_i\Big) = \pi_q\Big(\bigcap_{s\subset\mathcal{J}} \underset{\sim}{Y}_s\Big) \xrightarrow{\cong} \pi_q\underset{\sim}{\mathfrak{F}}(\Delta\mathcal{J},\emptyset).$$

<u>Proof</u>. Assume first that $A_i = \{*\}$ for all $i\in\mathcal{J}$, because the general case will follow from this case (see below). Write $Z_s(k)$ for $W(k)\wedge X_s$, to define an S-module $\underset{\sim}{Z}_s$ such that $\pi_q\underset{\sim}{Z}_s \cong \pi_q\underset{\sim}{Y}_s \cong h_q(X_s)$. We shall denote $\bigcap_{i\in s} X_i$ as X^s.

Consider the following commutative diagram for some 1-simplex Δij of $\Delta\mathcal{J}$:

$$\begin{array}{ccccc}
\cdots\ \pi_q\Omega(\underset{\sim}{Z}_i,\underset{\sim}{Z}^{ij}) & \longrightarrow & \pi_q(\underset{\sim}{Z}^{ij}) & \longrightarrow & \pi_q(\underset{\sim}{Z}_i)\ \cdots \\
\downarrow e & & & & \\
\pi_q\Omega(\underset{\sim}{Z}_{ij},\underset{\sim}{Z}_j) & & \downarrow ? & & \downarrow \cong \\
\downarrow \cong & & & & \\
\cdots\ \pi_q\underset{\sim}{\mathfrak{F}}(\Delta ij,e_i) & \longrightarrow & \pi_q\underset{\sim}{\mathfrak{F}}(\Delta ij) & \longrightarrow & \pi_q\underset{\sim}{\mathfrak{F}}(e_i)\ \cdots
\end{array}$$

The indicated isomorphisms are simply the renaming of a parameter, while e is induced by inclusion and ? is given by the same formula as b. But e is actually the map $h_{q+1}(X_i, X^{ij}) \rightarrow h_{q+1}(X_{ij}, X_j)$, which is an isomorphism by the general homology excision property for compact polyhedra. By the diagrammatic 5-lemma it follows that ? is an isomorphism.

Now let Δs be any simplex of $\Delta^{\mathcal{J}}$ and e_i some vertex not in Δs. Assume that $\pi_q(\underset{\sim}{Z}^s) \cong \pi_q \underset{\sim}{\mathfrak{F}}(\Delta s)$ holds. Form the commutative diagram

$$\begin{array}{ccccc}
\cdots\ \pi_q\Omega(\underset{\sim}{Z}_i, \underset{\sim}{Z}_i \cap \underset{\sim}{Z}^s) & \longrightarrow & \pi_q(\underset{\sim}{Z}_i \cap \underset{\sim}{Z}^s) & \longrightarrow & \pi_q(\underset{\sim}{Z}_i)\ \cdots \\
\downarrow e & & & & \\
\pi_q\Omega(\underset{\sim}{Z}_i \cup \underset{\sim}{Z}^s, \underset{\sim}{Z}^s) & & \downarrow ?' & & \downarrow \cong \\
\downarrow ?'' & & & & \\
\pi_q\Omega(\underset{\sim}{\mathfrak{F}}^*(\Delta s), \underset{\sim}{\mathfrak{F}}(\Delta s)) & & & & \\
\downarrow \cong & & & & \\
\cdots\ \pi_q \underset{\sim}{\mathfrak{F}}(\Delta(s \cup i), e_i) & \longrightarrow & \pi_q \underset{\sim}{\mathfrak{F}}(\Delta(s \cup i)) & \longrightarrow & \pi_q \underset{\sim}{\mathfrak{F}}(e_i)\ \cdots
\end{array},$$

where $\underset{\sim}{\mathfrak{F}}^*(\Delta s)$ is the same as $\underset{\sim}{\mathfrak{F}}(\Delta s)$ but for the system $(X; \mathcal{J}; \{X_i \cup X_s \mid s \subset \mathcal{J}\})$. (Thus, $\underset{\sim}{\mathfrak{F}}(\Delta s)$ are subspaces of $\underset{\sim}{\mathfrak{F}}^*(\Delta s)$.) The map e has again the form of an excision isomorphism $h_{q+1}(X_i, X_i \cap X^s) \longrightarrow h_{q+1}(X_i \cup X^s, X^s)$. To prove that ?" is an isomorphism, we apply the diagrammatic 5-lemma to the commutative diagram

$$\begin{array}{ccccc}
\cdots\ \pi_q\Omega(\underset{\sim}{Z}_i \cup \underset{\sim}{Z}^s) & \longrightarrow & \pi_q\Omega(\underset{\sim}{Z}_i \cup \underset{\sim}{Z}^s, \underset{\sim}{Z}^s) & \longrightarrow & \pi_q(\underset{\sim}{Z}^s)\ \cdots \\
\Big\downarrow \cong & & \Big\downarrow ?'' & & \Big\downarrow \cong \\
\cdots\ \pi_q\Omega\mathfrak{F}^*(\Delta s) & \longrightarrow & \pi_q\Omega(\mathfrak{F}^*(\Delta s), \mathfrak{F}(\Delta s)) & \longrightarrow & \pi_q(\mathfrak{F}(\Delta s))\ \cdots
\end{array}$$

where the isomorphism at the left is of the same type as the one at the right. Now that $?''$ is seen to be an isomorphism, the earlier diagram similarly shows that $?'$ is an isomorphism. This completes the inductive proof in case $A_i = \{*\}$ for all $i \in \mathcal{J}$.

For the general case recall that

$$Y_s(k-1) = \Omega(Z_s^1(k), Z_s^0(k)),$$

where $Z_s^1(k) = W(k) \wedge X_s$ and $Z_s^0(k) = W(k) \wedge A_s$. Write $\mathfrak{F}_\nu(K,L)(k) = \{\varphi : K \to Z(k) \mid \varphi(L) = *$, and for each $\Delta s \subset K$ $\varphi(\Delta s) \subset Z_s^\nu\}$, $(\nu = 0,1)$. From the definition of $\mathfrak{F}(K,L)$ it is evident that

$$\mathfrak{F}(K,L)(k-1) \cong \Omega(\mathfrak{F}_1(K,L)(k),\ \mathfrak{F}_0(K,L)(k))$$

by rearranging the priority of variables. So, we have the commutative diagram

$$\begin{array}{ccccc}
\cdots\ \pi_q\Omega(Z^{1\mathcal{J}}) & \longrightarrow & \pi_q\Omega(Z^{1\mathcal{J}}, Z^{0\mathcal{J}}) & \longrightarrow & \pi_q(Z^{0\mathcal{J}})\ \cdots \\
\Big\downarrow \cong & & \Big\downarrow \pi_q(\underset{\sim}{b}) & & \Big\downarrow \cong \\
\cdots\ \pi_q\Omega\mathfrak{F}_1(\Delta\mathcal{J}) & \longrightarrow & \pi_q\mathfrak{F}(\Delta\mathcal{J}) & \longrightarrow & \pi_q\mathfrak{F}_0(\Delta\mathcal{J})\ \cdots
\end{array},$$

in which the left and right isomorphisms have just been established, so that the map $\pi_q(\underset{\sim}{b})$ is an isomorphism by the diagrammatic 5-lemma. □

Corollary 3.1. For any general homology theory h and finite family $\mathcal{X} = \{(X_i, A_i) | i \in I\}$ of pairs of subpolyhedra of a compact polyhedron X, there is a corresponding spectral sequence $\{E^n_{r;q}, d^n_{r;q} : E^n_{r;q} \to E^{n+r}_{r;q-r+1} | n, q \in \mathbb{Z}, r \geq 1\}$ such that $E^n_{1;q} \cong C^n(\mathcal{X}; h_{q+n})$. Under this isomorphism $d^n_{1;q}$ corresponds to the cochain coboundary operator, and $E^n_{\infty;q}$ is isomorphic to $\dfrac{F^n h_q(X^{\mathcal{X}}, A^{\mathcal{X}})}{F^{n+1} h_q(X^{\mathcal{X}}, A^{\mathcal{X}})}$ for some filtration

$$\cdots \supset F^n h_q(X^{\mathcal{X}}, A^{\mathcal{X}}) \supset F^{n+1} h_q(X^{\mathcal{X}}, A^{\mathcal{X}}) \supset \cdots$$

of $h_q(X^{\mathcal{X}}, A^{\mathcal{X}})$ (containing both $h_q(X^{\mathcal{X}}, A^{\mathcal{X}})$ and $\{0\}$), where $(X^{\mathcal{X}}, A^{\mathcal{X}}) = (\bigcup_{i \in I} X_i, \bigcup_{i \in I} A_i)$. Moreover, the entire structure is functorial in $(X; \mathcal{X}; I)$.

Corollary 3.2. If X is a compact polyhedron and h a general homology theory, there is a spectral sequence $\{E^n_{r;q}, d^n_{r;q} : E^n_{r;q} \to E^{n+r}_{r;q-r+1} | n, q \in \mathbb{Z};\ r \geq 2\}$ with $E^n_{2;q} \cong H^n(X; S_{q+n})$, where S_{q+n} stands for the induced sheaf of the general homology presheaf $U \mapsto h_{q+n}(X, X-U)$. The group $E^n_{\infty;q}$ is isomorphic to $\dfrac{F^n h_q(X)}{F^{n+1} h_q(X)}$ for a suitable filtration

$$\cdots \supset F^n h_q(X) \supset F^{n+1} h_q(X) \supset \cdots$$

of $h_q(X)$ (containing both $h_q(X)$ and $\{0\}$).

Bibliography

[1] Cain, R. N., "The Leray Spectral Sequence of a Mapping for Generalized Cohomology", Comm. Pure and Appl. Math. 24J1971), 53-70.

[2] Cain, R. N., "A Spectral Sequence that Relates the General Homology of a Polyhedron to the Homology Presheaf", Department of Mathematics RR 71-20, C-MU, 1971.

[3] Cartan, H., and S. Eilenberg, Homological Algebra, Princeton University, 1956.

[4] Godement, R., Topologie Algébrique et Théorie des Faisceaux, Hermann, 1958.

[5] Whitehead, G. W., "Recent Advances in Homotopy Theory", lecture notes from the Holiday Symposium, December 1969, New Mexico State University.