APPROXIMATIONS FOR A CLASS OF
FUNCTIONAL DIFFERENTIAL EQUATIONS

by

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Abstract

Functional differential equations of the form

$$\ddot{u}(t) = -A(0)g(u(t)) - \int_0^t \dot{A}(t-T)g(u(T)) + F(t),$$

on a Hilbert space &, are considered. $A(t)$ is a family of bounded, symmetric, linear operators on M while g can be nonlinear and unbounded. Solutions are considered when $F(t)$ approaches a constant F_0 as t tends to infinity. Conditions are given which guarantee that the limits of these solutions will be the same as those of the problems,

$$A(0)g(u(t)) + \int_0^t \dot{A}(t-T)g(u(T))dT = F(t),$$

$$\ddot{u}(t) = -A(\infty)g(u(t)) - T(u(t))\dot{u}(t) + F(t).$$

The first of these is the abstract statement of the quasi-static approximation in viscoelasticity. In the second $T(u)$ is to be a linear map, with domain in &« for each fixed u. This is an approximation suggested by the Coleman-Noll theory of retardation. A short discussion of the connections with materials with memory is included.

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1. Introduction.

In a recent paper [8] we considered functional differential equations of the form,

$$\dot{u}(t) = -\int_0^t A(t-\tau)g(u(\tau))d\tau + f(t), \tag{1.1}$$

on a Hilbert space W. A(t) is a family of bounded linear operators on M while g can be nonlinear and unbounded. The object was to obtain conditions on A, g and f in order that all solutions decay to zero as t tends to infinity. The results were generalizations of those of Hannsgen [4] and Levin and Nohel [5].

The present paper is also concerned with equation (1.1) but from a different point of view. We introduce various equations which are simpler than (1.1) but have solutions which approximate those of (1.1). The original motivation for this study came from the theory of elasticity and is outlined in the last section of the paper. We wanted to investigate the validity of a procedure called the *quasi-static approximation* in the theory of viscoelasticity. At the same time, we wanted to develop a different kind of approx-

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imation which was suggested in some work of Coleman and Noll [2]. These ideas seemed of sufficient interest to warrant a presentation of them in the same general setting as in [8].

The functional equations which arise in elasticity theory are of second order. A certain class of them are of the form obtained by differentiating (1.1), that is,

$$\ddot{u}(t) = -A(0)g(u(t)) - \int_0 A(t-\tau)g(u(\tau))d\tau + \dot{f}(t) . \qquad (1.2)$$

The abstract quasi-static approximation amounts to dropping the "acceleration" term $\ddot{u}$ in (1.2) and considering the equation,

$$A(0)g(u(t)) + \int_0 A(t-\tau)g(u(\tau))d\tau = \dot{f}(t). \qquad (1.3)$$

It is clear that (1.3) is much simpler than (1.2). One has only to solve a linear Volterra equation and then invert g. In viscoelasticity the latter step will involve solving a static problem.

We call the procedure derived from Coleman and Noll's work the ~~slow-flow approximation~~. It involves replacing (1.2) by differential equations of the form,

$$\ddot{u}(t) = -A(\infty)g(u(t)) - \Gamma(u(t))\dot{u}(t) + \dot{f}(t) . \qquad (1.4)$$

For each fixed u, $\Gamma(u)$ is a linear map with domain in W. In [2] a particular Γ is suggested (see section 7). We show that for our purposes there are many possible Γ's. What is essential is the form of the first term on the right of (1.4). This illustrates the crucial role of the quantity $A(\infty)$.

We show that there is some validity in using (1.3) or (1.4) to approximate (1.2) if one is concerned with approach to steady state. We assume that $f(t)$ tends to a constant f_o as t tends to infinity. Then we give conditions which guarantee that all solutions of (1.2), (1.3) or (1.4) tend to a common limit.

Our results for (1.2) are minor variations of those of [8]. They depend on the concept of _strong positivity_ of the kernels $A(t)$. The treatment of (1.3) requires a new property of $A(t)$ which we call _invertibility_. We feel that the most interesting feature of our work is the close connection between these two notions. We show, in fact, that the condition for strong positivity, as given in [8],also yields invertibility. These ideas are discussed in section two. Equation (1.2) is considered in section three and the two kinds of approximations are discussed in sections four and five. Section six contains some examples.

The importance of the approximate equations is that they can be handled in cases in which the full functional equations have proved too difficult. One such example is given in section six. Thus it is worthwhile to know that in cases where all equations can be treated the approximations have some validity.

2, Properties of the Kernels A(t) .

We first review some notions from [8]. We denote by Cy[0,oo) the space of continuous functions from [0,oo) into &. Let $\tilde{A}$ = fA(t) : t ^ 0) be a strongly continuous, one parameter family of bounded, symmetric, linear transformations on W. For any veC^[o,oo) we set,

$$Q_{\tilde{A}}[v;T] = J_{O} (v(t), J_{O} A(t-T)v(T)dT)dt. \quad (2.1)$$

We say that $\tilde{A}$ is ~~positive~~ if Q~[v;T] is always non-negative and that $\tilde{A}$ is ~~strongly positive~~ if,

$$Q_{\tilde{A}}[v;T] \ \text{^}\ QJJ[V;T] \quad (2.2)$$

for some $\tilde{M}$ constructed as follows. $\tilde{M}$ is to be a semi-group generated by a symmetric linear operator M, having dense domain fi_{M}, and such that,

$$(M§,§)\ 1\ m||?||^2,\ m > 0 \ \text{for all}\ 5e*>_M. \quad (2.3)$$

We call $veC_K[0,oo)$ weakly stable if it is weakly bounded and weakly uniformly continuous on [0,oo). The basic result of [8] was the following theorem which was called the weak stability principle.

Theorem 2.1. If A i§. strongly positive, v is weakly stable and Q,[v;T] is bounded independently of T, then v(t) tends weakly to zero as t tends to infinity.

Sufficient conditions for strong positivity were discussed at length in [8]. The best results concern the Laplace transform of $\tilde{A}$

and are closely related to our needs for the quasi-static approximation. We say that $\tilde{A}$ is <u>invertible</u> if it has a Laplace transform $A^A(s)$ which exists in Re s > Q and satisfies,

$$A^A(s) = s'\text{-}S? + B^A(s), \qquad (2.4)_1$$

$$A^A(s) = s\hat{\ }Q + 0(|s|^2) \quad \text{as} \quad |s|\text{-*co}, \qquad <^{2'4)}_2$$

where P and Q are bounded, symmetric, linear transformations, which have bounded inverses $P\sim^1$ and $Q\sim^1$, and $B^A(s)$ is continuous in &e s ;> 0 and regular in &e s > 0.

We make the following basic assumption throughout the paper. This provides the necessary technical conditions for both strong positivity and invertibility.

(i) $A(t)GC^2[0,oo)$,

(ii) $A^{(k)}(t)€L_1[0,oo)$, k= 1,2,

(iii) $A(t) = A(oo) + B(t)$, $B(t)€L_1[0,Oo)$, **(A)**

(iv) $((-1)^k A^{(k)}(0)5,5)$ ^oc|!5||^2, a > o,

k = 0,1, for all 5e»,

(v) (A(oo)?,5) > p||5||^2, p > 0 for all 5e».

<u>Remark</u> 2.1. (1) The existence of A(oo) as a uniform limit of A(t) is implied by (ii) .

(2) Conditions (iv) and (v) imply that A(0) , A(0) and A(oo) have bounded inverses.

The integral here, and all subsequent ones, are in the Bochner sense.

(3) Conditions (A) imply all the properties of invertibility save for the existence of A^A(s)~^1. In particular (2.4)^ and (2.4)$_2$ hold with P = A(oo), Q = A(0). In fact, (2.4)$_2$ can be sharpened to

A^A(s) = s^AfO) + s"^2 Ȧ(O) + o(| s|"^2) as I sHoo. (2.5)

It is shown in [8] that Ã will be strongly positive, with M = cl for some constant c, provided that A^ satisfies the condition,

(5,Re B^A(iT)S) ^ (2° / (C^2 + T^2)) ||S||^2 for all ?€» and TeR. (2.6)

It is easy to see from (2.5) and (A) (iv) (k=l) that (2.6) will hold, for some c, provided that \r\ ;> T, T sufficiently large. Thus we have the following result.

Theorem 2.2. A sufficient condition for strong positivity is: Given any M there exists £ > 0 such that

(5,&e B^A(iT)5) ;> tlUi!^2 for all ?e» and \r\ £ M. (P)

The main new result in this section is contained in the following:

Theorem 2. 3. (P) JLS a^ sufficient condition for invertibility.

Proof. We observe first that it follows from (2.4)$_1$ and (2.4)$_2$, and successive approximations, that A^A(s)$^{-1}$ exists for &e s ^ 0, | s| > R and Be s ^_ 0, 0 < |s| <p for R sufficiently large and

p sufficiently small. Moreover we have,

$$A^A(s)"^1 = sA(O)"^1(I+O(s"^1)) \text{ for large } |s|, \qquad (2.7^\wedge$$

$$A^A(s)""^1 = sA(oo)"^1(I+O(s)) \text{ for small } |s|. \qquad <^{2}-^{7)}2$$

(Recall P = A(oo) and Q = A(0)).

Thus we need only establish that $A^A(s)\sim^1$ exists in the semi-annulus r_{pR}, Re s ^ *0,* p £ | s| <£ R. For any fixed s, a sufficient condition for the existence of $A^A(s)$ is the inequality

$$(5,\text{Re } A^A(s)\, 5) \ \wedge\ k||?||^2,\ k > o. \qquad (2.8)$$

Condition (P) clearly implies that (2.8) holds on the portion Re s = 0 of T_{pR}. Suppose *seT* $_{PR}$ s = *rj* + ir, rj > 0. Then we have, by (A) (v) ,

$$(\S\text{-},\text{Re } A^A(s)?) \ \wedge - \wedge - r\ ||5||^2 + (\S,\text{Re } B^A(s)\ ?).$$

Hence (2.8) holds for any s with Re s > 0 provided that (5,&e $B^A(s)$ ξ) is non-negative for any 5e& and any seF_{pR} with Re s > 0. But for any fixed 5 *(%,&&* B^(s) ?) is a harmonic function of (77,T). It is continuous in *r* ^> 0 and tends to zero as |s|-»oo. Moreover it is positive on rj = 0 by (P) and (2.5). Hence by the maximum principle it is positive in 77 > 0. This completes the proof of Theorem 2.3.

Remark 2.2. Equation $(2.4)_2$ shows that we may consider $sA^A(s)$ as defined on Re s *;> 0,* with the value A(oo) at s = 0. We may then interpret Theorem (2.3) as stating that $sA^A(s)$ has a bounded inverse

$\hat{A}(s)$ on $\text{Re } s \geq 0$, with,

$$\hat{A}(s) = A^{-1}(0)s + O(s^{-1}) \quad \text{as} \quad |s| \to \infty, \tag{2.9$_1$}$$

$$\hat{A}(s) = A(\infty)^{-1} + O(s) \quad \text{as} \quad |s| \to 0 . \tag{2.9$_2$}$$

Moreover $\hat{A}(s)$ is continuous in $\text{Re } s \geq 0$ and regular in $\text{Re } s > 0$.

It is shown in [8] that the following result holds:

Theorem 2.4. If $\dim H < \infty$ sufficient conditions for (P) are

(P') $(-1)^k A^{(k)}(t)$ positive definite for $t \geq 0$, $k = 0,1,2$,

Remark 2.3. (A_2) was the condition given by Hannsgen in R^1. We show in [8] that (P) is more general than (P').

3. The Functional Differential Equation.

We consider the equation

$$\ddot{u}(t) = -A(0)g(u(t)) - \int_0^t \dot{A}(t-\tau)g(u(\tau))d\tau + F(t). \tag{3.1}$$

We make the following assumptions on F:

(i) $F \in C^1[0,\infty)$, $|F^{(k)}(t)| \leq M < \infty$ for $t \in [0,\infty)$ $k = 0,1$

(ii) $F(t) = F(\infty) + R(t)$, $R \in L_1[0,\infty)$ (F)

(iii) $\int_0^t R(r)dr = L + \rho(t)$, $\rho \in L_1[0,\infty)$.

Remark 3.1. Most of conditions (F) are technical. The crucial fact is the presence of the term $F(\infty)$. Condition (i) $(k=1)$ implies that

F is uniformly continuous on $[0,\infty)$ and hence the fact that $R \in L_1[0,\infty)$ implies that $R(t) \to 0$ as $t \to \infty$.

Integration of (3.1) yields the equation,

$$\dot{u}(t) = -\int_0^t A(t-\tau)g(u(\tau))d\tau + f(t)\,,\quad f(t) = \dot{u}(0) + \int_0^t F(\tau)d\tau. \tag{3.2}$$

Conditions (F) imply that:

$$\text{(i)}\quad f \in C^2[0,\infty),\ |f^{(k)}(t)| \le M,\ k = 1,2,$$

$$\text{(ii)}\quad f(t) = F(\infty)t + K + \rho(t),\ \rho \in L_1[0,\infty). \tag{f}$$

Equation (3.2) was considered in [8] in the special case $F(\infty) = 0$. The assumptions on g were that there exists a functional $G(u)$, defined on $\mathcal{D}_g$, such that we have,

$$\frac{d}{dt} G(u(t)) = (g(u(t)),\dot{u}(t)), \tag{3.3}$$

for any $u(t)$ which is differentiable on $[0,\infty)$ with $u(t) \in \mathcal{D}_g$ for all t.* Moreover G is to satisfy the condition,

$$\|g(u)\| \le M(1+G(u)), \tag{G}$$

for some $M < \infty$ and for all $u \in \mathcal{D}_g$. (Note that (G) implies that G has an infinum which is greater than minus infinity).

It was shown in [8] that if (G) is satisfied, (f) is satisfied with $F(\infty) = 0$, $\hat{A}$ is strongly positive and u is a solution of (3.2) such that $g(u(t))$ is weakly stable, then $g(u(t))$ tends weakly

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* Note that (3.3) is satisfied if $g(u)$ is the gradient of a functional G.

to zero as t tends to infinity. In order to modify this result when $F(\infty) \neq 0$ we need two additional assumptions. These are:

$$G(u) \geq \varphi(\|u\|)\,\|u\| \quad \text{where} \quad \varphi(\rho)\to\infty \quad \text{as} \quad \rho\to\infty, \text{ for large } \|u\|, \tag{G_1}$$

$$\int_t^\infty B(\tau)\,d\tau \in L^1(0,\infty)\,. \tag{A_1}$$

Remark 3.2. Condition (G_1) seems to be essential. We do not know whether (A_1) is essential or not. It arose also in some considerations in [8].

Theorem 3.1. Suppose (F), (G), (G_1) and (A_1) are satisfied and $\tilde{A}$ is strongly positive. Let u be a solution of (3.1) such that $g(u(t))$ is weakly uniformly continuous. Then $g(u(t))$ tends weakly to $A(\infty)^{-1}F(\infty)$.

Proof. We reduce to the case studied in [8]. Let $p = A(\infty)^{-1}F(\infty)$ and write,

$$F(\infty) = \int_0^t A(t-\tau)\,d\tau = \int_0^\infty B(\tau)\,d\tau\,\beta + \left(\int_0^\infty B(\tau)\,d\tau\right)p,$$

$$\tilde{g}(u) = g(u) - p.$$

Then (3.2) can be rewritten as,

$$\dot{u}(t) = -\int_0^t A(t-\tau)\,\tilde{g}(u(\tau))\,d\tau + \psi(t) + \varphi(t), \tag{3.4}$$

where,

$$\psi(t) = \left(F(t) - \int_0^t B(\tau)\,d\tau\,p\right)', \quad \varphi(t) = F(t) + \int_t^\infty B(\tau)\,d\tau\,p. \tag{3.5}$$

We can now apply the results of [8] provided that we show that $\tilde{g}$ satisfies condition (G). We choose as a G functional for g the quantity $\tilde{G}(u) = G(u) - (P,u)$. Then it is easy to see from (G) and (G_1) that $\tilde{G}$ satisfies G. The proof of the result in [8] involves an energy integral obtained by multiplying (3.2) by g(u(t)) and integrating. This yields then the boundedness of $Q_{\tilde{A}}[g(u);T]$ and also that of G(u). But in the present case this means that $\tilde{G}(u)$ is bounded which, by (G), implies that $\tilde{g}(u)$ is bounded. Thus the result of [8] implies that $\tilde{g}(u)$ tends weakly to zero which is the conclusion of Theorem 3.1.

Theorem 3.1, like those of [8], is a conditional result. In section six we give some examples in which the weak uniform continuity can be verified.

The fact that g(u(t)) has a weak limit need not imply that u(t) has a limit. However, we do have the following obvious result:

Corollary 3.1. *If dim # < oo and $g^{-1}(u)$ exists as a continuous operator on & then the hypotheses of Theorem 3.1 imply that* $u(t) \to g^{-1}(A(\infty)^{-1}F(\infty))$.

4. The Quasi-Static Approximation.

We consider the equation,

$$A(0)g(u(t)) + \int_0^t A'(t-\tau)g(u(\tau))d\tau = F(t). \tag{4.1}$$

Theorem 4.1. Suppose condition (F) jLs satisfied and $\tilde{A}$ JLS invertible. Then any solution of (4.1) satisfies g(u(t))-*A~ ^{1}(oo)F(oo).

Proof. If u is a solution of (4.1) then the function £(t) = g(u(t)) is a solution of the linear Volterra equation,

$$A(O)£(t) + J_{o} A(t-T)C(T)dT = F(t). \quad (4.2)$$

It is a standard result that (4.2) has a unique solution. We will prove Theorem 4.1 by constructing this solution and showing that it satisfies

$$\lim_{t \to oo} C(t) = AfoorMco). \quad (4.3)$$

We begin by formally taking the Laplace transform of (4.2). This yields,

$$sA^{A}(s)£^{A}(s) = F^{A}(s). \quad (4.4)$$

From Remark (3.2), equation (4.4) yields,

$$C^{A}(s) = A(s)F^{A}(s). \quad (4.5)$$

We note that conditions (f) imply that $F^{A}(s)$ exists in fte s ^ 0, s ^ 0, is regular in Re s > 0 and satisfies,

$$F^{A}(s) = s'Mo) + s"^{2}\dot{F}(O) +0(|s|^{2}) \text{ as } |s|^oo, \quad (4.6)_1$$

$$F^{A}(s) = s'^{1}F(\infty) + R^(s), \quad (4.6)_2$$

where R^(s) is continuous in Be s ^ 0,

We now define a function $£^{A}(s)$ by the right side of (4.5). We see that it exists in Be s ^ 0, s / 0 and is regular in Be s > 0-.

Moreover, by (2.9) and (4.6), it satisfies,

$$\hat{C}(s) = s^{-1}A(0)^{-1}F(0) + O(s^{-2}), \text{ as } |s| \to \infty, \qquad (4.7)_1$$

$$\hat{C}(s) = s^{-1}A(\infty)^{-1}F(\infty) + \hat{M}(s), \qquad (4.7)_2$$

where $\hat{M}(s)$ is continuous in $\operatorname{Re} s \geq 0$. With the information above we can form the inversion integral,

$$(2\pi)^{-1} e^{ct} \int_{-\infty}^{+\infty} e^{i\tau t}\hat{C}(c+i\tau)\,d\tau, \qquad (4.8)$$

for any $c > 0$. The conditions given yield the fact that this integral is independent of c in $c > 0$ and that is represents a function $C(t)$ whose transform is $\hat{C}$. From (4.7) we can write this function as,

$$C(t) = A(\infty)^{-1}F(\infty) + (2\pi)^{-1} e^{ct} \int_{-\infty}^{+\infty} e^{i\tau t}\hat{M}(c+i\tau)\,d\tau. \qquad (4.9)$$

In the integral term of (4.9) we can pass to the limit $c = 0$. Then the Riemann-Lebesgue lemma shows that this term tends to zero as t tends to infinity. This yields (4.3). The fact that C is a solution of (4.2) follows from (4.4) and the convolution theorem.

5. Slow-Flow Approximations.

We consider differential equations of the form,

$$\ddot{u}(t) = -A(\infty)g(u(t)) - T(u(t))\dot{u}(t) + F(t). \qquad (5.1)$$

For each $u \in \mathcal{G}$, $T(u)$ is to be a linear map but it need not be bounded. We suppose that the domains $\mathcal{D}_{T(u)}$ all contain a common set $\mathcal{D}_T$. It is to be understood that a solution of (5.1) satisfies $\dot{u}(t) \in \mathcal{D}_T$

for all t. We impose on T the condition,

$$(A(\infty)^{-1}T(u)v,v) \geq \gamma\|v\|^2, \quad \gamma > 0 \ \text{ for all } \ u\in\mathcal{H}_g, \ v\in\mathcal{H}_r \qquad \text{(T)}$$

The condition *(T)* guarantees that the term $-T(u(t))\dot{u}(t)$ provides a "damping" mechanism.

We first establish a stability result for (5.1).

<u>Theorem</u> 5.1. <u>Suppose conditions</u> A (v), (G), (G_1), (F) <u>and</u> (T) <u>hold</u>. <u>Then any solution of</u> (5.1) <u>satisfies</u>,

(i) $\|u(t)\|$, $\|g(u(t))\|$, $\|\dot{u}(t)\|$ uniformly bounded on $[0,\infty)$,

(ii) $\dot{u}\in L_2[0,\infty)$.

<u>Proof</u>. We first eliminate the term $F(\infty)$ in (5.1) just as we did in section 3. We rewrite the equation as,

$$\ddot{u}(t) = -A(\infty)\tilde{g}(u(t)) - T(u(t))\dot{u}(t) + R(t), \qquad (5.2)$$

where $\tilde{g}(u) = g(u) - A(\infty)^{-1}F(\infty)$. Now we multiply (5.2) by $A(\infty)^{-1}\dot{u}(t)$ and obtain, from (G), (F) and the symmetry of $A(\infty)^{-1}$,

$$\frac{d}{dt}\left\{\frac{1}{2}(A(\infty)^{-1}\dot{u}(t),\dot{u}(t)) + \tilde{G}(u(t))\right\} = -(A(\infty)^{-1}T(u(t))\dot{u}(t),\dot{u}(t))$$

$$+ (\dot{u}(t),A(\infty)^{-1}R(t)) \leq -\frac{\gamma}{2}\|\dot{u}(t)\|^2 + \frac{1}{2\gamma}\|A(\infty)^{-1}R(t)\|^2. \qquad (5.3)$$

Remark 3.1 and (F)(ii) imply that $R\in L_2[0,\infty)$ thus we infer from (5.3) that $(A(\infty)^{-1}\dot{u}(t),\dot{u}(t))$ are bounded and that $\|\dot{u}(t)\|$ belongs to $L_1[0,\infty)$. Conclusion (i) then follows by (G_1), (G) and A (v) respectively.

Our asymptotic stability result is obtained from the following elementary proposition.

Lemma 5.1. (i) *Let* $u : R^+ \to W$ *be weakly uniformly continuous*. *Then* $u \in L_2^o[0,\infty)$ *if and only if* $u(t)$ *tends weakly to zero*.

(ii) *Let* $u : R^+ \to M$ *be differentiable and let* $\dot{u}$ *be weakly uniformly continuous*. *Then* $u(t)$ *tends weakly to zero only if* $\dot{u}(t)$ *tends weakly to zero*.

Theorem 5.1 and Lemma (5.1) yield the following conditional result.

Theorem 5.2. *Suppose conditions* A (v), (G), (G_1), (F) *and* (T) *hold*. *Let* u *be a solution of* (5.1) *such that* $g(u(t))$ *is weakly uniformly continuous and* $F(u(t))\dot{u}(t)$ *is weakly stable*. *Then* $\dot{u}(t)$ *and* $\ddot{u}(t)$ *tend weakly to zero.*

Proof. Lemma 5.1 (i) and Theorem (5.1) (ii) show that $\dot{u}(t)$ tends to zero weakly provided it is weakly uniformly continuous. But this follows from the fact that $\ddot{u}(t)$ is weakly bounded as one sees from (5.2), the boundedness of $\tilde{g}$ and the weak stability of $F(u(t))\dot{u}(t)$. (5.2) and the hypotheses also imply that $\ddot{u}(t)$ is weakly uniformly continuous, hence it tends weakly to zero by Lemma 5.1 (ii).

We conclude from Theorem (5.2) and Lemma 5.1 (i) applied to R that the quantity,

$$A(\infty)\tilde{g}(u(t)) + T(u(t))\dot{u}(t) \tag{5.4}$$

tends weakly to zero. If the second term in (5.4) tends to zero weakly we have the conclusion we want which is that $\tilde{g}(u(t))$ tends weakly to zero. In general, however, one cannot expect this even though $\dot{u}(t)$ tends weakly to zero. Thus we have the following result.

Theorem 5.3. *If the hypotheses of Theorem 5.2 hold and, in addition, $T(u(t))\dot{u}(t)$ tends weakly to zero then $g(u(t))$ tends weakly to $A(\infty)^{-1}F(\infty)$.*

Proof. Theorem (5.1) (i) and the hypothesis imply that the right side of (5.2) is (strongly) bounded. Hence $\|\ddot{u}(t)\|$ is bounded and therefore $\|\dot{u}(t)\|^2$ is uniformly continuous. Thus Theorem (5.1) (ii) implies $\dot{u}(t)$ tends strongly to zero which in turn implies that the second term in (5.4) tends strongly to zero.

Remark 5.1. Two important special cases in which Theorem (5.3) applies are:

(1) $F(u) = r_o$, a constant, r_o dense in W.

(2) $\dim W < \infty$, $F(u)$ continuous as a function of u.

6. Examples.

We present here three kinds of examples. These are essentially the same as those in [8] and the reader may find details in that reference. In all cases it will be seen that the quasi-static approximation is trivial and we will not discuss it. In the first two cases we are able to verify all the hypotheses and hence conclude that both the functional equations and the slow-flow approximations have the same steady state limits. In the third example we are not able to verify the weak stability condition for the functional equation but we indicate some slow-flow approximations which can be handled.

(I) Equations in R^n.

We recall that in this case condition (P') is sufficient for all our needs. We assume here that both g and T are defined and continuous on all of R^n. We assume further that $g(u) = \nabla_u G(u)$ where $G(u)$ satisfies (G) and (G_1). For $n \geq 2$ this is, of course, a drastic restriction on the form of (3.1), (4.1), and (5.1). The main point here is that the weak stability requirements can be verified directly from the equations themselves.

Theorem 6.1. Suppose $M = R^n$, (F), (G), (G_1) and (A_1) are satisfied and $\tilde{A}$ is strongly positive. If u is a solution of (3.1) then $g(u(t)) \to A(\infty)^{-1}F(\infty)$ as $t \to \infty$.

Proof. From Theorem (3.1) we need only show that $g(u(t))$ is uniformly continuous. Since g is continuous it suffices to prove that u is uniformly continuous and this in turn will follow if we can show that $\dot{u}$ is bounded. We differentiate (3.4) and obtain,

$$\ddot{u}(t) = -A(0)g(u(t)) - \int_0^t \dot{A}(t-\tau)g(u(\tau))d\tau + \dot{\varphi}(t) . \qquad (6.1)$$

In the proof of Theorem (3.1) we showed that $\tilde{g}(u(t))$ is bounded. Hence (6.1), (A)(ii) and (F) imply that $\ddot{u}(t)$ is bounded. Thus $\dot{u}(t)$ is uniformly continuous. We assert that the boundedness of u and the uniform continuity of $\dot{u}$ implies that $\dot{u}$ is weakly, and hence strongly, bounded. Suppose not. Then there exists an $\eta \in H$ and a sequence $t_n \to \infty$ such that $(\dot{u}(t_n), \eta) \geq n$. By the uniform continuity of $\dot{u}$ we can then find a $\delta > 0$ such that $(\dot{u}(t), \eta) > n$ for $t \in [t_n - \delta, t_n]$. Hence we would have, by the mean-value theorem, $(u(t_n) - u(t_n - \delta), \eta) > n\delta$ which contradicts the boundedness of u.

<u>Theorem</u> 6.2. <u>Suppose</u> $H = R^n$ <u>and</u> (Av), (G), (G_1), (F) <u>and</u> (F) <u>hold</u>. <u>If</u> u <u>is a solution of</u> (6.1) <u>then</u> $g(u(t)) \to A(\infty)^{-1}F(\infty)$ <u>as</u> $t \to \infty$.

<u>Proof</u>. From Theorems (5.1)-(5.3) and the continuity of g and T we see that it suffices to prove the uniform continuity of $\dot{u}$ and this follows immediately from (5.2) which shows that $\ddot{u}$ is bounded.

(II) <u>Linear Partial Differential Functional Equations</u>.

The equations considered here have the form,

$$u_{tt}(x,t) = a(0)Lu(x,t) - \int_0^t a(t-r)Lu(x,r)\,dr + F(x,t). \qquad (6.2)$$

The space H is $L^2(\Omega)$, where Ω is a bounded region in R^n. L is a strongly elliptic partial differential operator of the form,

$$Lu = \sum_{\substack{|\alpha| \le m \\ |\beta| \le m}} (-1)^{\alpha} D^{\alpha}(a^{\alpha\beta}(x) D^{\beta} u), \qquad (6.3)$$

where the coefficients $a^{\alpha\beta} = a^{\beta\alpha}$ are smooth functions of x. a is a scalar function so that $A(t) = a(t)I$. This will be strongly positive in M if a is strongly positive in R^1. It is required that the solution of (6.2) belong to H_o^m, the completion of $C_o^\infty(\Omega)$ under the norm,

$$\|u\|_m = \sum_{|\alpha| \leq m} \int_\Omega |D^\alpha u|^2 dx.$$

This is equivalent to zero Dirichlet boundary data.

We assume that Gårding's inequality is satisfied with the coefficient of $\|u\|_o^2$ equal to zero. That is if

$$B(u,v) = \int_\Omega Luvdx,$$

then

$$B(u,u) > c\|u\|_m^2 \qquad c > 0. \tag{6.4}$$

Remark 6.1. Condition (6.4) implies that the generalized Dirichlet problem,

$$Lu = \varphi, \quad \varphi \in L_2(\Omega), \tag{6.5}$$

has a unique generalized solution $u \in H_o^m$ and if φ and the $a^{\alpha\beta}$ are smooth this will be a classical solution.

We assume that F satisfies condition (F) (with $F(\infty)$ of course replaced by $F(x,\infty)$). Except for the modification to account for the term $F(x,\infty)$, the treatment of (6.2) is the same as in [8]. The role of $G(u)$ is played by $-\frac{1}{2} B(u,u)$. The weak uniform continuity is verified by differentiating the equation with respect to t.

The result is the following:

<u>Theorem</u> 6.3. <u>Let</u> a <u>be strongly positive in</u> R^1 <u>and</u> $a(\infty)$ <u>be positive</u>. <u>Suppose</u> (F) <u>and</u> (6.4) <u>hold</u>. <u>If</u> u <u>is a solution of</u> (6.2) <u>then</u> $Lu(\cdot,t)$ <u>tends weakly to</u> $a(\infty)^{-1}F(x,\infty)$.

<u>Remark</u> 6.2. The conclusion of Theorem 6.3 and standard embedding theorems can be used to show that the hypotheses of Theorem 6.3 imply that

$$u(.,t)—OC \text{ in } E£, \qquad (6.6)$$

where χ is the unique generalized solution of $Lu = a(\infty)^{-1}F(\chi,\infty)$. The argument is the same as that in [8]. This is an analog of Corollary (3.1).

The slow-flow approximations appropriate to (6.2) have the form,

$$u_{tt}(x,t) = -a(\infty)Lu(x,t) - ru_t(x,t) + F(x,t). \qquad (6.7)$$

We require these to be linear also which means F is a fixed linear operator with domain Ω_1. We choose for F another elliptic operator of the same form as L and also satisfying Gårding's inequality in the form,

$$(Mu,u) \text{ ^ } 5||u||\text{^}. \qquad (6.8)$$

Then (F) is satisfied as is condition (1) of Remark (5.1). Once again the weak stability can be established by differentiating the equation and one obtains the following result.

Theorem 6.4. Let a be strongly positive in $R^!$ and a(oo) be positive. Suppose (F), (6.4) and (6.8) hold. If u is a solution of (6.7) then Lu(•, t) tends weakly to a(oo)"^{1} F(x,oo). (Remark 6.2 again holds).

(C) A Nonlinear Partial Differential Functional Equation.

Consider the equation,

$$u_{fct}(x,t) = a(O)\text{-}^\wedge\text{-}\ a(u_x(x,t)) + \int_o \dot{a}(t-r)\text{-}^\wedge\text{-}\ a(u_x(x,T))dr + F(x,t), \quad (6.9)$$

with u(O,t) = u(l_5t) = 0. This equation arises in the next section in an elasticity context. The results of [8] are easily modified to yield the following result: Suppose (F) is satisfied and a is strongly positive in $R^!$. Assume that CJ satisfies the conditions,

(i) a(o) = 0, at (?) ^ m > 0,

(ii) |a(§)| £ m| ?| + K| 5|^r, 0 < r < 2.

Suppose u is a solution of (6.9) such that for any Tj€C$^\infty$°[O,l],

$$\int_o^1 CJ(U_x(x,t))n_x(x)dx \text{ is uniformly continuous in } [0,oo). \quad (6.10)$$

Then $-\frac{\partial}{\wedge}-$ ^(u_x> tends weakly to a(oo)~^{1}F(x,oo) as t->oo .

We are not able to verify condition (6.10) hence the result for (6.9) remains a conditional one. We note that the quasi-static approximation is completely trivial. One determines $-\frac{\partial}{\partial x}$ a(u_x) by solving a linear Volterra equation and then solves a simple two-point boundary-value problem for u.

We can also handle certain slow-flow approximations. The simplest one is,

$$u_{tt} = a(\infty)\frac{\partial}{\partial x}\sigma(u_x) + \Gamma u_{xxt} + F(x,t), \tag{6.11}$$

u(O,t) = u(l,t) , where Γ is a positive constant. This is an inhomogeneous version of the equation treated in [3]. By modifying the techniques there one can show that if (F) holds and (i) and (ii) hold for a then (6.11) has a unique classical solution for any u(x,O), u_t (x,0) which are sufficiently smooth and that any such solution satisfies,

$$\lim_{t\to\infty} \text{-J-JJ7}\; a(u_x(x,t)) = a(\infty)^{-1}F(x,\infty), \tag{6.12}$$

pointwise.

A little more complicated version of (6.11), which arises in section 7 is the equation,

$$u_{tt} = a(\infty)\frac{\partial}{\partial x}\sigma(u_x) + \frac{\partial}{\partial x}(\Gamma(u_x)u_{xt}) + F, \tag{6.13}$$

u(O,t) = u(l,t) = 0. The homogeneous version of this equation is treated in [7]. Again a modification of the techniques yields the same results as for (6.11) provided that Γ satisfies the condition,

$$0 \le m \le \Gamma'(\xi) \le M < \infty .$$

7. Materials with Memory.

We consider situations in elasticity in which stress and strain can be described by scalar functions s and u, respectively, of time. The theory of materials with memory makes the assumption that s and u are related by the constitutive law,

$$s(t) = \mathfrak{F}(u^t), \tag{7.1}$$

where u^t denotes the history of u, that is, $u^t(\tau) = u(t-\tau)$, $\tau \in [0,\infty)$ and $\mathfrak{F}$ is a functional. One obtains various theories by specifying Banach function spaces B in which the histories are to lie and then demanding that $\mathfrak{F}$ be continuous.

One class of history spaces which has been extensively studied [1] is that obtained by letting B consist of all measurable functions φ on $[0,\infty)$ with,

$$\|\varphi\| = |\varphi(0)| + \left(\int_0^\infty k(\tau)|\varphi(\tau)|^p d\tau\right)^{1/p} < \infty. \tag{7.2}$$

In (7.2) k is a positive integrable function. The ~~linear~~ functionals on such a B are easily described. They have the form ,

$$\mathfrak{F}(\varphi) = a(0)\varphi(0) + \int_0^\infty \dot{a}(\tau)\varphi(\tau)d\tau, \tag{7.3}$$

where

$$a(0) > 0; \ \dot{a}(\tau) = k^{1/p}K(\tau), \ K \in L^q \ \text{with} \ p^{-1} + q^{-1} = 1. \tag{7.4}$$

A related class of nonlinear functionals is obtained by letting,

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* This is one form of linear viscoelasticity.

$$\mathfrak{F}(\varphi) = a(0)a(cp(0)) + J_{0}^{\infty} \dot{a}(T)a(<p(T))dT. \quad (7.5)$$

It is known that functionals of the form (7.5) will be continuous and bounded on spaces B, with the norms (7.2), provided that g(5) is continuous and that the integral in (7.5) is defined on all of B ([6]). It follows from Hölder's inequality that the integral will exist if (7.4) is satisfied and g satisfies,

$$l^{a}(5) I \underline{<} LM|?|^{r}, \ r < p \ \text{ for large } \ \backslash<p\backslash . \quad (7.6)$$

We consider the motion of a unit mass on a filament in which stress and strain are related by a functional of the form (7.5). We assume there is an applied force F(t). This yields the functional differential equation

$$\ddot{u}(t) = -s(t) = -a(O)a(u(t)) - J_{o}^{\infty} \dot{a}(r)a(u(t-T))dT + P(t) . \quad (7.7)$$

An appropriate problem for (7.7) would include the specification of the initial history of u, that is $\overset{o}{u}(T) = u(-T)$, as an element of B. We specialize to the case where the filament is initially at rest, that is $u(T) \equiv 0$ Te(-oo,O]. Then equation (7.7) assumes the form (3.1) on $\# = R^{1}$, that is,

$$\ddot{u}(t) = -a(O)a(u(t)) - J_{o}^{t} \dot{a}(t-r)a(u(r))dr + F(t) . \quad (7.8)$$

Another problem which can be described within the same general framework is that of one-dimensional elastic motion of a bar. If u(x,t) is displacement at time t of a section which in the equilibrium configuration is at position x, then the strain is given by

$u_{,}(x,t)$. If we assume that stress and strain are related in the same way as above and that the bar is at rest up to time $t = 0$ and clamped at the ends, we obtain the problem ($u_{tt} = \text{div}\, \sigma + F$) *

$$u_{tt} = a(0)\sigma'(u_x) + \int_0^t a'(t-\tau)\sigma(u_x)d\tau + F(x,t) \qquad (7.9)$$

$u(0,t) = u(1,t) = 0$, where F is the applied force. Note that this is equation (6.9).

The quasi-static approximation in elasticity consists of dropping the time derivatives in (7.8) and (7.9). Our theorems give an indication of the validity of this procedure. The idea for the slow-flow approximation arises as follows. Coleman and Noll [2] discussed the possibility of expanding the functional $\mathfrak{G}$ in (7.1) in a Taylor series. The first two terms of the expansion are obtained as follows. We expand $u(t-T)$ in a Taylor series and keep two terms; that is we write

$$u^t \approx u(t)\chi_1 - \dot{u}(t)\chi_2, \qquad (7.10)$$

where $\chi_1(T) \equiv 1$, $\chi_2(T) \equiv T$. Then if $\mathfrak{G}$ has a Frechét derivative we will have,

$$\mathfrak{G}(u^t) = \mathfrak{G}(u(t)\chi_1) - \dot{u}(t)\delta\mathfrak{G}(u(t)\chi_1|\chi_2) + O(\|\dot{u}(t)\chi_2\|). \qquad (7.11)$$

Now $\mathfrak{G}(u(t)\chi_1)$ and $\delta\mathfrak{G}(u(t)\chi_1|\chi_2)$ are functions of $u(t)$, which we denote by $M(u(t))$ and $N(u(t))$ respectively. Thus if we keep only the first two terms on the right side of (7.11) we have, approximately, c(t) = M(u(t)) + N(u(t))u̇(t) . (7.12)

If we substitute this approximation into the functional equation,

ü(t) = -a(t) = -^(uS

we obtain the differential equation,

ü(t) = -M(u(t)) - N(u(t))u̇(t). (7.13)

The above procedure when applied to the functional (7.5) is easily seen to lead to the approximation,

t ~ r°°. . .
3(u) ~a(O)a(u(t)) + J [a(T)a(u(t)) - a(T)Ta' (u(t))u(t)]dT
o .

= a(co)CT(u(t)) + T(u(t))u(t) (7.14)

where

T(u(t) = -(i å(T)TdT)a« (u(t)). (7.15)
o

Thus we are led to the slow-flow approximations,

ü(t) = -a(co)a(u(t)) + r(u(t))u̇(t) + F(t) , (7.16)

and

u^ = a(oo)-^~ a(u) + -^- r(u)u . + P, (7.17)
tt v dx v x dx v x xt

for (7.8) and (7.9) respectively. Our work also gives some indication of the validity of these equations. In particular it shows that we may expect correct limiting values from (7.17) if we replace $F(u_x)$ by a positive constant. This corresponds to the viscous damping in what is called a Kelvin material.

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