OSCILLATORY PROPERTIES OF

COMPLEX DIFFERENTIAL SYSTEMS

by

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Abstract

Let $A(z)$ be an $n \times n$ matrix whose elements are analytic functions of a complex variable z in a simply-connected domain D. The differential equation

$$\frac{dw}{dz} = A(z) w$$

is said to be nonoscillatory in D if it does not possess a nontrivial solution vector $w = (w_1, \ldots, w_n)$ such that $w_k(z_k) = 0$, $k = 1, \ldots, n$, where the z_k are points of D. The equation is called suborthogonal in D if, for any $z, \zeta \in D$ and any nontrivial solution w, $\text{Re}\{w(z) w(\zeta)\} > 0$.

The problem studied in this paper is that of establishing connections between these properties (or their absence) and certain simple properties of the coefficient matrix A. A number of criteria are obtained.

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1. Ihe systems to be discussed in this paper are of the form

$$\text{(1.1)} \qquad \frac{dw}{\ } = A(z)w,$$

where $A(z)$ is an nxn matrix whose elements are analytic func-
tions in a simply-connected domain D in the complex plane,
and w is a vector whose n components are analytic in D.
Tfhe system (1.1) is said to be nonoscillatory in D if every
nontrivial solution vector w has at least one component which
does not vanish in D. Evidently, this property may also be de-
fined by the requirement that, for arbitrary complex constants
$a_1, \ldots _n c$ and arbitrary $z_1, \ldots _n z$ in D, (1.1) should have
exactly one solution $w = (w-, \ldots, w)$ for which $w, (z,) = ou,$

J^{**-} $jL, \ll\ll, n*$

A related property, which is likewise useful in the study of
the oscillatory behavior of the solutions of (1.1), is that of
suborthogonality. The equation is said to be suborthogonal in D
if, for any nontrivial solution w and for any two points $£, r\backslash$
in D, $\text{Re}\{w(£)w(rj)\} > 0$ (where the expression in braces is the
scalar product of the two vectors in question). For $n = 2$, sub-
orthogonality evidently implies nonoscillation; for $n > 2$, the
relation between the two concepts is less obvious.

In formulating the nonoscillation condition of this section we shall assume that the domain D is bounded by a contour dD and that the coefficient matrix A is analytic in D + SD.   This is done in the interest of more concise formulation and is not an essential restriction.   Our condition will be stated in terms of the Hölder norm $\|A\|_p$ (p$\triangle$>l) of the coefficient matrix A, induced by the norm $\|p\|_p = [|P_1|^P + ... + |P_n|^P]^{1>/p}$ of the vector $\beta = (\beta_1, ..., \beta_n)$.

Theorem JL-JL   Let   D be a. simply-connected domain in the complex plane, bounded by a contour dD. If the matrix A if analytic in D + dD and

$$(1.2) \qquad \int_{\partial D} \|A\|_p |dz| < 2TT(P-1)^P [_P \sin \wedge]^{-1} = c_{p,} \quad (p>1),$$

then the system (1.1) Jjs, nonoscillatory in D.

For p = 2, (1.2) reduces to the condition

$$(1.3) \qquad \int \|A\|_2 |dz| < ir,$$

recently obtained by B. Schwarz [9] , who also showed that the constant TT is the best possible (the fact that the system is non-oscillatory if the left-hand side of (1.3) is bounded by 1 had previously been proved by W. J. Kim [3]).   Whether condition (1.2) is sharp for other values of p is an open question.   It is easily confirmed that $c_p = c_q$ if $p^{"1} + q^{"1} = 1$, and that c_p decreases from ir to 2 if p increases from 2 to oo .   Thus, $c_p > 2$ except for p = l,oo.   In the two limiting cases, condition (1.2) is contained in a recent result of Schwarz [8] according to which the inequality

(1.4)
$$\int_{\partial D} \||A\|| \, |dz| < 2,$$

where $\||A\||$ is the matrix norm induced by an arbitrary absolute vector norm, is sufficient to guarantee the nonoscillation of equation (1.1) in D. A short proof of this result will be found at the end of this section.

If D is transformed into another domain D' by a conformal mapping $z \to z'$, (1.1) is transformed into an equation of the same form and the integral on the left-hand side of (1.4) remains invariant [7]. It is therefore sufficient to prove Theorem 1.1 for the case in which D is the unit disk. The situation is further simplified by a result of W. J. Kim [3] who showed that if (1.1) is oscillatory in $|z| \leq 1$, then there exists a solution w of (1.1) such that every component of w has a zero on a circle $|z| = r$ $(0 < r \leq 1)$.

By (1.1), we have
$$\|w'\|_p = \|Aw\|_p \leq \|A\|_p \|w\|_p$$

and therefore, if ρ is a positive continuous function on $|z| = r$,
$$\int_{|z|=r} \rho (\|w'\|_p)^p |dz| \leq \int_{|z|=r} \rho (\|A\|_p)^p (\|w\|_p)^p |dz|.$$

In particular, if we set $\rho = (\|A\|_p)^{1-p}$ and introduce a new (real) variable s by setting $ds = \|A\|_p |dz|$, we obtain

(1.5)
$$\int_o^L (\|\tfrac{dW}{ds}\|_p)^p ds \leq \int_o^L (\|W\|_p)^p ds,$$

where $W(s) = w(z(s))$,

(1.6)
$$L = \int_{|z|=r} \|A\|_p |dz|,$$

and, since the analytic vector w is single-valued in the unit
disk, $W(s+L) = W(s)$.

If $W_{\bar{1}}.,\ldots,W_n$ are the components of W and if a_k denotes
the zero of W, on $|z| = r$, the periodicity of W enables us
to write (1.5) in the form

$$\sum_{k=1}^{n} \int_{\alpha_k}^{\alpha_k+L} |w_k'|^p \, ds \leq L \, B \sum_{k=1}^{n} \int_{a_k}^{a_k+L} |w_k|^p \, ds,$$

where the prime now denotes differentiation with respect to s.
Setting $R_k = |w_k|$ and observing that $|R_k'| \leq^{\wedge} I^{W \wedge U}$ we find that
the real functions R_k satisfy the inequality

$$(1.7) \qquad \sum_{k=1}^{n} \int_{a_k}^{a_k+L} |R_k'|^p \, ds \leq \sum_{k=1}^{n} \int_{a_k}^{a_v+L} R_k^p \, ds.$$

and the conditions $R_k(a_k) = R_k(a_k+L) = 0$.

We now use an inequality of Hardy, Littlewood and Pólya
[2,Chap.VII], according to which

$$(1.8) \qquad \int_o^a f^p \, ds \leq [ir^{-1}ap \sin_p J]^p (p-1)^{-1} \int_o^a (f^T)^p \, ds \sim (2a \, c_-^{''X})^p \int_o^a (f')^p \, ds,$$

if $f(0) = 0$, f is continuous and nondecreasing in [0,a] and
$f' \in L_p[0,a]$. (In [2], p is an even integer, but the proof given
there is valid for any p > 1). The monotonicity assumption for
f' is not essential. If this assumption is dropped and f' is
replaced by $|f'|$ on the right-hand side of (1.8), the inequality
remains true, since f may be replaced by the function

$$F(s) = \tfrac{1}{2} \int_o^s [f(t) + |f'(t)|] \, dt,$$

for which $f(s) \leq F(s)$, $0 <\leq F'(s) \leq |f'(s)|$, and which satisfies
the conditions under which (1.8) holds.

Applying this generalized version of (1.8) (with $a = \frac{L}{2}$) to the functions $f(s) = R_k(\alpha_k + s)$ and $f(s) = R_k(L + \alpha_k - s)$, respectively, and adding the results, we obtain

$$\int_{\alpha_k}^{\alpha_k + L} R_k^p ds \leq (Lc_p^{-1})^p \int_{\alpha_k}^{\alpha_k + L} |R_k'|^p ds, \quad k = 1, \ldots, n,$$

and a comparison with (1.7) shows that $Lc_p^{-1} \geq 1$, i.e., by (1.6),

$$\int_{|z|=r} \|A\|_p |dz| \geq c_p.$$

But the norm $\|A\|_p$ is a subharmonic function in the closed unit disk [7], and for a subharmonic function S the mean value

$$\frac{1}{2\pi r} \int_{|z|=r} S |dz|$$

is a nondecreasing function of r. Hence, the assumption that (1.1) possesses an oscillatory solution in $|z| \leq 1$ leads to the inequality

$$\int_{|z|=1} \|A\| |dz| \geq c_p.$$

Condition (1.2) is thus sufficient to guarantee the nonoscillation of (1.1) in $D + \partial D$. This establishes Theorem 1.1.

We end this section with a short proof of the result of Schwarz, quoted above, according to which (1.4) guarantees the non-oscillation of equation (1.1) in D if $\|\|A\|\|$ is the matrix norm induced by an arbitrary absolute vector norm $\|\|u\|\|$, i.e., a norm for which $\|\|u\|\| \leq \|\|v\|\|$ if $u = (u_1, \ldots, u_n)$, $v = (v_1, \ldots, v_n)$ and $|u_k| \leq |v_k|$, $k = 1, \ldots, n$. As before, it is sufficient to consider the unit disk and to show that, if (1.4) holds, (1.1) cannot have a nontrivial solution $w = (w_1, \ldots, w_n)$ such that $w_k(\alpha_k) = 0$, where

$|a_1| = .., = |a_n| = r < 1$. Suppose, then, there exists such a
solution, and denote by $G(z,£)$ the Green's matrix with the fol-
lowing properties: $G(z,£)$ is diagonal; if $z = re^{it}$, $£ = re^{is}$
and $p_{k'}$ is the solution of $a_k = e^{i£}$ such that $p^ £ s < p^ + .2TT$,
then the diagonal elements G_k of G are $G_{,k} = 1$ if $p._k \leq^ t < s$
and $G_{,k} = -1$ if $s <_L t \leq^ £_k + 2TT$. It follows from this defini-
tion (and the fact that $^w v(^a>) ^= ^o$) that, on the circle $|z| = r$,
w is a solution of the integral equation

$$2w(\zeta) = I_{J|z|=r} G(z,C) A(z)w(z)\,dz.$$

By the triangle inequality,

$$2\|w(C)\| = \| jGAwdz\| \leq J\| GAw\| |dz|.$$

Since $\|\ \|$ is an absolute vector norm, we have $\|G\| \leq \|1\| = 1$
and thus $\|GAw\| ^ \|A\|\ \|W\|$. if $£$ is so chosen that
$M = \|w(C)\| = \max_{|z|=r} \|w(z)\|$, it follows that $2M ^ M \int \|A\| |dz|$,
i.e.,

$$2 1J_{|z|=r} \|A\| |dz|.$$

Using, in the same manner as before, the fact that a norm of an
analytic vector is subharmonic, we obtain the required contradic-
tion to (1.4).

2.   In the present section we derive two sufficient conditions for suborthogonality, the first referring to a general simply-connected domain and the second to the unit disk.

Theorem 2.1.   Let D and A have the same meaning as in Theorem 1.1, and let $\|A\|$ denote the Euclidean norm of A.   If

$$(2.1) \qquad \int_{\partial D} \|A\| \, |dz| < \pi,$$

then equation (1.1) is suborthogonal in $D + \partial D$.   The constant π in (2.1) is the best possible.

Theorem 2.2.   Let A be an analytic $n \times n$ matrix in $|z| < 1$, and let $\|A\|$ denote the Euclidean norm of A.   Let $p(r)$ be a function with the following properties: $p(r) \geq 0$ and $(1-r^2)p(r)$ is non-increasing on $[0,1)$;

$$(2.2) \qquad \int_0^1 p(r) \, dr < \frac{\pi}{4}.$$

If

$$(2.3) \qquad \|A(z)\| \leq p(|z|), \qquad |z| \leq 1,$$

then equation (1.1) is suborthogonal in $|z| \leq 1$.   The constant $\frac{\pi}{4}$ in (2.3) is the best possible.

We begin the proofs of these results with the remark that an equation (1.1) which fails to be suborthogonal in the closure $D + \partial D$ of a (not necessarily simply-connected) domain D must possess a nontrivial solution vector w such that $\mathrm{Re}\{w(z_0)w(\zeta_0)\} = 0$, where z_0 and ζ_0 are points of ∂D.   Indeed, our assumption implies the existence of points z and ζ in $D + \partial D$ such that $\mathrm{Re}\{w(z)w(\zeta)\} = 0$ for a nontrivial solution w.   For constant ζ, $\mathrm{Re}\{w(z)w(\zeta)\}$ is a harmonic function of z in $D + \partial D$, and it

follows from the maximum principle that there exists a point z_Q on SD such that $Re\{w(z_Q) w(£) \} = 0$. Since $Re\{w(z_Q) w(£) \}$ is a harmonic function of £, a repetition of this argument shows that there exists a point $£_Q$ of SD such that $Re\{w(z_Q) w(£_o)\} = 0$, as asserted.

We next remark that it is sufficient to consider the case in which the solution $w = (w_{\bar{1}},...,w_n)$ just mentioned is such that $w_k(z_Q) = o$, $k = 1,...,n-1$, $Im\{w_n(z_Q) \} = Re\{w_n(£_Q) \} = 0$. To establish the truth of this assertion,, we define the (constant) unitary matrix Q by $Qw(z_Q) = (0,0,. . . .,0, ||w(z_Q) ||)$ and consider the vector function $v = Qw$. By (1.1) , v is a solution of the equation $v^T = Bv_5$ where $B = QAQ\sim"^{1}$. Since $||B|| = ||A||$, the conditions imposed on A in Theorems 2.1 and 2.2 apply equally to B. By the definition of v, $v_k (z_o) = 0$, $k = 1,...,n-1$ and $lm\{v_n(z_o)\} = 0$. Since $v(z_Q)v(C_o) = Qw(z_Q) \cdot Qw(£_Q) = w(z_Q-)w(C_Q)$, it follows from $Re\{w(z_Q) W(£_Q) \} = 0$ that $Re(v_n(£_o) \} = 0$. Ihe solution v of $v' = Bv$ thus has the indicated behavior at the points z_o, $£_o$ of (3D if (1.1) is not suborthogonal in $D + dD$.

We now prove Theorem 2.1. In view of the preceding remarks it is sufficient to show that, under the assumptions made, the equation cannot have a nontrivial solution $w = (w.. ,. . . .,w^n)$ such that $w_k(z_o) = O$, $k = 1,...,n-1$, $Im\{w_n(z_o) \} = Re\{w_n(£_o)) = 0$, where $|z_o| = |£_o| = 1$. (Again, we only need consider the case in which D is the unit disk.) Suppose, then, there exists a solution with these properties. The points z_o and $£_o$ divide the circle $|z| = 1$ into two arcs C_1 and C_2, one of which -- say C_1 -- is such that

$$(2.4) \qquad Y = \int_{C_1} N ||dz| \; \pounds_{\pm} \int_{|z|=1} ||A|| \, |dz| .$$

By (1.1), $||w'||^2 \; \pounds \; ||A||^2||W||^2$, or, if we introduce on $|z| = 1$ a real parameter t by $dt = ||A|||dz|$, $||\dot{w}||^2 \; \pounds \; ||w||^2$, where the dot denotes differentiation with respect to t. Setting $t = 0$ at one of the ends of C_1 -- say at z_0 --, we then have

$$(2.5) \qquad \int_0^Y ||\dot{w}||^2 dt \; \pounds \; \int_0^{-\!\!\!»Y} ||w||^2 dt,$$

or, with $w_k = u_k + iv_k$ ($v^\wedge$, v_k real),

$$\sum_{k=1}^n [\int_{o*}^{rY} \dot{u}_k^2 \, dt - \int_o^{rY} u_k^2 \, dt] + EM \sum_{k=1}^n \int_o^{rY} \dot{v}_k^2 \, dt - \int_o^{rY} v_k^2 , dt] \; \pounds \; 0.$$

Hence, there must exist an index $''k$ for which either

$$(2.6) \qquad \int^{rY} \dot{}^2 dt \; \wedge \; \int^{rY} {}^2 dt$$

or

$$(2.6\text{M} \qquad \int_o^{rY} \dot{v}_k^2 \, dt \; \wedge^{'} \; \int_o^{rY} v_k^2 \, dt.$$

By assumption, all the functions $u_1, \ldots, u_n$, $v_1, \ldots, v_n$ vanish either for $t = 0$ or for $t = y$. By classical results [cf., e.g.,1] this implies

$$\int_o^{pY} u_k^2 dt \leq (2\gamma\pi^{-1})^2 \int_o^{pY} \dot{u}_k^2 dt,$$

and a similar inequality for v_K. Combining this with whichever of the two inequalities (2.6), (2.6^f) is satisfied, we find that $2Y \geq \pi$. Since y is subject to the inequality (2.4), this conflicts with (2.1). Hence, if (2.1) holds, equation (1.1) is sub-orthogonal in $|z| \wedge 1$.

The fact that the constant π in (2.1) is the best possible follows from the example used by Schwarz [9] to show that his nonoscillation criterion is sharp. In this example the left-hand side of (2.1) has the value $\pi + \epsilon$, where ϵ is an arbitrarily small positive number, and it is easily seen that the equation in question is not only oscillatory but also fails to be sub-orthogonal.

Turning now to the proof of Theorem 2.2., we note that we may assume that A is analytic in $|z| \leq 1$ (replacing z by ρz, where ρ is in $(0,1)$ and sufficiently close to 1). As before, we have to show that under the assumptions made, (1.1) cannot have a nontrivial solution $w = (w_1,\ldots,w_n)$ such that every one of the functions $\text{Re}\{w_k\}$, $\text{Im}\{w_k\}$, $k = 1,\ldots,n$, vanishes at either z_0 or ζ_0, where $|z_0| = |\zeta_0| = 1$. If there is such a solution, we draw through the points z_0 and ζ_0 the (unique) circle which is orthogonal to $|z| = 1$, and we denote by C the arc of this circle which is in $|z| < 1$. We may assume (replacing, if necessary, z by βz where $|\beta| = 1$, β constant) that C is symmetric with respect to the imaginary axis. The transformation $z \to (t+ir)(1-irt)^{-1}$ (with a suitable real r, $-1 < r < 1$) carries C into a section of the real axis in the t-plane and maps $|z| \leq 1$ onto $|t| \leq 1$. It is easily confirmed that $|z| > |t|$ if $z \in C$ and that, for all $|z| < 1$,

$$(2.7) \qquad \frac{|dz|}{1-|z|^2} = \frac{|dt|}{1-|t|^2} .$$

In the t-plane, the system (1.1) becomes

$$(2.8) \qquad \frac{d\eta(t)}{dt} = \frac{dz}{dt} A[z(t)]\eta(t)$$

$$= B(t)\eta(t) ,$$

where $\eta(t) = w[z(t)]$. Since the points $z = z_0, \zeta_0$ are carried into the points $t = \pm 1$, (2.8) has a solution $\eta = (\eta_1, \ldots, \eta_n)$ such that every one of the functions $\text{Re}\{\eta_k\}$, $\text{Im}\{\eta_k\}$, $k = 1, \ldots, n$, vanishes for either $t = 1$ or $t = -1$. By (2.7), (2.8) and (2.3), we have, for $z \in C$,

$$(1-|t|^2)\|B(t)\| = (1-|z|^2)\|A(z)\| \leq (1-|z|^2)p(|z|)$$

$$\leq (1-|t|^2)p(|t|),$$

where the last inequality follows from $|z| > |t|$ and the assumption that $(1-r^2)p(r)$ is nonincreasing for $r \in [0,1]$. Hence, $\|B(t)\| \leq p(|t|)$, where $B(t)$ is the coefficient matrix of equation (2.8) and $t \in (-1,1)$. Accordingly,

$$\|\eta'\| \leq \|B\|\|\eta\| \leq p(|t|)\|\eta\|, \quad t \in (-1,1),$$

and, with $ds = p(|t|)dt$, $s(-1) = 0$, $\dot{\eta} = \dfrac{d\eta}{ds}$,

(2.9)
$$\int_0^{\gamma} \|\dot{\eta}\|^2 ds \leq \int_0^{\gamma} \|\eta\|^2 ds,$$

where

(2.10)
$$\gamma = \int_{-1}^{1} p(|t|)dt = 2\int_0^1 p(t)dt.$$

Treating (2.9) by means of the procedure applied to (2.5), we find that $2\gamma \geq \lambda$, where γ is the constant (2.10). Since this inequality, obtained under the assumption that (1.1) is not suborthogonal in $|z| < 1$, is incompatible with (2.2), this proves the main assertion of Theorem 2.2.

Examples showing that (2.1) is the best possible constant of its kind can be constructed in all cases in which $p(z)$ is an even analytic function in $|z| < 1$, which is positive for real z and

and such that $|p(z)| \leq L \, p(|z|)$ (e.g., p will have these properties

if $p(z) = \sum\limits_{m=o}^{\infty} a_m z^{2m}$, $a_m \neq 0$). For even n, we set $A = p(z) C$,

where C is a constant matrix whose only nonzero elements $c_{r,s}$

are $c_{r,r+1} = 1$, $r = 1,\ldots,n-1$, and $c_{n1} = (-1)^{n/2}$. ihe equation

$w = Aw$ then has a solution all of whose components are of the form

$\pm \sin a(z)$ or $\pm \cos a(z)$, where $a(z) = \int^{z} p(t)dt$. If there is

equality in (2.2), i.e., if $\int_{-1}^{1} p(t)dt = \frac{\pi}{2} z$, then all components

vanish at either $z = 1$ or $z = -1$. Hence $w(1)w(-1) = 0$, and

the equation is not suborthogonal in $|z| \leq 1$. A trivial modifi-

cation of this example will produce an equation which fails to be

suborthogonal in $|z| < 1$. To obtain an example for odd n, we

define a matrix A_1 by adding a row and a column of zeros to the

matrix A just considered. If $w = (w_1,\ldots,w_n)$ was the solution

in the previous case, the equation $v' = A_1 v$ has the solution

$v = (w_1,\ldots,w_n,0)$, and we evidently have $\|A_1\| = \|A\|$ and

$v(1)v(-1) = 0$.

3. It is natural to conjecture that the assumptions of Theorem 2.2 are also sufficient to guarantee the nonoscillation of (1.1) in $|z| < 1$. However, all that we are able to prove is that the equation will be nonoscillatory if the constant ^ in (2.2) is replaced by a suitable smaller constant.

__Theorem__ 3.1. __Let__ A __be^ am analytic__ nxn-__matrix in__ $|z| < 1$, __and let__ p(r) (CXJ:<1) __b£ a positive function such that__ $(1-r^2)p(r)$ __is nonincreasing and__

(3.1) $$\sin \int_o^x p(r)\,dr < [2(n-1)]^{-\frac{1}{2}}.$$

(3.2) $$||A(z)|| \leq p(|z|)$$

$(|z| < 1)$ __, then equation__ (1.1) __is^ nonoscillatory in the unit disk__.

We note that, for n = 2, condition (3.1) is equivalent to (2.2). For n > 2, (3.1) is the more stringent condition.

Ihe proof of Theorem 3.1 utilizes the following geometric lemma.

__Let__ $u^{v'} = (u_{k_1}, \ldots, u_{kn})$, k = 1,...,n, n $\geq$ 2 __be__ (__constant__) __complex unit vectors. If__ $u^{\wedge} = 0$ __for__ k = 1,...,n, __then there exist integers__ k,m __such that__ $1 \leq k < m \leq n$ __and__

(3.3) $$|u^{(k)}u^{(m)}| \pounds \,^{\wedge} f.$$

For n = 2, (3.3) is trivial since $u^{(1)}$ and $u^{(2)}$ are, in this case, orthogonal. Suppose that, for n > 2, (3.3) fails for all permissible pairs k,m, i.e.,

$$\frac{n-2}{n-1} < |\sum_{\nu=1}^{n} u_{k\nu}\bar{u}_{m\nu}|$$

for $k \neq m$, $1 \leq k,m \leq n$. Since $u_{kk} = u_{mm} = 0$, this implies

$$2\left(\frac{n-2}{n-1}\right) < \sum_{\nu \neq k,m} |u_{k\nu}|^2 + \sum_{\nu \neq k,m} |u_{m\nu}|^2$$

$$= 2 - |u_{km}|^2 - |u_{mk}|^2,$$

i.e.,

$$\frac{2}{n-1} > |u_{km}|^2 + |u_{mk}|^2.$$

Adding over all $\frac{1}{2}n(n-1)$ pairs k,m (and observing that $u_{kk} = 0$, $k = 1,\ldots,n$), we obtain

$$n > \sum_{k,m=1}^{n} |u_{km}|^2 = n.$$

Hence, there must exist a pair k,m for which (3.3) holds.

Suppose now that (1.1) has an oscillatory solution $w = (w_1,\ldots,w_n)$, i.e., suppose there exist points $z_1,\ldots,z_n$ in $|z| < 1$ such that $w_k(z_k) = 0$, $k = 1,\ldots,n$. If $u(z)$ is defined by $w(z) = u(z)\|w(z)\|$, we then may apply the lemma to the unit vectors $u(z_k)$. Accordingly, there exist subscripts k,m such that $|u(z_k)u(z_m)| \leq (n-2)(n-1)^{-1}$.

Next, we subject z to a Möbius transformation $z \to t$ which carries the points z_k, z_m into two points t_1, t_2 on the real axis in the t-plane. As shown in the proof of Theorem 2.2, this can be done in such a way that the coefficient matrix of the transformed equation $\eta' = B\eta$ satisfies the inequality $\|B(t)\| \leq p(|t|)$ if (3.2) holds. We note that for the function v defined by $\eta(t) = v(t)\|\eta(t)\|$ we have $|v(t_1)v(t_2)| = |u(z_k)u(z_m)| \leq (n-2)(n-1)^{-1}$.

We set $\eta = \alpha + i\beta$, $B = B_1 + iB_2$ (α, β, B_1, B_2 real), and denote the real 2n-dimensional vector (α, β) by γ. If the real $2n \times 2n$ matrix D is defined by

$$D = \left(\begin{matrix} B_r & B_2 \\ B_2 & B_1 \end{matrix} \right) >$$

the equation $TJ' = BI7$ is equivalent to the real equation $y' = {}^D Y;$, and we have $||B|| = ||D||,$ $||T|| = ||y||.$ If 6 is a constant unit vector in E^{2_n} and a is defined by $y = a||y||,$ then, as shown in [5], the function $x = 6a$ is subject to the inequality

$$\frac{-\%}{\sqrt{1-x^*}} \quad 1 \quad ||D||.$$

Since $6a$ can be written in the form $Re\{cv\}$, where c is a complex n-dimensional unit vector, and since $||p|| = ||B||,$ this is equivalent to

$$j\dot{r}^{\varphi'}_{,} \wedge i\mathbb{N},$$

where $<p = Re\{cv\}.$ Integrating this inequality from t_1 to t_2 (where the subscripts are so chosen that $t_1 < t_2$), and setting $c = v(t_1)$, we obtain

$$| \text{arc cos } Refv(t_1) v(t_2)] | \pounds \int_{t_1}^{t_2} ||B||dt \pounds \int_{-1}^{1} ||B||dt \pounds 2\int_{o}^{1} p(t)dt.$$

Since $| Re\{v(t_1) v(t_2)\} | \pounds |v(t_1)v(t_2)| \wedge (n-2)(n-1) \sim^1$, this implies

$$\text{arc cos } \hat{\,}y^{\angle} \wedge 2j^{|} \int_{o}^{1} p(t)dt,$$

i.e.,

$$\sin J \int_{o}^{1} p(t)dt \hat{\,}\_ [2(n-1)]^{-\frac{1}{2}}.$$

The assumption that (1.1) is oscillatory in $|z| < 1$ has thus led to an inequality which is incompatible with (3.1), and Theorem 3.1 is proved.

4.   In this final section we derive two necessary conditions for the suborthogonality of analytic systems (1.1) in the unit disk. Necessary conditions for the nonoscillation of such systems can be found in a recent paper by M. Lavie [4].

Theorem 4.1. *Let* $\Lambda(A) = \max \alpha A \alpha$ *for* $\|\alpha\| = 1$, *i.e., let* $\Lambda(A)$ *be the largest eigenvalue of the Hermitian part of the matrix* A. *If the analytic system* (1.1) *is suborthogonal in* $|z| < 1$, *then*

$$(4.1) \qquad\qquad \Lambda[A(z)] \leq \frac{1}{1-|z|^2} \, .$$

The inequality is sharp for all $|z| < 1$.

Theorem 4.2. *If* (1.1) *is suborthogonal in* $|z| < 1$, *then*

$$(4.2) \qquad\qquad \|A(z)\| \leq \frac{\sqrt{2}}{1-|z|^2} \, .$$

We begin with the remark that it is sufficient to prove both (4.1) and (4.2) for $z = 0$. To see this, we map the unit disk onto itself by the transformation $z \to t$, where $z = (t+\alpha)(1+\overline{\alpha}t)^{-1}$ and $\alpha (|\alpha|<1)$ is a constant. In this mapping, (1.1) is transformed into the equation $\frac{d\sigma}{dt} = B(t)\sigma$, where

$$w(z) = \sigma(t), \quad B(t) = \frac{1-|\alpha|^2}{(1+\overline{\alpha}t)^2} \, A\left[\frac{t+\alpha}{1+\overline{\alpha}t}\right].$$

Since $B(0) = (1-|\alpha|^2)A(\alpha)$ and the transformation does not affect the suborthogonality of the equation, this establishes our assertion.

To prove Theorem 4.1, we observe that it follows from the suborthogonality of the equation that $\mathrm{Re}\{w(\beta z)w(\beta\overline{z})\} > 0$ for any $|z| < 1$, $|\beta| = 1$. Using the power series expansion of the vector w,

we have $w(pz) = w(o) + pzw'(o) + 0(z^2)_f$ and a similar expression for $w(p''\bar{z})$. Hence

$$h(z) \equiv w(kz)w(k\bar{z}) = ||w(o)||^2 + 2zRe\{pw»(o)w(o)\} + 0(z^2).$$

The (scalar) function $h(z)$ is analytic in $|z| < 1$ and has there a positive real part. By a classical result, this implies $|h^T(o)| \pounds 2 Refh(o)\}$. Thus, if $\arg p$ is so chosen that $pw'(o)w(o) \wedge 0$, $|w'(o)w(o)| 11||w(o)||^2$, or, by (1.1),

$$|w(o)A(o)w(o)| \pounds ||w(o)||.$$

Since $w(o)$ may be taken to be an arbitrary non-zero vector, we obtain $A[A(o)] \pounds 1$. As remarked above, this is equivalent to (4.1).

To show that the inequality (4.1) is exact, we consider the equation $w^T = Aw$ with $A = (1-z^2)^{-1}I$, where I is the nxn unit matrix. The general solution of this equation is $w = f(z)a$, where $f(z) = [(1+z)(1-z)^{-1}]^{1/2}$ and a is an arbitrary cons-**tant** vector. The equation is suborthogonal in $|z| < 1$. Indeed, $Re\{w(z)w(\pounds)\} = ||a||^2 Re\{f(z)f(\pounds)\}$, and it is easily confirmed that the latter expression is positive for $|z| < 1, |\pounds| < 1$. Since $||A(z)|| = |1-z^2|^{-1}$, we have equality in (4.1) for all $z \in (-1,1)$.

Theorem 4.2 is a consequence of the following result.

If $w(z)$ is an analytic vector in $|z| < 1$ such that $Re\{w(z)w(\pounds)\} > 0$ for $|z| < 1, |\pounds| < 1$, then

(4.3) $$||w'(o)|| \leq \sqrt{2} ||w(o)||.$$

The constant $\sqrt{2}$ is the best possible.

Indeed, if (1.1) is suborthogonal in $|z| < 1$ then (4.3) shows that, for any nontrivial solution w, $\|A(o)w(o)\| \leq \sqrt{2}\, \|w(o)\|$. Since $w(o)$ is arbitrary, we have $\|A(o)\| \leq \sqrt{2}$, and this is equivalent to (4.2).

To prove (4.3), we note that, for any t in the unit disk, $\text{Re}\{w(tz)w(\bar{t}z)\} > 0$. If $w(z) = a_o + a_1 z + \ldots$ is the power series expansion of w, we thus have

$$0 < \text{Re}\{\frac{1}{2\pi}\int_{|z|=1} [\sum_{k=o}^{\infty} a_k(zt)^k][\sum_{k=o}^{\infty} a_k(\bar{z}t)^k]|dz|\}$$

$$= \text{Re}\{\sum_{k=o}^{\infty} \|a_k\|^2 t^{2k}\}$$

for $|t| < 1$. Since the function in braces has a positive real part, it follows from the result quoted above that $\|a_1\|^2 \leq 2\|a_o\|^2$. Because of $a_o = w(o)$, $a_1 = w'(o)$, this proves (4.3).

We now construct an example which shows that the inequality (4.3) is sharp. We denote by $P(z)$ a polynomial of degree $n - 1$, and we define the vector $w = (w_1, \ldots, w_n)$ by $w_k = P(\omega^k z)$, $k = 1, \ldots, n$, where $\omega = \exp(2\pi i n^{-1})$. If $P(z) = b_o + b_1 z + \ldots + b_{n-1} z^{n-1}$, it follows from the properties of the roots of unity that

$$w(z)w(\zeta) = n \sum_{k=o}^{n-1} |b_k|^2 (z\bar{\zeta})^k.$$

Accordingly, we shall have $\text{Re}\{w(z)w(\zeta)\} > 0$ if the polynomial $R(z) = |b_o|^2 + |b_1|^2 z + \ldots + |b_{n-1}|^2 z^{n-1}$ has a positive real part for $|z| < 1$. A polynomial with this property is

$$R(z) = 1 + 2 \sum_{k=1}^{n-1} \frac{n-k}{n} z^k.$$

Indeed, $R(z)$ is the n-th Fejér mean of the partial sums of the
function $f(z) = (1+z)(1-z)^{"1}$, for which $\text{Re}(f(z)) > 0$ in
$|z| < 1$, and the Fejér means of a function with positive real
part share this property [6]. We thus may set $b^{\wedge}[2-2kn^{-1}]^{1/2}$,
$k = 1,\ldots,n-1$, $b_o = 1$. *The* corresponding vector w will then
satisfy $\|w(o)\|^2 = n$, $\|w»(o)\|^2 = 2n(1-n\sim^1)$, i.e.,

$$\|w'(o)\| = \sqrt{2(1-)} \ \|w(o)\|,$$

and this shows that the constant VT in (4.3) cannot be replaced
by a smaller number.

This, of course, does not imply that the constant $\backslash f\sim 2$ in
(4.2) is also the best possible. All we can say is that the true
constant is ≥ 1, since $A(A) \pounds \|A\|$ and (4.1) is a sharp inequality.

References

[1] Courant, R., and D. Hilbert, <u>Methods</u> <u>of</u> <u>Mathematical</u> <u>Phy</u>-<u>sics</u>, Vol.1, New York, Interscience, 1953.

[2] Hardy, G. H., J. E. Littlewood, and G. Pólya, <u>Inequalities</u>, Cambridge University Press, 19

[3] Kim, W. J., "Disconjugacy and disfocality of differential systems", J. Math. Anal. Appl. <u>26.</u>(1969), 9-19.

[4] Lavie, M., "Some function-theoretic aspects of disconjugacy of linear differential systems", Trans. Am. Math. Soc. <u>143</u> (1969), 153-171.

[5] Nehari, Z., "Oscillation theorems for systems of linear equations, Trans. Am. Math. Soc. <u>139</u>(1969), 339-347.

[6] Pólya, G., and G. Szegö, <u>Aufgaben</u> <u>und</u> <u>Lehrsatze</u> <u>aus</u> <u>der</u> <u>Analysis</u>, New York, Dover, 1945.

[7] Schwarz, B., "Norm conditions for disconjugacy of complex differential systems", J. Math. Anal. Appl. <u>218</u>(1969), 553-568.

[8]_____, "Mappings of domains by components of solutions of differential systems", to appear.

[9]_____, "Curves on the unit sphere and disconjugacy of differential systems", to appear.