A ONE-SIDED FUBINI THEOREM
FOR GOWURIN MEASURES

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1. Introduction

Recently much interest has been shown in the notion of Gowurin measures (for example see [3],[4],[5], and [6]). They have been used to give a neat integral representation for bounded linear operators on the space of continuous functions defined on a compact Hausdorff space with values in a normed linear space (see [3],[4],[8]). In [3] there was obtained an integration theory with respect to such measures for functions defined on compact spaces with values in a normed linear space. In [6] there was obtained a generalization of a Riesz representation theorem contained in [2]. The main purpose of our paper is to further the study of Gowurin measures by developing a Fubini type theorem for such measures.

It must be emphasized that in our Fubini type theorem for Gowurin measures the so-called iterated integrals cannot be interchanged. In other words, the order of integration may not be reversed which is definitely not the case with the usual Fubini Theorem. Also we want to emphasize that we are talking about finitely additive set functions whose functional values are bounded linear operators.

Our first step will be to define a product measure. Then we will obtain some results similar to the usual results for the cross product of two measures.

Let X,Y and Z be normed linear spaces, let H be a compact Hausdorff space and let f be a scalar valued function on $H \times H$. We will represent elements in $H \times H$ by pairs (s,t) and elements in X by x. Let K_1 and K_2 be Gowurin measures defined on the Borel field of H. The range of K_1 is to be in $B(X,Y^{**})$ and the range of K_2 is to be in $B(Y^{**},Z^{**})$ where $B(S_1,S_2)$ denotes all bounded linear operators from S_1 to S_2 (X^{**} is the bidual of X). Assuming that the scalar valued function f is continuous the first theorem shows that

$$\int_H dK_2(t)\int_H dK_1(s)(f(s,t)\cdot x) = \int_{H\times H} d(K_1\times K_2)(f(s,t)\cdot x).$$

Now suppose

$$\int_H dK_2(t)\int_H dK_1(s)(f_n(s,t)\cdot x) = \int_{H\times H} d(K_1\times K_2)(f_n(s,t)\cdot x)$$

where $\{f_n\}_{n\in N}$ denotes some sequence of scalar valued functions defined on $H \times H$. If f is now in some sense the limit of the functions f_n, we ask when can we write

$$\int_H dK_2(t)\int_H dK_1(s)(f(s,t)\cdot x) = \int_H d(K_1\times K_2)(f(s,t)\cdot x).$$

A partial answer to this question is given here as described in the next paragraph.

In [8] the notions of "convergence almost everywhere" and "convergence in measure" were given. An example was also given showing that "convergence almost everywhere" does not imply "convergence in measure" which of course is contrary to the usual arrangements. We extend results in this area by showing that the limit case of the Fubini theorem holds if the following three conditions are satisfied: (1) f_n converges $K_1 \times K_2$ a.e. to f; (2) convergence a.e. implies convergence in measure; (3) K_2 is <u>linearly non-zero</u> relative to K_1.

2. <u>Definitions and Notations</u>

As above X,Y,Z will denote normed linear spaces and X^{**}, Y^{**}, Z^{**} will denote the corresponding bidual spaces. Also H will denote a compact Hausdorff space and Σ will denote the Borel field of H.

If S is any non-empty set then [S] will denote the collection of finite subsets of S. If S is any normed linear space then $|\cdot|_S$ will denote the norm on S. If X and Y are any two normed linear spaces, then B(X,Y) denotes the class of bounded linear operators from X to Y.

A finite collection $P = P_n = \{e_1, \ldots, e_n\}$ of elements of Σ is a <u>partition</u> of H if H is the union of the e_i and if the e_i are pairwise disjoint. If f maps H to X and if $\epsilon > 0$, then the partition is called an ϵ-<u>partition of</u> H <u>with respect</u> to f if the diameter of f(e) is $\leqq \epsilon$ for all **e** in P. The partition P is an <u>essential partition of</u> H if for each i, $e_i \neq \emptyset$ and $e_1 \cup \ldots \cup e_i$ is a closed G_δ (for some ordering on the sets in P).

Let λ be a finitely additive set function defined on the Borel field Σ of the compact Hausdorff space H, having its values in some space B(X,Y) (X and Y normed linear spaces). The X-Gowurin <u>constant for</u> λ <u>over</u> H is defined to be

$$W(X,\lambda:H) = \sup|\Sigma\lambda(e)x|_Y$$

where the supremum is taken over all (finite) partitions P_n of H, $P_n = \{e_1,\ldots,e_n\}$, and for any choice of $F_n \in [X]$ (where F_n is a subset of n elements of X) with $e\in P_n$, $x\in F_n$ and $|x| \leq 1$. The finitely additive set function λ is called a Gowurin measure if this supremum $W(X,\lambda:H)$ is $\leqq \infty$.

Thus if λ is a Gowurin measure then

$$|\Sigma\lambda(e)x|_Y \leqq W(X,\lambda:H)\sup|x|_X$$

for all (finite) partitions P_n of H and for all finite collections F_n in $[X]$ where $e\in P_n$ and $x\in F_n$.

In this paper our finitely additive set functions λ will have their functional values in $B(X,Y^{**})$ and $B(Y^{**},Z^{**})$. Hence we will refer to the Gowurin constants simply as W_λ. When we wish to consider a particular $E\in\Sigma$, then by $W_\lambda(E)$ we will mean $\sup|\Sigma\lambda(e)x|_Y$ where the supremum is taken over all (finite) partitions $P(E) = P_n$ of E by subsets of Σ and for all $F_n\in[X]$ with $e\in P(E)$, $x\in F_n$ and $|x| \leq 1$. This $W_\lambda(E)$ will be referred to as the Gowurin constant of λ on E.

The space of summable functions relative to λ has been defined in [3]. It has been shown that all continuous functions on H are summable.

If h is a function from H to X then h is called measurable if $h^{-1}(G)$ is an element of the Borel field Σ of H for all open sets G in X. The function h is called summable (integrable) if it is measurable and if it satisfies the following condition: Suppose for each $\epsilon > 0$ there exists an ϵ-partition $P(\epsilon)$ of H with respect to h by subsets of Σ (that is, for all $e \in P(\epsilon) \subset \Sigma$, $|h(s_1)-h(s_2)|_X \leqq \epsilon$ for all s_1 and s_2 in e). Suppose also there exists a fixed element y^{**} of Y^{**} such that for any refinement $P = \{e_1,\ldots,e_n\}$ of $P(\epsilon)$ then $|y^{**} - \sum_{i=1}^{n} \lambda(e_i)h(t_i)|_{Y^{**}} < \epsilon$ where t_i is any point in e_i for each i. The point y^{**} is called the <u>integral of</u> h <u>relative to</u> λ and is denoted by $\int_H d\lambda \cdot h$.

Actually in [3] the definition of summability is restricted to "essential ϵ-partitions". However, an argument in [7] demonstrates that the definition can be made in terms of a partition of H with the subsets from Σ.

If f is a scalar-valued function defined on H, then for $x \in X$, $f \circ x$ will denote the function from H to X defined by $(f \circ x)(t) = f(t) \cdot x$ for all $t \in H$.

Elementary properties shown in [3] will be used in the arguments. In particular, we will use for h, a summable function from H to X,

$$|\int_H d\lambda \cdot h| \leq W(\lambda)\, |h|_C$$

where

$$|h|_C = \sup_{t \in H} |h(t)|_X .$$

3. <u>The Fubini Theorems</u>

For the compact space H, let K_1 and K_2 be two Gowurin measures defined on the Borel field Σ of H. The range of K_1

is in $B(X,Y^{**})$ and the range of K_2 is in $B(Y^{**},Z^{**})$. Let $A \times B$ be a measurable rectangle in $\Sigma \times \Sigma$ and let $x \in X$. We can now define the <u>product</u> <u>measure</u> as

$$(K_1 \times K_2)(A \times B)x = K_2(B)[K_1(A)x].$$

Then $(K_1 \times K_2)(A \times B)$ is in $B(X,Z^{**})$.

Let W_1 and W_2 denote the Gowurin constants of K_1 and K_2 respectively. Our first step is to extend $K_1 \times K_2$ as a finitely additive set function to the field $\mathcal{E}$ generated by measurable rectangles. Moreover it shall be shown that $K_1 \times K_2$ is Gowurin on $\mathcal{E}$.

<u>Lemma</u> 1. <u>The product measure</u> $K_1 \times K_2$ <u>extends uniquely to</u> $\mathcal{E}$, <u>the field generated by the collection of measurable rectangles</u>.

<u>Proof</u>. Let $\mathcal{G}_o$ be the set of bounded complex valued functions defined on $H \times H$ and let f be in $\mathcal{G}_o$ such that for each fixed t in H and for each x in X the function $f(s,t) \cdot x$ is K_1-integrable as a function of s and such that $\int_H dK_1(s)(f(s,t) \cdot x)$ is an integrable function of t relative to K_2. Let $\mathcal{G}_1$ be all functions g of $\mathcal{G}_o$ such that $g \circ f$ is in $\mathcal{G}_o$ for all f in $\mathcal{G}_o$.

It is clear that $\mathcal{G}_1$ is linear and closed under products. Let $\mathfrak{F}$ be the collection of all subsets of $H \times H$ such that if $F \in \mathfrak{F}$ then χ_F is in $\mathcal{G}_1$ where χ_F is the characteristic function of F. Since $\mathcal{G}_1$ is closed under products, it follows that $\mathfrak{F}$ is closed under intersections. It is also immediate that $\mathfrak{F}$ is closed under complementation. Consequently $\mathfrak{F}$ is a field. It is obvious that every measurable rectangle is in $\mathfrak{F}$, since

$$\int_H dK_1(s)\chi_{A\times B}(s,t)(f(s,t)\cdot x) = \chi_B(t)\int_A dK_1(s)(f(s,t)\cdot x)$$

and the product of $\chi_B(t)$ with any function of t which is K_2 integrable is again K_2 integrable.

Now for any $F\in\mathfrak{F}$ and for any x in X define

$$\mu(F)x = \int_H dK_2(t)[\int_H dK_1(s)(\chi_F(s,t)\cdot \mathbf{x})].$$

Clearly $\mu(F)$ is in $B(X,Z^{**})$. In fact $\|\mu(F)\| \leqq W_2\cdot W_1$. It is now necessary to check that μ coincides with $K_1\times K_2$ over measurable rectangles $A\times B$. Let x be in X then

$$\mu(A\times B)x = \int_H dK_2(t)[\int_H dK_1(s)\chi_A(s)(\chi_A(s)\cdot(\chi_B(t)\cdot x))]$$

$$= \int_H dK_2(t)(\chi_B(t)\cdot K_1(A)x)$$

$$= K_2(B)[K_1(A)x] = (K_1\times K_2)(A\times B)\mathbf{x}.$$

Of course this holds for all x in X so $\mu(A\times B) = (K_1\times K_2)(A\times B)$. By elementary properties of the integral, μ is finitely additive on $\mathfrak{F}$ and hence on the field $\mathcal{E}$ generated by the measurable rectangles. It remains to show that this extension is unique.

Let λ be another finitely additive measure which coincides with $K_1\times K_2$ on $\mathcal{E}$. Let $\mathfrak{B}_o$ be all functions f of $\mathfrak{G}_o$ which are λ-integrable and such that

$$\int_{A\times B} d\lambda(s,t)(f(s,t)\cdot x) = \int_B dK_2(t)[\int_A dK_1(s)(\ f(s,t)\cdot x)]$$

for all $A\times B$ in $\Sigma\times\Sigma$. Let $\mathfrak{B}_1$ be all functions g of $\mathfrak{B}_o$

such that $g \circ f$ is in $\mathcal{B}_o$ for all f in $\mathcal{B}_o$. Let $\mathfrak{F}'$ be all subsets of $H \times H$ such that if F is in $\mathfrak{F}'$ then χ_F is in $\mathcal{B}_1$. Again $\mathfrak{F}'$ is a field containing $\mathcal{E}$. Thus for all F in $\mathcal{E}$,

$$\lambda(F)\cdot x = \int_{H\times H} d\lambda(s,t)(\chi_F(s,t)\cdot x) = \int_H dK_2(t)[\int_H dK_1(s)(\chi_F(s,t)\cdot x)]$$
$$= \mu(F)x.$$

Thus $\lambda = \mu$ on $\mathcal{E}$. This completes the proof of the Lemma.

<u>Lemma</u> 2. <u>The product measure</u> $K_1\times K_2$ <u>is Gowurin over</u> $\mathcal{E}$.

<u>Proof</u>. It shall be shown in fact that if W is the Gowurin constant of $K_1\times K_2$ then $W \leq W_1\cdot W_2$. Let $\{E_1,\ldots,E_n\}$ be any partition of $H \times H$ and let $\{x_1,\ldots,x_n\}$ be a finite collection of elements of X such that $E_i\in\mathcal{E}$ and $|x_i|_X \leqq 1$ for $i=1,\ldots,n$. Then

$$\left|\sum_{i=1}^{n} K_1\times K_2(E_i)\cdot x_i\right|_{Z^{**}} = \left|\sum_{i=1}^{n}\int_H dK_2(t)[\int_H dK_1(s)(\chi_{E_i}(s,t)\cdot x_i)]\right|_{Z^{**}}$$
$$= \left|\int_H dK_2(t)[\int_H dK_1(s)(\sum_{i=1}^{n}\chi_{E_i}(s,t)\cdot x_i)]\right.$$
$$\leqq W_2\cdot W_1 \sup_t \sup_s \left|\sum_{i=1}^{n}\chi_{E_i}(s,t)\cdot x_i\right|_C$$
$$\leqq W_2\cdot W_1 \operatorname{Max}|x_i|_X .$$

We are now in a position to show one of our main theorems.

<u>Theorem</u> 1. <u>For every continuous scalar-valued function</u> f <u>on the compact space</u> $H \times H$,

$$\int_H dK_2(t)[\int_H dK_1(s)(f(s,t)\cdot x)] = \int_{H\times H} d(K_1\times K_2)(s,t)(f(s,t)\cdot x).$$

<u>Proof</u>. First it will be shown that $f(s,t)\cdot x$ is an integrable

function from $H \times H$ to X relative to $K_1 \times K_2$. Let $\epsilon > 0$. By an argument similar to the corresponding argument used in [3] it can be shown that for any two ϵ-partitions $\{e_1, \ldots, e_n\}$ and $\{e_1', \ldots, e_m'\}$ of $H \times H$ into subsets of $\mathcal{E}$

$$(1) \qquad \left| \sum_{i=1}^{n} (K_1 \times K_2)(e_i)(f(s_i,t_i)\cdot x) - \sum_{j=1}^{m} (K_1 \times K_2)(e_j')(f(s_j',t_j')\cdot x) \right|_{Z^{**}} \leqq W \max_{i,j} |f(s_i,t_i) - f(s_j',t_j')| \cdot |x|_X$$

where $(s_i,t_i) \in e_i$, $(s_j',t_j') \in e_j'$, $i = 1, \ldots, n$; $j = 1, \ldots, m$ and W is the Gowurin constant of $K_1 \times K_2$ (as in Lemma 2). By the compactness of H and the continuity of f one can find a finite partition $\{e_1, \ldots, e_n\}$ of $H \times H$, e_i in $\mathcal{E}$, $i = 1, \ldots, n$, such that $|f(s_i,t_i) - f(u_i,v_i)| < \epsilon$ if $(s_i,t_i) \in e_i$ and $(u_i,v_i) \in e_i$, $i = 1, \ldots, n$. Now let $(s_i,t_i) \in e_i$ and let $x_i = f(s_i,t_i)\cdot x$ for each $i = 1, \ldots, n$. Then as ϵ approaches zero, $\sum_{i=1}^{n} \chi_{E_i} \cdot x_i$ will approach $f(s,t)\cdot x$. Since Z^{**} is complete, inequality (1) shows that $\sum_{i=1}^{n} (K_1 \times K_2)(e_i)(f(s_i,t_i)\cdot x)$ converges to some limit say z^{**} in Z^{**}. Inequality (1) also shows that the limit does not depend on $\{e_1, \ldots, e_n\}$ and the choice of (s_i,t_i) in e_i, $i = 1, \ldots, n$, provided $\{e_1, \ldots, e_n\}$ forms an $\mathcal{E}$-partition. Therefore $f(s,t)\cdot x$ is integrable and

$$\int_{H \times H} d(K_1 \times K_2)(s,t)(f(s,t)\cdot x) = z^{**}.$$

Since f is a continuous function, $f(s,t)\cdot x$ is integrable relative to K_1 (see [3]). Since K_1 is Gowurin, $\int_H dK_1(s)(f(s,t)\cdot x)$ is a continuous function of t. Hence it is K_2-integrable. Thus all the integrals are well defined.

The equality holds for all functions of the form $\chi_{A\times B}(s,t)\cdot x$ and thus for all functions of the form $\sum_{i=1}^{n} \chi_{A_i}^{\circ} (\chi_{B_i}(s,t)\cdot x)$. We have seen that $f(s,t)\cdot x$ is the limit in the norm of functions of the form $\sum_{i=1}^{n} \chi_{A_i}^{\circ} (\chi_{B_i}(s,t)\cdot x_i)$. Thus the equation holds for all functions $f(s,t)\cdot x$ where f is continuous. This completes the proof of the theorem.

In [7] the notions of convergence a.e. and convergence in measure were defined for Gowurin measures. They are stated here for completeness. Let λ be a Gowurin measure defined on Σ. Suppose $E\in\Sigma$. Then E is a <u>Gowurin negligible set</u> (or often times more specifically a λ <u>negligible set</u>) if the Gowurin constant of λ on E is zero, that is $W_\lambda(E) = 0$. A property $P(t)$ holds for almost all t in H if the set of points for which $P(t)$ is not true is a λ negligible set. Let the sequence $\{f_n\}_{n\in N}$ and the function f be measurable functions from H to X. The sequence $\{f_n\}_{n\in N}$ <u>converges</u> a.e. <u>in Gowurin measure</u> (or <u>converges</u> λ a.e.) to f if the set of points s in H for which the sequence $\{f_n(s)\}$ does not converge to f is a λ negligible set. The sequence $\{f_n\}_{n\in N}$ <u>converges in Gowurin measure</u> (or <u>converges in</u> λ-<u>measure</u>) to f if for every $\epsilon > 0$ there exists a natural number N such that for all $n \geqq N$ the set $E_n = (s\in H : |f_n(s)-f(s)|_X > 1/n\}$ has Gowurin constant less than ϵ and $E_n\in\Sigma$.

An example is given in [7] which shows convergence a.e. in Gowurin measure does not imply convergence in Gowurin measure. In this respect we would like to prepare the way for the second main result stated in the Introduction. We first give some elementary results which are helpful and of interest in themselves.

Lemma 3. *Assume that* λ *is a Gowurin measure from* Σ *to* $B(X,Y^{**})$. *If* $\{A_n\}_{n\in N}$, $A_n \in \Sigma$ *for all* $n \in N$, *is any sequence such that the intersection of the sets* A_n *is a* λ *negligible set and for every* $\epsilon > 0$ *there exists an integer* K *such that* $\bigcap_{n=1}^{K} A_n$ *has Gowurin constant less than* ϵ, *then "convergence* a.e. *in Gowurin measure", implies "convergence in Gowurin measure".*

The proof for the above is straightforward. Hence it is omitted.

Proposition 4. *Let* K *be a Gowurin measure from* Σ *to* $B(X,Y^{**})$. *If* K *is weakly countably additive on* Σ (t^*K *is countably additive for all* $t^* \in B(X,Y^{**})^*$), *then "convergence* K a.e." *implies "convergence in* K *measure".*

Proof. Let $\{E_n\}_{n\in N}$ be a sequence of sets in Σ such that $E_{n+1} \subset E_n$ for all n. By Lemma 3 if $E = \bigcap_{n=1}^{\infty} E_n$ and if $W_K(E) = 0$ it is sufficient to show that for every $\epsilon > 0$ there exists N such that $n \geq N$ implies $W_K(E_n) \leq \epsilon$. By the Pettis Theorem (see [2]) the weakly countably additive Gowurin measure K is countably additive on Σ. Since $W_K(E) = 0$ it follows that $K(A) = 0$ for all $A \subset E$ and $A \in \Sigma$. Now define the function v_K on Σ by $v_K(L) = \sup \sum_{i=1}^{n} |K(e_i)|$ where $L \in \Sigma$, where the sup is taken over all finite Σ partitions $\{e_1,\ldots,e_n\}$ of L and where $|K(e_i)|$ is the norm of $K(e_i)$ as an element of $B(X,Y^{**})$. Following similar arguments to [1] it can be shown that v_K is a countably additive function on Σ and also $v_K(E) = 0$. Thus,

$v_K(E_n) = \sum_{i=n}^{\infty} v(E_i - E_{i+1})$. Hence for all m large enough $|v_K(E_m)| < \epsilon$. Choose such an m and let $\{e_{1m}, \ldots, e_{nm}\}$ be a Σ partition of E_m. Let $\{x_{1,m}, \ldots, x_{n,m}\}$ be a collection of points in X with $|x_{i,n}|_X \leq 1$ for $i = 1, \ldots, n$. Then

$$\left| \sum_{i=1}^{n} K(e_{i,m}) \cdot x_{i,m} \right| \leq v_K(E_m) < \epsilon$$

and hence $W_K(E_n) < \epsilon$ for all n sufficiently large. This completes the proof.

It is now necessary to define the notion of a section and to show some of the usual properties of sections. Let E be in $\mathcal{E}$. Then for t in H define $E_t = \{s \in H | (s,t) \in E\}$. If $E = A \times B$ then $E_t = A$ if $t \in B$ and $E_t = \emptyset$ if $t \notin B$. It is clear from this that for all $E \in \mathcal{E}$, E_t is in Σ.

Proposition 5. *If K_1 and K_2 are as defined previously then for every $x \in X$ and $E \in \Sigma$*

$$\int_H dK_2(t)[K_1(E_t)x] = (K_1 \times K_2)(E)x.$$

Proof. Let $g(t) = K_1(E_t)x$ for all t in H. We shall show that the function g from H to Y^{**} is integrable relative to K_2. Let $E = A \times B$. Now $g(t) = \chi_B(t)(K_1(A)x)$ and $\int_H dK_2(t)g(t) = \int_B dK_2(t)[K_1(A)x] = K_1 \times K_2(A \times B)x$. Now let E be a disjoint union of measurable rectangles. The above computations generalize trivially to E. This completes the proof.

It is now necessary to define a natural property which will enable one to deduce that if E is a $K_1 \times K_2$ negligible set then the set of t for which E_t is not K_1 negligible is a K_2 negligible set. With this property a limit case of the

Fubini theorem will hold.

Definition. Let K_1 and K_2 be as defined above. The measure K_2 is said to be linearly non-zero relative to K_1 if for every relation $\sum_{i=1}^{n} K_1(e_i)x_i \neq 0$ with $\{e_1,\ldots,e_n\}$ and $\{x_1,\ldots,x_n\}$ such that $e_i \in \Sigma$; e_i are pairwise disjoint; $x_i \in X$, $|x_i| \leq 1$ for all i; whenever C is in Σ with $W_{K_2}(C) \neq 0$ there exists a partition $\{c_1,\ldots,c_m\}$ of C and there exist vectors $x_{i,j} \in X$, $|x_{i,j}| \leq 1$, $c_j \in \Sigma$, $j = 1,\ldots,m$; $i = 1,\ldots,n$ such that $\sum_{i,j} K_2(c_j)K_1(e_i)x_{i,j} \neq 0$.

To show the designated form of the Fubini Theorem we need the following lemma and corollary.

Lemma 6. Let K_2 be linearly non-zero relative to K_1. If E in $\mathcal{E}$ is $K_1 \times K_2$ negligible then the set C of points t for which E_t is not K_1 negligible is K_2 negligible.

Proof. If E is $K_1 \times K_2$ negligible then the Gowurin constant of $K_1 \times K_2$ on E is zero. Now let C be the set of points t for which E_t is not K_1 negligible. Hence for each t in C there exists a partition $\{e_1,\ldots,e_n\}$ of E_t and a finite set of points $\{x_1,\ldots,x_n\}$ of X, $|x_i| \leq 1$ for all i, such that $\sum_{i=1}^{n} K_1(e_i)x_i$ is not equal to zero. By hypothesis K_2 is assumed to be linearly non-zero relative to K_1. If we assume also the Gowurin constant of K_2 on C is non-zero then there exists a partition $\{c_1,\ldots,c_m\}$ and there exist vectors $x_{i,j}$ in X, $|x_{i,j}| \leq 1$, $c_j \in \Sigma$, $j = 1,\ldots,m$, $i = 1,\ldots,n$, such that $\sum_{i,j} K_2(c_j)K_1(e_i)x_{i,j} \neq 0$. Now for an elementary set $E = A \times B$ we have that the collection $\{e_i \times c_j\}_{i,j=1}^{n,m}$ are pairwise disjoint sets contained in E which contradicts the fact that the Gowurin constant of $K_1 \times K_2$ on E

is zero. Since the proof generalizes immediately to the union of disjoint rectangles we have that the Gowurin constant of K_2 on C must be zero that is, C is K_2 negligible.

Corollary. Let K_2 be linearly non-zero relative to K_1. If the sequence $\{f_n\}_{n\in N}$ converges $K_1 \times K_2$ a.e. to the function f, then except for t in a K_2 negligible set the sequence $\{f_n(s,t)\}_{n\in N}$ converges K_1 a.e. to $f(s,t)$ as a function of s.

Proof. Let E be the set of ordered pairs (s,t) in $H \times H$ for which $f_n(s,t)$ does not converge to $f(s,t)$. Now E_t by definition is the set of points s in H for which $f_n(s,t)$ does not converge to $f(s,t)$. Since E is $K_1 \times K_2$ negligible, the Lemma says that except for the points t in a K_2 negligible set A, E_t is K_1 negligible. But this means that except for the points t in A, the sequence $\{f_n(s,t)\}_{n\in N}$ converges K_1 a.e. to $f(s,t)$ as a function of s.

We can now give the limit case of our Fubini Theorem.

Theorem 2. Let K_1 and K_2 be as above. Assume also the following hypotheses:

(1) Gowurin convergence a.e. implies convergence in Gowurin measure.

(2) The measure K_2 is linearly non-zero relative to K_1.

(3) The sequence $\{f_n\}_{n\in N}$ of scalar-valued functions on $H \times H$ converges $K_1 \times K_2$ a.e. to the scalar-valued function f (on $H \times H$) where f is integrable relative to $K_1 \times K_2$ and $\int_H dK_1(s)(f(s,t)\cdot x)$ is K_2 integrable as a function of t.

(4) There exists a scalar-valued function g (on $H \times H$) integrable relative to $K_1 \times K_2$ such that $|f_n - g| < B$

where B *is some constant and* $\int_H dK_1(s)(g(s,t)\cdot x)$ *is* K_2 *integrable as a function of* t.

(5) *For each* n,

$$\int_H dK_2(t)\int_H dK_1(s)(f_n(s,t)\cdot x) = \int_{H\times H} d(K_1\times K_2)(f_n(s,t)\cdot x).$$

Under such hypothesis then

$$\int_H dK_2(t)\int_H dK_1(s)(f(s,t)\cdot x) = \int_{H\times H} d(K_1\times K_2)(f(s,t)\cdot x).$$

Proof. Let E be the set of pairs (s,t) for which the sequence $\{f_n(s,t)\}_{n\in N}$ does not converge to $f(s,t)$. Statement (3) says that E is a $K_1\times K_2$ negligible set. Statement (2) permits the application of the Corollary to Lemma 6, which says that the sequence $\{f_n(s,t)\}_{n\in N}$ converges K_1 a.e. as a function of s for all t in the complement of some K_2-negligible set A. Hence by (1), the sequence $\{f_n(s,t)\}$ converges in K_1-measure to $f(s,t)$ for all t in the complement of A. Now for all x in X we have that $|f_n(s,t)\cdot x - g(s,t)\cdot x| \leq B|x|$. Utilizing a dominated convergence theorem from [8] we can conclude that the sequence $\{\int_H dK_1(s)(f_n(s,t)\cdot x)\}_{n\in N}$ converges to $\int_H dK_1(s)(f(s,t)\cdot x)$ for all t in the complement of A. Again by (1) it follows that this sequence of integrals converges in K_2-measure to $\int_H dK_1(s)(f(s,t)\cdot x)$. By properties in [3] we have

$$\int_H dK_1(s)(g(s,t)\cdot x) - \int_H dK_1(s)(f_n(s,t)\cdot x)\,|_{Y^{**}} \leq W_1 B|x|.$$

Applying again the dominated convergence theorem from [8] we have that the sequence $\{\int_H dK_2\int_H dK_1(f_n(s,t)\cdot x)\}_{n\in N}$ converges to

$\int_H dK_2 \int_H dK_1 (f(s,t)\cdot x)$. On the other hand, the sequence $\int_{H\times H} d(K_1\times K_2)(s,t)(f_n(s,t)\cdot x)$ converges to $\int_{H\times H} d(K_1\times K_2)(s,t)(f(s,t)\cdot x)$. Hence by (5), our conclusion holds. This completes the proof of the theorem.

4. Conclusions and Remarks

One of the conditions in the hypothesis of Theorem 2, requires that Gowurin convergence a.e. implies convergence in Gowurin measure. Of course for the usual measure theoretical considerations this is the case. We have given in Proposition 4 through the assistance of Lemma 3 a condition under which the above requirement will hold. The Pettis theorem (see [2], page 318) of course is of fundamental importance here. It assures that the weakly countably additive Gowurin set function is countably additive on Σ.

Proposition 5 is necessary for the calculus of sections. The notion of one Gowurin measure being linearly non-zero relative to another is interesting. As seen in the proof of the theorem it is necessary to relate the Gowurin constants of the two measures.

It must be emphasized how these theorems are 'one sided'. We cannot interchange the roles of K_1 and K_2. For an interesting survey of the work being done in vector measures, the book [1] by N. Dinculeanu is recommended.

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