RESOLUTION AND THE CONSISTENCY
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Abstract

It is shown by a purely syntactic argument how the completeness of resolution in type theory implies the consistency of type theory with axioms of extensionality, descriptions, and infinity. In this system the natural numbers are defined, and Peano's Postulates proved; indeed, classical analysis and much more can be formalized here. Nevertheless, Gödel's results show that the completeness of resolution in type theory cannot be proved in this system.

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Resolution and the Consistency of Analysis

Peter B. Andrews*

§1. Introduction.

In [2] we formulated a system ft, called a Resolution system, for refuting finite sets of sentences of type theory, and proved that ft is complete in the (weak) sense that every set of sentences which can be refuted in the system 3 of type theory due to Church [5] can also be refuted in ft. The statement that ft is in this sense complete is a purely syntactic one concerning finite sequences of wffs. However, it is clear that there can be no purely syntactic proof of the completeness of ft, since the completeness of ft is closely related to Takeuti's conjecture [9] (since proved by Takahashi [8] and Pravitz [7]) concerning cut-elimination in type theory. As Takeuti pointed out in [9] and [10], cut-elimination in type theory implies the consistency of analysis. Indeed, Takeuti's conjecture implies the consistency of. a formulation of type theory with an axiom of infinity; in such a system classical analysis and much more can be formalized. Hence, to avoid a conflict with Gödel's theorem, any proof of the completeness of resolution in type theory must involve arguments which cannot be formalized in type theory with an axiom of infinity. Indeed, the proof in [2] does involve a semantic argument.

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Nevertheless,, it must be admitted that anyone who does not find the line of reasoning sketched above completely clear will have difficulty finding a unified and coherent exposition of the entire argument in the published literature. We propose to remedy this situation here.

We presuppose familiarity with §2 (The System 3) and Definitions 4.1 and 5.1 (The Resolution System ft) of [2], and follow the notation used there. In particular,, Q stands for the contradictory sentence $\forall p_o p_o$. To distinguish between formulations of JJ with different sets of parameters,, we henceforth assume IT has no parameters, and denote by $3(A^1, \ldots, A^n)$ a formulation of the system with parameters $A^1, \ldots, A^n$. If H is a set of sentences^)i {- B shall mean that *B* is derivable from some finite subset of W in system S. The deduction theorem is proved in §5 of [5] .

We shall incorporate into our argument Gandy's results in §3 of [6] with some minor modifications. We also wish to thank Professor Gandy for the basic idea (attributed by him to Turing) used below in showing the relative consistency of the axiom of descriptions. (This idea is mentioned briefly at the top of page 48 of [6].)

We shall have occasion to refer to the following wffs:

The set $\mathcal{E}$ of axioms of extensionality;

$$E^{o}: \quad \forall p_o \forall q_o \, . \; p_o \equiv q_o \supset . \; p_o = q_o .$$

$$E^{(\alpha\beta)}: \quad \forall f_{\alpha\beta} \forall g_{\alpha\beta} \, . \; \forall x_\beta [f_{\alpha\beta} x_\beta = g_{\alpha\beta} x_\beta] \supset . \; f_{\alpha\beta} = g_{\alpha\beta}$$

The axiom of descriptions for type α:

$$D^{\alpha}: \quad \forall f_{o\alpha} \, . \; \exists_1 x_\alpha f_{o\alpha} x_\alpha \supset f_{o\alpha} [\iota_{\alpha(o\alpha)} f_{o\alpha}] .$$

An axiom of infinity for type α:

$$J^{\alpha}: \quad \exists r_{o\alpha\alpha} \forall x_\alpha \forall y_\alpha \forall z_\alpha \, . \; \exists w_\alpha r_{o\alpha\alpha} x_\alpha w_\alpha \wedge$$
$$\sim r_{o\alpha\alpha} x_\alpha x_\alpha \wedge \, . \sim r_{o\alpha\alpha} x_\alpha y_\alpha \vee \sim r_{o\alpha\alpha} y_\alpha z_\alpha \vee r_{o\alpha\alpha} x_\alpha z_\alpha .$$

We let G denote the system obtained when one adds to ?($\iota_{\iota(o\iota)}$) the axioms $\mathcal{E}, D^{\iota}$, and J^{ι}. (Description operators and axioms for higher types are not needed, since Church showed [5] that they can be introduced by definition. This matter is also discussed in [3]).

In §4 we shall show how the natural numbers can be defined, and Peano's Postulates can be proved, in G. The basic ideas here go back to Russell and Whitehead [11], of course, but our simple axiom of infinity is not that of Principia Mathematica, but is due to Bernays and Schönfinkel [4]. The natural numbers can be treated in a variety of ways in type theory (e.g., as in [5]), but we believe that the treatment given here has certain advantages of simplicity and naturalness. The simplicity of the axiom of infinity J^1 is essential to our program in §3.

Once one has represented the natural numbers in G, one can easily represent the primitive recursive functions. (With minor changes in type symbols, the details can be found in Chapter 3 of [1].) Syntactic statements about wffs can be represented in the usual way by wffs of G via the device of Gödel numbering. Thus there is a wff Consis of G whose interpretation is that G is consistent, and by Gödel's theorem it is not the case that h_G Consis. Nevertheless, much of mathematics can be formalized in G.

The completeness theorem for ft (Theorem 5.3 of [2]) is also a purely syntactic statement, and hence can be represented by a wff R of G. After preparing the ground in §2 with some preliminary results, in §3 we shall show that by using the completeness of ft we can prove the consistency of G. This argument will be purely syntactic, and could be formalized in G, so h_n [R 3D Consisi . Thus it is not the case that h_rR^ so any proof of the completeness of resolution in type theory must transcend the rather considerable means of proof available in G. Of course such a proof can be formalized in transfinite type theory or in Zermelo set theory.

§2. Preliminary Definitions and Lemmas.

We first establish some preliminary results which will be useful in §3. The reader may wish to postpone the proofs of this section and proceed rapidly to §3.

In presenting proofs of theorems of 3 (and extensions of 3), we shall make extensive use of proofs from hypotheses and the deduction theorem. Each line of a proof will have a number, which will appear at the left hand margin in parentheses. For the sake of brevity, this number will be used as an abbreviation for the wff which is asserted in that line. At the right hand margin we shall list the number(s) of the line(s) from which the given line is inferred (unless it is simply inferred from the preceding line). We use "hyp" to indicate that the wff is inferred with the aid of one or more of the hypotheses of the given line. Thus in

(.1)	⊢A	
(.2)	B h B	hyp
(.3)	B h C	.1,.2
(.4)	D h C	.l,hyp

the hypothesis B is introduced in line *.2,* and £ is inferred from B and the theorem A in line .3; C is also inferred from A and a different hypothesis D in line .4. However,, if the wffs B and C are long, we may write this proof instead as follows:

(.1) h A

(.2) .2 h P hyp

(.3) .2 h C .1, .2

(.4) D h.3 -l, hyp

A generally useful derived rule of inference is that if ft is a set of hypotheses such that tf t- 3xA and #, AhJBj, where x does not occur free in £ or any wff of it, then M h B, We shall indicate applications of this rule in the following fashion:

(.17) H h SxA ...

(.20) M, .20 H A choose x (.17)

(.23) K_v20|- B ...

(.24) W h B .17,.23

If the wff A is long, we might write step (.17) as follows:

(.17) Jt h 3x.20

We shall present only abstracts of proofs, omitting many steps and using familiar laws of quantification theory, equality, and 7\-conversion quite freely. We shall usually omit type symbols on occurrences of variables after the first.

DEFINITION. For each wff A of $3T(t_{oi(o(oi))}, , , ,)$, let #A be the wff of U which is the result of replacing the primitive constant $t_{o1(o(o1))}^{*}$ everywhere by the wff

$$[Af_{o(ot)}Az_i . \ 3x_{o1} . \ f_{o(o1)}x_{o1} \ A x_{o1}z_i].$$

LEMMA 1, E°, E°V ff # D°[1].

Proof: First note that # D^{Ot} conv Vf $_{o(ot)}$x. 3$_1$x$_{ot}$fx =)

f [Az$_t$. 3x$_{ot}$. fx A xz]

(.1) .1 h 3-x f , x hyp
lot o(ot) oi **

(.2) .1,,.2 h $f_{o(ot)}x_{oi}$ A Vu_{oi} . fu 3 u = x choose x(•1)

(.3) .1,..2 h $x_{ot}z_i$ = 3 x_{ot} . $f_{o(ot)}$x x A x z .2

(.4) E°, .1, .2 I- Vz_t . x_{ot} z_i = Sx_{ot} . $f_{o(ot)}$x A xz .3 ,E°

(.5) E°,E^{Ot},.1,.2 h x_{Qi} = [Az^ 3x_{ot}. $f_{o(ot)}$x A xz] .4 ,E^{Ot}

(.6) E°,E^{Ot}, .1 , .2 h $f_{Q(ot)}$ [Az^ ax_{Qi}. fx A xz] .2,.5

(.7) E°,E^{Ot},.1 H .6 .1,.6

(.8) E°,E^{Ot}h # D^{Ot} .7

LEMMA 2. J[1] h J°[1]

Proof: We assume J^X.

(.1) ,1 h $Yx_t Vy_t Vz_t$, 8w_t r_{ott}xw A ~ rxx A. ~rxy V ~ ryz V rxz

choose r_{ott}

Let K o(ot)(ot) be [Au o/V ot- a V o t S A' ~ 3s t U ot S t V

3 V u ot s t A V V v ot fc t 3 r o t t s t t t]'

We shall establish in lines (.11), (.16) and (.31) that K has the properties necessary to establish J O1. To attack (.11) we consider two cases, (.2) and (.5).

(.2) .2 h ~ 3s t xs Ot hyp (case 1)

(.3) .2 H Kx o. ol TAt x . t 1 = t x] .2, def. of K

(.4) ,2 1- 3w Ot K X Qt W .3

(.5) .5 (- 3s t X ot S t hyp (case 2)

(.6) ,5, .6h X Qi S t choose s (.5)

(.7) .1, .5, .6, .7 h r- ·sw OIi 1 x choose w \ (.1)

(.8) .1, .5, .6, .7 h Kx UX [At u . w x = t x] .6, .1, def. of K

(.9) .1, .5, .6, .7 1- 3w Q t K X Q ^ W .8

(.10) .1, .5 H .9 .9, .1, -5

(ii) .1 V 3w Kx w .4, .10

I • ±±) ot oi ^

Next we attack (.16). The proof is by contradiction.

(.12) .12 h Kx Ot x Ot hyp

(.13) .12 I- 3s . x s A Vt . xt 3 r st .12, def. of K

v I r · · Ot t Ot t

(.14) .12 h 3s r ss .13 (instantiate t with s)

X »JXX

(.15) $.1 \vdash \forall s_\iota \sim r_{o\iota\iota}ss$ .1

(.16) $.1 \vdash \sim Kx_{o\iota}x_{o\iota}$ .14, .15

Finally we attack (.31).

(.17) $.17 \vdash Kx_{o\iota}y_{o\iota} \wedge Ky_{o\iota}z_{o\iota}$ hyp

(.18) $.17 \vdash \exists t_\iota y_{o\iota}t \wedge \exists t_\iota z_{o\iota}t$ .17, def. of K

(.19) $.17 \vdash \sim \exists s_\iota x_{o\iota}s \vee \exists s_\iota . xs \wedge \forall q_\iota . y_{o\iota}q \supset r_{o\iota\iota}sq$
.17, def. of K

In (.20) and (.21) we consider the two possibilities set forth in (.19).

(.20) $.17, \sim \exists s_\iota x_{o\iota}s \vdash Kx_{o\iota}z_{o\iota}$ .18, hyp, def. of K

(.21) $.17, .21 \vdash \exists s_\iota . x_{o\iota}s \wedge \forall q_\iota . y_{o\iota}q \supset r_{o\iota\iota}sq$ hyp

(.22) $.17, .21, .22 \vdash x_{o\iota}s_\iota \wedge \forall q_\iota . y_{o\iota}q \supset r_{o\iota\iota}sq$
choose s (.21)

(.23) $.17 \vdash \exists q_\iota .24$ .17, .18, def. of K

(.24) $.17, .24 \vdash y_{o\iota}q_\iota \wedge \forall t_\iota . z_{o\iota}t \supset r_{o\iota\iota}qt$ choose q (.23)

(.25) $.17, .21, .22, .24, z_{o\iota}t_\iota \vdash r_{o\iota\iota}sq \wedge r_{o\iota\iota}qt$ hyp, .22, .24

(.26) $.1, .17, .21, .22, .24, z_{o\iota}t \vdash r_{o\iota\iota}st$ .1, .25

(.27) $.1, .17, .21, .22, .24 \vdash \forall t_\iota . z_{o\iota}t \supset r_{o\iota\iota}st$ .26

(.28) $.1, .17, .21, .22, .24 \vdash Kx_{o\iota}z_{o\iota}$ .18, .22, .27, def. of K

(.29) .1,.17,.21 ⊢ .28 .23,.21, .28

(.30) .1,.17 ⊢ .28 .19,.20,.29

(.31) .1 ⊢ ~ $Kx_{o\iota}y_{o\iota}$ V - $Kyz_{o\iota}$ V Kxz .30

(.32) .1 ⊢ J°ι .11,.16,.31

(.33) J* ⊢ J°ι .32

We next repeat Gandy's definitions in [6] with some minor modifications.

DEFINITION. By induction on γ, we define wffs $Mod_{o\gamma}$ and $M_{o\gamma\gamma}$ for each type symbol γ.

$A_\gamma \overset{M}{=} B_\gamma$ stands for $M_{o\gamma\gamma}A_\gamma B_\gamma$.

$Mod_{o\kappa}$ stands for $[\lambda x_\kappa \exists p_o p_o]$ for $\kappa = o, \iota$.

M_{ooo} stands for $[\lambda p_o \lambda q_o . p_o \equiv q_o]$.

$M_{o\iota\iota}$ stands for $[\lambda x_\iota \lambda y_\iota . x = y]$.

$Mod_{o(\alpha\beta)}$ stands for $[\lambda f_{\alpha\beta} . \forall x_\beta \forall y_\beta . Mod_{o\beta}x_\beta \wedge Mod_{o\beta}y_\beta \wedge x_\beta \overset{M}{=} y_\beta \supset . Mod_{o\alpha}[f_{\alpha\beta}x_\beta] \wedge . f_{\alpha\beta}x_\beta \overset{M}{=} f_{\alpha\beta}y_\beta]$.

$M_{o(\alpha\beta)(\alpha\beta)}$ stands for $[\lambda f_{\alpha\beta}\lambda g_{\alpha\beta} . \forall x_\beta . Mod_{o\beta}x_\beta \supset . f_{\alpha\beta}x_\beta \overset{M}{=} g_{\alpha\beta}x_\beta]$.

LEMMA 3. $\vdash_{\mathfrak{I}} x_\alpha \overset{M}{=} x_\alpha \wedge . x_\alpha \overset{M}{=} y_\alpha \supset y_\alpha \overset{M}{=} x_\alpha \wedge . x_\alpha \overset{M}{=} z_\alpha \wedge z_\alpha \overset{M}{=} y_\alpha \supset x_\alpha \overset{M}{=} y_\alpha$.

Proof: by induction on α.

DEFINITION. For each wff A of 3, A^T is the result of replacing II , x by [Af . Vx . Mod x 3 f x .] every-
o(ooc) oa a oa a oa or
where in A.

LEMMA 4. If A 9 . . . _, A , and B are sentences of JT such that A^1, . . ., A^n h_{ff} B, then $(A^1)^T$, . . . , $(A^n)^T h_{ff} B^T$.

Proof: This is an immediate consequence of Theorem 3.26 of [6], since Gandy's full translation Q of jQ is C when C is a sentence. Our modifications of Gandy's definitions do not injure the proof.

LEMMA 5. h ~ Mod[M z 1 .
o OQQ Ct

Proof: Mod[$M_{o\alpha\alpha}$ z_α] is equivalent to

$Vx_a Vy_a$[Mod x_a A Mod y_Q A x i y 3 .Mod 1%^^] A

•$M_{oaa} z_a x_a$ H $M_{oaa} V_a$]

This is readily proved using the definition of Mod_{oo} and Lemma 3.

LEMMA 6. \-rr(E')T for each E in 6.

Proof: $(E^{O})^{T}$ is equivalent to

Vp_o[Mod p_o D Vq_o Mod q_o3 . [p_o = q_o] r> Vf_{oo}. Mod f_{oo}

⊃ .$f_{oo}p_o$ ⊃ $f_{oo}q_o$],

which is easily proved using the definition of Mod f_{oo}.

(E^Q/3) T is equivalent to

$$\forall f_{\alpha\beta} [\text{Mod } f \Rightarrow \forall g_{Q/3}. \text{ Mod } g =. \forall_{X/3} [\text{Mod } x \Rightarrow \forall h_{Qa}. \text{ Mod } h \supset. h[fx] \supset h.gx]$$

$$\supset \forall k_{o(aj3)} \bullet \text{ Mod } k \Rightarrow. kf \Rightarrow kg],$$

which we prove as follows:

(.1) .It- Mod $f_{\alpha\beta}$ A Mod $g_{Q/3}$ hyp

(.2) ,2(- $\forall x_\beta$[Mod x => $\forall h_{oQ}$. Mod h=> .h[fx] 3 h.gx hyp

(.3) .3 h Mod $k_{Q(a/3)}$ hyp

(.4) h $Mod_{o(oa)}$- M_{oaa}- $f_{a\theta} x_p$ LenunaS

(.5) .2, $Mod_{X/3}$ V- $[M_{o\alpha\alpha}. f_{\alpha\beta}x_\beta][f_{\alpha\beta}x_\beta] \supset. [M_{o\alpha\alpha}. f_{\alpha\beta}x_\beta].g_{\alpha\beta}x_\beta$

.2,.4 (instantiate $h_{o\alpha}$ with M[fx])

(.6) $\vdash$ $M_{o\alpha\alpha}[f_{\alpha\beta}x_\beta][f_{\alpha\beta}x_\beta]$ Lemma 3

(.7) .2, Mod $_{X/3}$H $f_{Q/3}x_3$ ^ $g_{a/3}x_0$ .5,.6

<'8> -2 »• f_{a3} "$\overset{M}{g}$ a|B ,7,def- Of $M_{o(cO)(a/S)}$

(.9) .1,.2,.3 h $k_{o(a/3)}f_{a/3}$ - $k_{o(a0)}g_{a3}$

.3,def. of Mod $k_{o(\alpha\beta)}$ .1, .8

(.10) h $(E^{a3})^T$ .9

LEMMA 7. h ~ Mod $r_{6 Ott}$.

Proof: Mod z is equivalent to

∀x, Vy^Mod x^ A Mod y A x^ = y^ => .Mod[zQ x] A .z x s z y]

so h Vz Mod z .
01 ot

Mod r is equivalent to
on M

Vx Vy [Mod x A Mod y Ax = y D .Modfr x]

A Vw . Mod w z> .r x w ≡ r y w] ,

which is easily proved.

LEMMA 8. J1 H ^ (JX)T.

Proof: (Jt)T is equivalent to

3r [Mod r A Vx . Mod x => Vy . Mod y ⊃

Vz . Mod z 3 . 3w [Mod w A r x w 1 A

~ r x x A . ~ r x y v ~ r y z V r x z l .

This is easily derived from Jl with the aid of Lemma *1.*

<u>DEFINITION</u>, Let 9 be the substitution

S x1 ... xn 9 i.e.^ the simultaneous substitution of A for all
...A

free occurrences of x^i for $1 \leq i \leq n$, where $x^1, \ldots, x^n$ are distinct variables and A^i has the same type as x^i for $1 \leq i \leq n$. If B is any wff, we let $\theta * B$ denote $\eta[[\lambda x^1 \ldots \lambda x^n\ B]A^1 \ldots A^n]$. If θ is the null substitution (i.e., $n = 0$), then $\theta * B$ denotes ηB.

Note that if x_α and y_β are distinct variables, $[[\lambda x_\alpha \lambda y_\beta\ B]A_\alpha C_\beta$ conv $[[\lambda y_\beta \lambda x_\alpha\ B]C_\beta A_\alpha]$, so the definition above is unambiguous. Clearly, if there are no conflicts of bound variables, $\theta * B$ is simply $\eta\ \theta\ B$, the η-normal form of the result of applying the substitution θ to B.

From the definition it is evident that if B conv C, then $\theta * B = \theta * C$.

§3. The Consistency of $\mathcal{G}$.

THEOREM. $\mathcal{G}$ is consistent.

Proof: The proof is by contradiction, so we suppose $\mathcal{G}$ is inconsistent. Thus

(1) $J^{\iota}, \mathcal{E}, D^{\iota} \vdash_{\mathfrak{J}(\iota_{\iota(o\iota)})} \Box$.

(2) $J^{o\iota}, \mathcal{E}, D^{o\iota} \vdash_{\mathfrak{J}(\iota_{o\iota(o(o\iota))})} \Box$.

Proof: Replace the type symbol t by the type symbol (ot) everywhere in the sequence of wffs which constitutes a proof of Q whose existence is asserted in step 1. By checking the axioms and rules of inference of JT one easily sees that a proof of Q] satisfying the requirements of step 2 is obtained.

(3) $J^{ot}, \mathcal{E}, \# D^{ot} \vdash_{\mathfrak{J}} U.$

Proof: The replacement of $\underset{\sim}{A}$ by $\# \underset{\sim}{A}$ everywhere in the proof whose existence is asserted in step 2 yields a proof satisfying step 3, possibly after the insertion of a few applications of the rule of alphabetic change of bound variables.

(4) $J^{\circ} \backslash \ \mathcal{E} \vdash_{\mathfrak{J}} \square$ by Lemma 1,

(5) $J \backslash \ \mathcal{E} \vdash_{\mathfrak{J}} \square$ by Lemma 2,

(6) $(J^1)^T, \{(E^{\gamma})^T \mid E^{\gamma} \in e) H_3 D$

Proof: by Lemma 4, since $\vdash_{\mathfrak{J}} \square^T \supset \square$.

(7) $(J^l)^T$ 1-jD by Lemma 6.

(8) J^l h$_{\mathfrak{J}}$ D by Lemma 8.

We next introduce parameters $\bar{r}_{ot\ t}$ and $\bar{g}_{\wedge 11}$.

Let 5 = {Vx r x [g x], Vx - r x x , Vx Vy Vz .
t o n t ^ 1 1 i t o t t t. i t t t
*

~ r x y v ~ r y z V r x z } .
Oil It OHM I Ot t I t

(9) Ph 3(7 g >• •
o t t t t

Proof: J[1] H ~x [J by (8), and ^ H ~r~ --. J[1] .
$\mathfrak{I}(r,g)$ $\mathfrak{I}(r,g)$

do) ^ I - R D

Proof: This follows from (9) by the completeness of resolution in type theory, i.e. Theorem 5.3 of [2]. The proof of this theorem is the one non-syntactic step in our present proof of the consistency of G.

(11) It is not the case that 5 h F].

Proof: An 77-wff of the form $\bar{r}_{ott}A_tB_t$ will be called <u>positive</u> if the number of occurrences of $\bar{g}_{tt}$ in ^A_t is strictly less than the number of occurrences of $\bar{g}_{tt}$ in B_t and otherwise <u>negative</u>. An 77-wff of the form $\sim\bar{r}_{ott}A_tB_t$ will be called positive iff $\bar{r}_{ott}A_tB_t$ is negative,, and negative iff $\bar{r}_{ott}A_tB_t$ is positive.

Let # be the set of wffs G having one of the following six forms:

(a) Vx^ r x[g x]

(b) Vx ~ r x x
~*i ^ *~

(c) VxVyVz [^ r x y v ~ r y z v r x z] where x , y , and
^t - t - t ~~ ~ - - i ~ r
z are distinct variables.
•* t

(d) $V y_t V z_t$ [~ r $A_t y$ V ~ r y z V r $A_t z$] where y_t and z_t are distinct from one another and from the free variables of A_t.

(e) $V z_t$ [~ r $A_t B_t$ V ~ r $B_t z$ V r $A_t z$] where z_t is distinct from the free variables of A_t and of B_i.

(f) G is a disjunction of wffs, each of the form r j\ B, or ~ r A B9 at least one of which is positive.

Let C be the set of wffs C such that for each substitution 6, 0 * c is in 3.

We assert that if p \- R C, then C € C. Clearly p c C, so it suffices to show that C is closed under the rules of inference of *SI.* For each rule of inference of ft and any substitution 0, we show that 0 * E *e* 5 for any wff E derived from wff(s) of C by that rule.

Suppose M V A and N V ^ ^A are in *C,* and M V .N is obtained from them by cut. Then 0 * [M V A] and 0 * [N V ^ A] must each have form (f) . (For 0 * [N V - A] = [(0 * N) V ~ (0*A)J;

even if $\underset{\sim}{N}$ is null, this cannot have any of the forms (a)-(e), so $\theta * \underset{\sim}{A}$ must have the form $\bar{r}\, \underset{\sim}{B}_{\iota} \underset{\sim}{C}_{\iota}$.) $\theta * [\underset{\sim}{M} \vee \underset{\sim}{A}] = [(\theta * \underset{\sim}{M}) \vee \theta * \underset{\sim}{A}]$; if $\theta * \underset{\sim}{A}$ is negative, $\theta * \underset{\sim}{M}$ must contain a positive wff (so $\underset{\sim}{M}$ cannot be null), so $\theta * [\underset{\sim}{M} \vee \underset{\sim}{N}]$ does also. If $\theta * \underset{\sim}{A}$ is positive, then $\theta * [\sim \underset{\sim}{A}]$ is negative, so $\theta * \underset{\sim}{N}$ must contain a positive wff, so $\theta * [\underset{\sim}{M} \vee \underset{\sim}{N}]$ does also, and hence has form (f).

Suppose $\underset{\sim}{D}$ is in $\mathfrak{C}$, and $[\lambda \underset{\sim}{x}_{\alpha} \underset{\sim}{D}] \underset{\sim}{B}_{\alpha}$ is obtained from $\underset{\sim}{D}$ by substitution. Let ρ be the substitution $S^{\underset{\sim}{x}_{\alpha}}_{\underset{\sim}{B}_{\alpha}}$, and let $\theta \circ \rho$ be the substitution which is the composition of θ with ρ (i.e., $(\theta \circ \rho) * \underset{\sim}{C} = \theta * (\rho * \underset{\sim}{C})$ for each wff $\underset{\sim}{C}$). Then $\theta * [[\lambda \underset{\sim}{x}_{\alpha} \underset{\sim}{D}] \underset{\sim}{B}_{\alpha}] = \theta * \eta[[\lambda \underset{\sim}{x}_{\alpha} \underset{\sim}{D}] \underset{\sim}{B}_{\alpha}] = \theta * (\rho * \underset{\sim}{D}) = (\theta \circ \rho) * \underset{\sim}{D} \in \mathfrak{F}$ since $\underset{\sim}{D} \in \mathfrak{C}$, so $[[\lambda \underset{\sim}{x}_{\alpha} \underset{\sim}{D}] \underset{\sim}{B}_{\alpha}] \in \mathfrak{C}$.

Suppose $\underset{\sim}{D} \in \mathfrak{C}$ and $\underset{\sim}{E}$ is derived from $\underset{\sim}{D}$ by universal instantiation. Thus $\underset{\sim}{D}$ has the form $\underset{\sim}{M} \vee \Pi_{o(o\alpha)} \underset{\sim}{A}_{o\alpha}$, where $\underset{\sim}{M}$ may be null. By considering the null substitution we see that $\eta \underset{\sim}{D} \in \mathfrak{F}$, so $\underset{\sim}{D}$ has the form $\Pi_{o(o\iota)} \underset{\sim}{A}_{o\iota}$ and $\underset{\sim}{E}$ has the form $\underset{\sim}{A}_{o\iota} \underset{\sim}{x}_{\iota}$. It is easily checked by examining forms (a)-(e) that if $\underset{\sim}{H}$ is any wff obtained from a wff of $\mathfrak{F}$ by universal instantiation, then $(\theta * \underset{\sim}{H}) \in \mathfrak{F}$. But $(\eta \underset{\sim}{A}_{o\iota}) \underset{\sim}{x}_{\iota}$ is obtained from $\eta \underset{\sim}{D}$ by universal instantiation, so $\theta * \underset{\sim}{E} = \theta * [(\eta \underset{\sim}{A}_{o\iota}) \underset{\sim}{x}_{\iota}]$ is in $\mathfrak{F}$.

The verification that $\mathfrak{C}$ is closed under the remaining rules of inference of $\mathfrak{R}$ is trivial, so our assertion is proved.

Now [U is not in C, so it is not the case that ^|-^D.

(12) The contradiction between (10) and (11) proves our theorem.

§4. The Natural Numbers in G.

We shall define the natural numbers to be equivalence classes of sets of individuals having the same finite cardinality. We let *o* denote the type symbol (o(ot)). cr is the type of natural numbers.

DEFINITIONS.

0 stands for [Ap Vx ~ p x] .

S stands for [An , N Ap . 3x . p x A
&& O(Ol) rOl I rOt t
n o(ot) N [At i . t i ^ x t A p ^oi t]] .

$N_{o\sigma}$ stands for [An_{σ} $Vp_{o\sigma}$. [$p_{o\sigma}0_{\sigma}$ A Vx_{σ}. $p_{o\sigma}x_{\sigma}$ D $p_{o\sigma}S_{\sigma\sigma}x_{\sigma}$] 3 $p_{o\sigma}n_{\sigma}$].

Vx A stands for Vx [N x *ZD* A] .

kx_{σ} A stands for 3x $_{\sigma}$[$N_{o\sigma}x_{\sigma}$ A A] .

Thus zero is the collection of all sets with zero members, i.e., the collection containing just the empty set [$Ax_{t}Q$] . S represents the successor function. If n, , xx is a finite cardinal (o(oi)) (say 2), then a set $p_{o\iota}$ (say [a, b, c}) is in Sn iff there is an individual (say c) which is in $p_{o\iota}$ and whose deletion from $p_{o\iota}$ leaves a set ({a, b}) which is in n. $N_{o\sigma}$ represents the set of natural numbers, i.e., the intersection of all sets which contain 0 and are closed under S.

We now prove Peano's Postulates (Theorems 1, 2, 3, 4, and 7 below.) In this section $\vdash B$ means B is a theorem of $\mathfrak{a}$.

1 $\vdash N_{o\sigma}0_{\sigma}$ by the def. of N.

2 $\vdash \forall x_{\sigma}.\ N_{o\sigma}x_{\sigma} \supset N_{o\sigma}.\ S_{\sigma\sigma}x_{\sigma}$

Proof:

(.1) $Nx_{\sigma}, .1 \vdash p_{o\sigma}0 \wedge \forall x_{\sigma}.\ px \supset p.\ Sx$ hyp

(.2) $Nx_{\sigma}, .1 \vdash p_{o\sigma}x_{\sigma}$ .1,hyp,def. of N

(.3) $Nx_{\sigma}, .1 \vdash p_{o\sigma}.\ Sx_{\sigma}$ .1,.2

(.4) $Nx_{\sigma} \vdash N.\ Sx_{\sigma}$ .3, def. of N.

3 The Induction Theorem

$\vdash \forall p_{o\sigma}.\ [p_{o\sigma}0_{\sigma} \wedge \dot{\forall} x_{\sigma}.\ p_{o\sigma}x_{\sigma} \supset p_{o\sigma}.\ S_{\sigma\sigma}x_{\sigma}] \supset \dot{\forall} x_{\sigma}p_{o\sigma}x_{\sigma}$

Proof: Let $P_{o\sigma}$ be $[\lambda t_{\sigma}.\ Nt \wedge p_{o\sigma}t]$.

(.1) $.1 \vdash p_{o\sigma}O \wedge \forall x_\sigma.\ Nx \supset .\ px \supset p.\ Sx$ hyp

(.2) $Ny_\sigma \vdash [P\ O \wedge \forall x_\sigma.\ Px \supset P.\ Sx] \supset Py_\sigma$ hyp, def. of N

(.3) $.1 \vdash P\ O$ def. of P, .1, Theorem 1

(.4) $.1 \vdash \forall x_\sigma.\ Px \supset P.\ Sx$ def. of P, .1, Theorem 2

(.5) $.1,\ Ny_\sigma \vdash Py_\sigma$ .2,.3,.4

(.6) $.1 \vdash \dot{\forall} y_\sigma py_\sigma$ .5, def. of $\dot{\forall}$, P

4 $\vdash \dot{\forall} n_\sigma.\ S_{\sigma\sigma}n_\sigma \neq O_\sigma$

Proof by contradiction:

(.1) $.1 \vdash Sn_\sigma = O$ hyp

(.2) $\vdash O_\sigma[\lambda x_\iota \square]$ def. of O

(.3) $.1 \vdash Sn_\sigma[\lambda x_\iota \square]$ .1,.2

(.4) $.1 \vdash \exists x_\iota \square$ .3, def. of S

(.5) $\vdash Sn_\sigma \neq O$ .4

(.6) $\vdash \dot{\forall} n_\sigma.\ Sn \neq O$ .5, def. of $\dot{\forall}$

Our first step in proving Theorem 7 is to show that if we remove any element from a set of cardinality Sn we obtain a set of cardinality n.

$$5 \vdash \dot{\forall} n_\sigma \forall p_{o\iota} . \sim p_{o\iota} w_\iota \wedge S_{\sigma\sigma} n_\sigma [\lambda t_\iota . \ t_\iota = w_\iota \vee p_{o\iota} t_\iota] \supset n_\sigma p_{o\iota}$$

The proof is by induction on n. First we treat the case $n = 0$.

(.1)	$.1 \vdash \sim p_{o\iota} w_\iota \wedge S\, 0\, [\lambda t_\iota .\ t = w \vee pt]$	hyp
(.2)	$.1 \vdash \exists x_\iota .3$	.1, def. of S
(.3)	$.1,.3 \vdash [x_\iota = w_\iota \vee p_{o\iota} x] \wedge 0[\lambda t_\iota .\ t \neq x \wedge .\ t = w \vee pt]$	choose x (.2)
(.4)	$.1,.3 \vdash \sim .\ w_\iota \neq x_\iota \wedge .\ w = w \vee p_{o\iota} w_\iota$	.3, def. of 0
(.5)	$.1,.3 \vdash w_\iota = x_\iota$	.4
(.6)	$.1,.3 \vdash \forall t_\iota .\ p_{o\iota} t \equiv .t \neq x_\iota \wedge .\ t = w_\iota \vee pt$	.1,.5
(.7)	$.1,.3 \vdash p_{o\iota} = [\lambda t_\iota .\ t \neq x_\iota \wedge .\ t = w_\iota \vee pt]$	.6, E^o, $E^{o\iota}$
(.8)	$.1,.3 \vdash 0\ p_{o\iota}$	.3,.7
(.9)	$\vdash \forall p_{o\iota} .\ \sim p_{o\iota} w_\iota \wedge S\, 0\, [\lambda t_\iota .\ t = w \vee pt] \supset 0\, p$	.2,.8

Next we treat the induction step

(.10)	$.10 \vdash Nn_\sigma \wedge \forall p_{o\iota} .\ \sim pw_\iota \wedge Sn[\lambda t_\iota .\ t = w \vee pt] \supset np$	(inductive) hyp

(.11) .11 h ~ p w A [SSn] [At . t = w V pt] hyp
Ot t (7 t

(.12) .11 h $3x_t$.13 .11, def. of S

(.13) .11, .13 h [x = w V p x] A. Sn [At . t ^ x A. t = wvpt]
t tOl Q I
choose x (.12)

From (.11) we must prove [Sn]p. We consider two cases in (.14) and (.17).

(.14) .14 H x_t = w_t hyp (case 1)

(.15) .11, .13, .14 h p = [At . . t ^ x . A. t = w V pt]
vJ X X X x

.11, .14

(.16) .11, .13, .14 h [Sn]p .13, .15

CT OX

In case 2 we shall use the inductive hypothesis.

(.17) .17 h x / w hyp (case 2)

x x
(.18) .17 h [At . t ^ x A. t = w V p t] =
X X X OI
[At . t = w V. t ^ x A p t] .17

(.19) .11, .13, .17 h Sn [At . t = w V. t ^ x A p t] .13, .18
0 1 X X OX

(.20) .10, .11, .13, .17 h n [At . t ^ x Ap t] .10, .11, .19
CT t t OX

(.21) .11, .13, .17 I- px .13, .17
Ot t

(.22) .10, .11, .13, .17 h [Sn]p def. of S, .20, .21
<? ot

(.23) .10,.11 h $[Sn_{\sigma}]p_{Ot}$ .16,.22,.12

(.24) .10 H Vp_{OI} . ~ pw_{1} A $[SSn_{\sigma}]$ $[At_{I}$. t = w V pt] n $[Sn_{G}]p$
.23

This completes the induction step. The theorem now follows from .9 and .24 by the Induction Theorem.

It will be observed that so far in this section we have not used the axiom of infinity J^{l}. We shall use it in proving the next theorem,, which will also be used to prove Theorem 7.

6 h_{1} V $\dot{n}_{G}$. n $p_{a\ Ol}$ ⊃ $\exists w_{t}$ ~ $p_{Ot}w_{t}$

(.1) . 1 I- Vx Vy Vz . 3w r xw A ~ rxx A . ~ rxy V ~ ryz V rxz
choose r (J^{l})

Let $P_{O\sigma}$ be $[An\ Vp_{\sigma}\ _{Ot}$ np z> $3z_{t}Vw_{t}$. $r_{Ott}zw$ 3 ~ pw] .

We may informally interpret rzw as meaning that z is below w. Thus Pn means that if p is in n, then there is an element z which is below no member of p. We shall prove $\dot{V}n_{\sigma}Pn$ by induction on n.

(.2) $0p_{Qt}$ h - $P_{ot}w_{t}$ def. of 0

(.3) |- P 0 .2, def. of P

Next we treat the induction step.

(.4) .4 $\vdash Nn_\sigma \wedge Pn$ (inductive) hyp

(.5) .5 $\vdash Sn_\sigma p_{o\iota}$ hyp

(.6) .5 $\vdash \exists x_\iota$ .7 .5, def. of S

(.7) .5,.7 $\vdash p_{o\iota} x_\iota \wedge n_\sigma[\lambda t_\iota .\ t \neq x \wedge pt]$ choose x (.6)

(.8) .4,.5,.7 $\vdash \exists z_\iota$ .9 .4, def. of P,.7

(.9) .4,.5,.7,.9 $\vdash \forall w_\iota .\ r_{o\iota\iota} z_\iota w \supset .\ w = x_\iota \vee \sim p_{o\iota} w$
choose z (.8)

Thus from the inductive hypothesis we see that there is an element z which is under nothing in p - {x}. We must show that there is an element which is under nothing in p. We consider two cases, (.10) and (.14).

(.10) .10 $\vdash \sim r_{o\iota\iota} z_\iota x_\iota$ hyp (case 1)

(.11) .4,.5,.7,.9,.10 $\vdash r_{o\iota\iota} z_\iota w_\iota \supset w \neq x_\iota$ .10

(.12) .4,.5,.7,.9,.10 $\vdash \forall w_\iota .\ r_{o\iota\iota} z_\iota w \supset \sim p_{o\iota} w$ .9,.11

(.13) .4,.5,.7,.9,.10 $\vdash \exists z_\iota$ .12 .12

Next we consider case 2, and show that x is under nothing in p.

(.14) $.14 \vdash r_{o\iota\iota} z_\iota x_\iota$ hyp (case 2)

(.15) $.1, .14, r_{o\iota\iota} x_\iota w_\iota \vdash r_{o\iota\iota} z_\iota w_\iota$ .14, hyp, .1

(.16) $.1, .4, .5, .7, .9, .14, r_{o\iota\iota} x_\iota w_\iota \vdash w_\iota = x_\iota \vee \sim p_{o\iota} w$ .9, .15

(.17) $\vdash w_\iota = x_\iota \supset . \; r_{o\iota\iota} xw \supset rxx$

(.18) $.1 \vdash \sim r_{o\iota\iota} x_\iota x$ .1

(.19) $.1, .4, .5, .7, .9, .14 \vdash \forall w_\iota . \; r_{o\iota\iota} x_\iota w \supset \sim p_{o\iota} w$ .16, .17, .18

(.20) $.1, .4, .5, .7, .9, .14 \vdash \exists z_\iota \forall w_\iota . \; r_{o\iota\iota} zw \supset \sim p_{o\iota} w$ .19

(.21) $.1, .4, .5 \vdash .20$ .13, .20, .8, .6

(.22) $.1 \vdash Nn_\sigma \wedge Pn \supset P\, Sn$ .21, def. of P

(.23) $.1 \vdash \dot{\forall} n_\sigma P n_\sigma$ .3, .22, Theorem 3

Having finished the inductive proof, we proceed to prove the main theorem.

(.24) $.24 \vdash Nn_\sigma \wedge n\, p_{o\iota}$ hyp

(.25) $.1, .24 \vdash \exists z_\iota \forall w_\iota . \; r_{o\iota\iota} zw \supset \sim p_{o\iota} w$ .23, .24, def. of P

(.26) $.1 \vdash \forall z_\iota \exists w_\iota r_{o\iota\iota} zw$ .1

(.27) $.1, .24 \vdash \exists w_\iota \sim p_{o\iota} w_\iota$ .25, .26

(.28) $.1 \vdash \dot{\forall} n_\sigma .\ np_{o\iota} \supset \exists w_\iota \sim p_{o\iota} w_\iota$ .27

(.29) $\vdash .28$ J^ι

7 $\vdash \dot{\forall} n_\sigma \dot{\forall} m_\sigma .\ S_{\sigma\sigma} n_\sigma = S_{\sigma\sigma} m_\sigma \supset n_\sigma = m_\sigma$

Proof:

(.1) $.1 \vdash Nn_\sigma \wedge Nm_\sigma \wedge Sn = Sm$ hyp

(.2) $.2 \vdash n_\sigma p_{o\iota}$ hyp

(.3) $.1, .2, \vdash \exists w_\iota \sim p_{o\iota} w_\iota$ .1,.2, Theorem 6

(.4) $.1, .2, .4 \vdash \sim p_{o\iota} w_\iota$ choose w (.3)

(.5) $.1, .2, .4 \vdash p_{o\iota} = [\lambda t_\iota .\ t \neq w_\iota \wedge .\ t = w \vee pt]$
.4, E^o, $E^{o\iota}$

(.6) $.1, .2, .4 \vdash n_\sigma [\lambda t_\iota .\ t \neq w_\iota \wedge .\ t = w \vee p_{o\iota} t]$ .2,.5

(.7) $.1, .2, .4 \vdash Sn_\sigma [\lambda t_\iota .\ t = w_\iota \vee p_{o\iota} t]$ .6, def. of S

(.8) $.1, .2, .4 \vdash Sm_\sigma [\lambda t_\iota .\ t = w_\iota \vee p_{o\iota} t]$ .1,.7

(.9) $.1, .2, .4 \vdash m_\sigma p_{o\iota}$ .1,.4,.8, Theorem 5

(.10) $.1 \vdash n_\sigma p_{o\iota} \supset m_\sigma p$ .3,.9

(.11) $.1 \vdash m_\sigma p_{o\iota} \supset n_\sigma p$ proof as for .10

(.12) $.1 \vdash \forall p_{o\iota} . \, n_\sigma p \equiv m_\sigma p$ .10, .11

(.13) $.1 \vdash n_\sigma = m_\sigma$ .12, E^o, $E^{o\iota}$

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