T-ORTHOGONALITY AND NONLINEAR FUNCTIONALS ON TOPOLOGICAL VECTOR SPACES

by

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T-Orthogonality and Nonlinear Functionals on Topological Vector Spaces

K. Sundaresan* and 0. P. Kapoor

In recent years the problem of concretely representing a class of nonlinear functionals on Banach spaces has received considerable attention. Suppose B is a Banach space equipped with an orthogonality relation i c B x B. Denoting (x,y)€i by x i y, a real valued function F on B is said to be orthogonally additive if x x y implies F(x+y) = F(x) + F(y). For example when B is a vector lattice^a natural orthogonality relation is the lattice theoretic one: x ±. y if |x| A |y| = 0. •tfie problem of representing orthogonally additive functions on normed vector lattices of measurable functions has been dealt in Drewnowskii and Orlicz [l],Mizel and Sundaresan [2], Friedman and Katz [4], Koshi [5], and several others. If B is the Hilbert space L~[O,1] with the usual concept of orthogonality i.e. x ±2 Y if t^ie inner product (x,y) = 0*, the problem of representing orthogonally additive functionals has been considered by Pinsker [3]. If B is an arbitrary Banach space there are several orthogonality relations which are generalisations of the usual concept of orthogonality when B is a Hilbert space. One such concept of considerable geometric and analytic interest is the following. Let (B,|| ||) be a Banach space. If x,yeB, x x^ y if ||x+Ay|| ^ ||x||

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for all real values of A. Tfre problem of representing orthogonally additive functionals on B with respect to the relation JL^ has been dealt in Sundaresan [7] .

None of the preceding concepts of orthogonality extend to arbitrary topological vector spaces. We introduce here a useful orthogonality concept in an arbitrary topological vector space. Let E be an Hausdorff topological vector space and T : E —>E , where E is the dual of E, be a linear mapping. If x,yeE, then x is T-orthogonal to y if Tx(y) , denoted by, (Tx,y) = 0. In the present paper the problem of characterizing T-orthogonally additive functionals on a topological vector space is dealt.

In the next section we recall briefly the basic terminology and establish few results useful in the subsequent discussion. In section 3 we discuss T-orthogonally additive functionals when T-orthogonality is not symmetric. In section 4 we consider the same problem when T-orthogonality is symmetric.

2. Throughout the paper E is a Hausdorff Topological vector space on the real field R_o E is the vector space of continuous linear functionals on E. To avoid trivialities we always assume that dim E ^ 2. If T : E—> E is a linear mapping and x,y€E, then x is T-orthogonal to y or briefly x x y, when T is understood, if (Tx,y) = 0. T-orthogonality is said to be symmetric, if (Tx,y) = o implies (Ty,x) = O_o A vector X is said to be T-isotropic or simply isotropic

if (TXjX) = O, *Tfhe* operator T is said to be symmetric if (Tx,y) = (Ty,x) for all x,yeE. If x,y,z,... are vectors in E, the span of x,y,z,... is denoted by [x,y,z,...].

We conclude this section with a few useful lemmas.

Lemma 1. If T : E —* E is a linear mapping such that T-orthogonality is symmetric and if there is a nonisotropic vector, then T is symmetric.

Proof. Let y,zeE. Suppose (Ty,z) *£* (Tz,y). Since the relation x is symmetric (Ty_5z) ^ 0 ^ (Tz,y). If y X y it is verified that there is a real number *a* ^ θ_9 such that y i y + az. Hence y + az x y. Thus a(Tz,y) = - (Ty,y) = a(Ty,z). Hence (Ty,z) = (Tz,y). If z \ z it is verified similarly that (Ty,z) = (Tz,y) • Let now y J. y and z x z. Let x be a vector such that x \ x. The preceding observation shows that (Tx,p) = (Tp,x) for all peE. Further since x \ x either x + y or x - y is not isotropic. Hence (T(x+y) *,z)* = (Tz, (x+y)) or (T(x-y) ,z) = (Tz, (x-y)) . Thus (Ty,z) = (Tz,y) and T is a symmetric mapping.

Lemma 2. If T : E —> E is a linear mapping and if the rank of T is an odd integer, then there is at least one non-isotropic vector.

Proof. Suppose every vector is isotropic.

The hypothesis of the lemma implies there exists a (2K+1)-dimensional subspace E^{2K+1} of E, for some positive integer K, such that $T(E^{2K+1})$ is also (2K+1)-dimensional. Thus if T_1 is the restriction of T to E^{2K+1} > ^{T}i roight be considered as a linear isomorphism on E^{2K+1} to E^{2K-J-1} such that the innerproduct $(T_1x,x) = 0$ for all $x \in E^{2K+1}$. Thus there exists continuous nonvanishing tengential vector field on the sphere in E^{2K+1}, contradicting Poincar^ -Brouwer theorem, Dugundji [t].

Lemma 3. If T : E—*E is a 1-dimensional linear mapping then the following two statements are equivalent.

(1) T-orthogonality is symmetric$_4$

(2) There is a nonisotropic vector x such that x J- y implies Ty = 0.

Proof. Let x \ x. Let yeTx" ^{1}(0) . Then (1) implies y a. x. Since T is 1-dimensional and Tx ^ 0, Tye [Tx] . Let Ty = ATx. Then since y ± x it is verified that either, A = 0 or (Tx,x) = 0. Since x X x_5 A = 0. Hence Ty = 0. Thus (1) implies (2). Conversely suppose (2) holds and X€E such that x i y implies Ty = 0. Since Tx / 0, Tx" ^{1}(0) is a subspace of codimension 1. Thus each £eE determines uniquely a real number ?\ and a vector h, x J. h, such that £ = Ax + h. Thus if $£_i = A_i x+h_i$, i = 1,2, then $£_1$ ± $£_2$ if and only if ^i^2 = ^ since $Th._1 = 0$. Hence JL is symmetric.

Remark 1. From the proof of the preceding lemma it is clear that (2) could as well be replaced by "for every nonisotropic vector x, x i y implies Ty = 0".

3. Let T : E -^ E be a linear mapping such that i is not symmetric. Let the rank of T = 1. Then from lemma 2 it is inferred that there is a nonisotropic vector. Let x be one such vector. Let M = Tx~^{1} (0). If y,zeM then since Tx ^ 0 and rank T = 1, Ty,Tze[Tx]. Since (Tx,z) =0 it is verified that y j. z. In particular for all yeM, y x y. Now if F is a continuous T-orthogonally additive functional on E then the preceding observation implies that F is homogeneous and additive on M. Thus F|M is a continuous linear functional on M. Since i is not symmetric it is inferred from lemma 3 that there is a vector yeM such that Ty ^ 0. Since M is a subspace and TyG[Tx] we can as well assume that Ty = Tx. •Thus x - y i x. Hence if F(?\x) = cp (A) then since *~h* (x-y) x *ix* for all pairs of real numbers A,u it is verified from the orthogonal additivity of F and linearity of F on M that cp(A+^) = cp(A) + cp(fi). Since F is a continuous function, cp : R—>R is a continuous additive function. Thus cp is linear. Now if £GE and £ = Ax + y, yeM, then F(Ax+y) = cp (A) + F(y). Since cp is linear on R it follows that FGE . Since every linear functional on E is orthogonally additive it is proved

that under the above hypothesis on T that a continuous function $F : E \rightarrow R$ is T-orthogonally additive if and only if F€E •

Next we proceed to the case when rank $T > 1$. First we deal the case of dim $E = 2$ or 3.

Proposition 1. If dim $E = 2$ or 3 and if $T : E \rightarrow E$ is a linear mapping such that rank $T > 1$ and T-orthogonality is not symmetric, then every continuous orthogonally additive functional on E is linear.

Proof. Let dim $E = 2$. Suppose that $e_1, e_2 \in E$ such that $e_1 \perp e_2$ but $e_2 \not\perp e_1$. Thus e_1, e_2 are linearly independent* Since the rank $T = 2$, $Te_1 \neq 0$. Hence $(Te^\wedge, e_2) = 0$ implies that $(Te_1, e_1) \wedge 0$. Thus there is a real number $a \neq 0$ such that $ae_1 + e^\wedge$ I e_1. Hence if A,u are two real numbers then A(ae_1+e_2) i lie^ Hence F(Aae$_x$ + Ae$_2$ + lie^ = F(A ($ae_1 + e_2$)) + F(|ie$_1$). Since e.^ ± e_2, $F((l\backslash BL+\backslash i)e_1+'\backslash e_2)$ = F^Aa+t-Oe^ + F(Ae$_2$). Thus P((>.a+|i)e$_1$) + F(Ae$_2$) = F(A(ae$_1$+e$_2$)) + FC^ej^). Hence F is additive on $[e_1]$. Since F is continuous F is homogenous on [e^] . Further noting that $ae_1 + e_2$ x e_1, $e_1 \setminus ae_1 + e_2$ it is verified as above that F is homogenous on $ae_1 + e^\wedge$. Since $ae_1 + e_2$ i e_1., the T-orthogonal additivity of F at once implies that F is linear.

Next we proceed to the case when dim $E = 3$. Let the rank $T = 2$ and e^e^eE such that $e_1 \perp e_2$ and $e_2 \setminus e_1$. If (Te^^^e_1) ^ 0 or $(Te_2, e_2) \neq 0$ y then as in the preceding case it is verified that F is linear on [e^e^ . If $(Te_1, e_1) = 0$ and $(Te_2, e_2) = 0$, then F is homogeneous on $[e_1]$

and $[e_2]$ • Since $e_1^* \times e_2$ is linear on the subspace $[e_1, e_2]$. Thus in either case F is linear on $[e_1, e_2]$. Now if Te_1, Te_2 are linearly independent then since the rank T = 2 there exists a vector $e_3 \notin [e_1, e_2]$ such that $Te_3 = 0$. Since $e_3 \times e_3$ F is homogenous on $[e_3]$. Further since $e_3 \perp [e_1, e_2]$ and F is linear on $[e_1, e_2]$ it is verified that F is a linear functional. If Te_1, Te_2 are linearly dependent then either $Te_1 = 0$ or $Te_1 = \lambda Te_2$, $\lambda \neq 0$. Since $(Te_2, e_2) \neq 0$ there is a vector $x \in [e_1, e_2]$ such that $e_2 \perp x$. Thus if $Te_1 = 0$ then $e_1 \times x$. If $x \perp e_2$ or $x \times e_1$ then as in the case of dim E = 2 it is verified that F is homogenous in $[x]$. Since $[e_1, e_2] \times x$, F is a linear functional. If $x \perp e_2$ and $x \times e_1$, then, since $e_2 \times e_1$, $x + e_2 \perp e_1$. However since $e_1 \times e_2$, $e_1 \times x + e_2$. Once again F is verified to be homogenous on $[x+e_2]$. Since $x \times e_2$ and F is homogenous on $[e_2]$ it is verified that F is homogenous on $[x]$. Thus F is linear. Next suppose $Te_1 \neq 0$. Then since $Te_1 = \lambda Te_2$ for some $\lambda \neq 0$, and $e_2 \times e_1$ there is a vector $x \in [e_1, e_2]$ such that $[e_1, e_2] \times x$. If $x \times [e_1, e_2]$ then once again F is homogenous on $[x]$ and F is a linear functional. If $x \times [e_1, e_2]$ since the rank T = *2,* $(Tx, x) \neq 0$. Further since $Te_1 \neq 0$, and $e_1 \times [x, e_2]$ it follows that $(Te_1, e_1) \neq 0$. Since $e_1 \times x$, $x \times e_1$ and $(Te_1, e_1) \neq 0 \neq (Tx, x)$ it is verified that (*) there is a real number $a \neq 0$, such that $x + ae_1 \times x + ae_1$ or $x + ae_1 \times x - ae_1$. In the case of the first alternative, F is homogenous on $[x+ae_1]$. Then since $x \times e_1$ and F is homogenous

on [e.┴]9 it is verified that F is homogenous in [x].
Thus F is linear. If x + ae1 x x - ae,┴ then if ~K,\A are two real numbers F((A+fi)x + Aaex - l-lae^ = FfMx+ae^) + F(|i(x-ae1)) = F(Ax) + F(|ax) + F(AaeJ. - F(|iae┴), since A(x+aej x |i(x-ae┴).,
Since x x e^ F((A+|j)x + Aae-^ae^ = F((A+|i)x) + FtAae^fiae^ .
From the preceding equations it is verified that F((A+|a) x) = = F(Ax) + F(p.x) after noting that F is homogenous on [e.┴].
Since F is continuous, F is homogenous on [x]. Hence F is a linear functional completing the proof in the case rank T = 2.

Next suppose dim E = 3, and rank T = 3. Since T-orthogonality is not symmetric there exist linearly independent vectors ei>eo such that e,┴ JL e2 and e2 X e,┴. Thus as in the case of dim E = 2 it is verified that F is linear on [e^,e2]. Suppose there is a vector e3^[e15e2] such that e3 x '•el'e2"'# If el ^ e3 or e2 "^ e3 tlien F ^s homogenous on [e3] and F is a linear functional. Next let e,┴ I e^ and e2 JL e3 or equivalently *[e^^e^]* J- e^. Since e3 I *[e*┴,e2]* e3^[e1,e2] and rank T = *3,* (Te3,e3) / 0. Similarly since el x e2 ^ el X e3 ^t ^s ver^fied that (Te^^^e,)┴^ 0. Thus since e┴ x e3, e3x e┴ there is a nonzero real number a such that either ae,┴ + e3 x ae.┴ + e3 or e3 + ae,┴ x e3 - ae,┴.
Thus as in the case of (*) in the preceding paragraph it follows that F is homogenous on [e3]. Hence F is a linear functional.
Next suppose there is no vector e3^[e┴,e2] such that e3 x [e┴,e2]. Since rank T = 3, there is a vector x / 0 such that x x [e┴,e2] and x^[e┴]. Since such a vector

$x \in [e_1, e_2]$ there are real numbers $a, b, b \neq 0$ such that $ae_1 + be_2 \perp e_2$, and $ae_1 + be_2 \perp e_1$. Thus since $e_1 \perp e_2$ and $e_2 \not\perp e_1$ it is verified that $(Te_2, e_2) = 0 = (Te_1, e_1)$. Hence we are in the case $e_1 \perp e_1$, $e_2 \perp e_2$, $e_1 \perp e_2$ and $e_2 \not\perp e_1$. Since $e_1 \perp [e_1, e_2]$, and Te_1, Te_2 are linearly independent there is a vector $e_3 \notin [e_1, e_2]$ such that $e_2 \perp e_3$. Identifying linear functionals f on E with points in E by the mapping $f \leftrightarrow \sum_{i=1}^{3} f(e_i)e_i$ it is verified that there are real numbers a_3, b_1, c_1, c_2, and c_3 such that $Te_1 = a_3e_3$, $Te_2 = b_1e_1$, $Te_3 = \sum_{i=1}^{3} c_ie_i$. Since the rank $T = 3$, a_3, b_1, c_3, c_2 are nonzero real numbers. Thus $e_3 \perp e_3$ while $e_3 \not\perp e_3$. Hence F is homogenous on $[e_3]$. Further it is verified that $e_3 \perp c_2e_3 - c_3e_2$ and $c_2e_3 - c_3e_2 \perp e_3$. Hence F is linear on $[e_3, c_2e_3 - c_3e_2]$. Now since $e_3 \perp [e_3, c_2e_3 - c_3e_2]$ and F is homogenous on $[e_3]$ it follows that F is linear on E.

Next we proceed to the main theorem of this section.

Theorem 1. Let E be a real Hausdorff topological vector space and $T : E \to E^*$ be a linear mapping such that the T-orthogonality is not symmetric. Then every continuous orthogonally additive functional on E is linear.

Proof. In view of the introductory comments in this section we may assume that rank $T \geq 2$. Since the range of T is of dimension at least 2, and orthogonality is not symmetric we claim that there exist two vectors $e_1, e_2 \in E$ such that $e_1 \perp e_2$,

$e_2 \times e_1$ and $-Te_1, Te_2$ are linearly independent. For let x,y be two vectors such that x i y, and y \ x. If Tx,Ty are linearly dependent let peE be such that Tp,Ty are linearly independent. If y \ p then since y \ x there exists a real number a such that y JL p + ax. If p + a x i y then since x x y, p i y. Thus y X p and p i y and Tp,Ty are linearly independent. Next if p + ax \ y, then p + ax,y are vectors of the required type. If y x p, then if p \ y, p,y have the desired properties. If p ± y then p + x i y and p 4- x,y have the desired properties. Thus there exist vectors e-)*e? as claimed. Let now xeE ~ fei^e9-'#. Consider the linear mapping T| [x_5e_1, e_2]=0^. Then applying proposition 1 to T_1 and the function F it follows F| [x, e_1, e_2] is linear. Tliis also implies in particular that F is linear on [x] for all xeE. Next let x,y be two linearly independent vectors, x*Y^tei*e?] • If x JL y(yJ-x) F is verified to be linear on [x,y] from the preceding observation. Next if x \ y and y \ x, then if (Tx,x) ^ 0 or (Ty,y) ^ 0 it is possible to find a real number a such that x x x + ay or y x y + ax, Then in either case as before F is linear on the span of [x_3y] . If (Tx,x) = 0 = (Ty,y), then since (Tx,y) / 0 / (Ty,x) it is verified that there is a real number a such that x + ay x y + x, once again verifying F is linear on [x,y]. Thus in any case F is linear on [x,y]. Hence F is a linear functional.

4. We discuss here the case when the T-orthogonality is symmetric. We note that if F : E—>R is orthogonally additive then the even and odd parts F_1, F_2 of F are also orthogonally additive. This is verified from the equations F-^x) =-j [F (x) +F(-x)] and $F_2(x)$ = $\frac{1}{2}$[F(X)-F(-X)].

As in the preceding sections we assume that dim E ^> 2. Further we note that if dim T = 1 then as observed in lemma 2 there is a xeE such that (Tx,x) ^ 0. Now as in the case when T-orthogonality is not symmetric, dim T = 1 (see first paragraph in section 3) it is verified that if F is a orthogonally additive functional on E and M = Tx" ^{1}(0) then F|M is linear. Since E = M © [x] it is verified that F determines a unique continuous function cp : R—>R, cp(0) =0 such that F(Ax+y) = cp(A) + *I* (y) if yeM and F|M = *I.* Conversely if *leE* and cp : R—^R is a continuous function, cp(O) =0, then the function F : E—^R defined by F(£) = cp(A) + *t* (y), if £ = Ax + y, yeM, determines a continuous orthogonally additive function. The preceding fact is verified by noting that for y,ZGM, Ax + y JL *XK* + z if and only if AjJ, = 0 since orthogonality is symmetric and y ± z.

We proceed to discuss the case when rank T ^ 2.

Proposition 2. Let dim E = 2. If T : E—>E* is a linear mapping, rank T = 2, and if T-orthogonality is symmetric, then a continuous function F : E—>R is even and orthogonally additive if and only if F(X) = c(Tx,x) for some real number c.

Proof. If (Tx,x) = 0 for all X€E, then since F is even orthogonally additive functional it follows that F(x) = F(-x), and F(x) + F(-x) = F(0) = 0. Thus F (x) =0 for all xeE.

Next if for some x (Tx,x) ^ 0, then from lemma 1 it is inferred that T is a symmetric mapping. Let e.^ be a vector such that e^^ \ e^ Then there is a vector e_2, $e_2M^e]\_]$ such that e_1 i e_2. Since T is of rank 2, Te_2 / 0« Thus e_2 J- e_1 implies e_2 \ e_2. Hence we can assume that there are real numbers a ^ 0 / *h,* such that Te_1 = ae-j^ and $Te_2 = be_2$. We can assume without loss of generality that a > 0. It is verified that $^x i^e i_1 + {}^x o^e 2$ ± ^l^e l + ^2^e 2 ııxıf and on^Y ^f a x l y l + b x 2 Y 2 = ∘ * Now if b > ∘ tl:ien there are vectors x,y,x€ [e^ ,ye [e_2] such that (Tx,x) = 1 = (Ty,y) . If b < 0 then there are vectors x,y,as above such that (Tx,x) = 1 = - (Ty,y) . For such a pair x,y, for all real numbers K, K(x+y) I K(x-y) or K(x+y) -L K(x+y) according as b > 0 or b < 0. Since F is even and Kx ± Ky, it is verified from the orthogonal additivity of F that F(Kx) = F(Ky) or F(Kx) = -F(Ky). Now it is verified that there is a real number c such that for all K, F(Kz) = c(TKz,Kz) where z = x or z = y, noting that F(Kx) = F(Ky) and F(Kx) = -F(Ky) according as (Tx,x) = (Ty,y) or (Tx,x) = -(Ty^y). Let now 4 be an arbitrary vector in E. Let £ = Ax + |iy. Then from the orthogonal additivity of F it follows that

$$F(Ax+|-iy) = F(Ax) + F(f-iy) = c(TAx,Ax) + c(T\mu y,\mu y)$$
$$= c(T(Ax+|jy),\lambda x+\mu y).$$

Hence **F(£)** = $c(T\xi,\xi)$.

Theorem 2. Let dim E ^ 2 and T : E—>E be a linear mapping such that rank T ^> 2. If T-orthogonality is symmetric^then a continuous real valued function F on E is even and orthogonally additive only if there is a real number c such that for all $\xi \in E$,

$$F(\xi) = c(T\xi,\xi).$$

Proof. If (Tx,x) = 0 for all xeE ^then since x ± x for all x, F is linear on [x] . Since F is also even F(x) =0 for all xeE and it follows that F(x) = c(Tx,x) for all x, where c is an arbitrary real number.

Next let x be a vector such that (Tx,x) ^ 0. Let F be a continuous orthogonally additive function. Let M = Tx $^{-1}$(0) . There exists a yeM such that (Ty,y) / 0. For let every vector in M be isotropic. Since the rank T ^> 2 there is a vector peM such that Tp&jTx] • Thus there exists a zeM such that p X z. Now p + zeM. Since p + z i p + z, (Tp,z) + (Tz,p) = 0 , since every vector in M is isotropic. Since the mapping T is symmetric the preceding equation implies p i z contradicting the choice of z. Thus there is a vector yeM with (Ty,y) ^ 0. Let T_1 = T| [x,y]. Since (Ty,y) ^ 0 and (Tx,y) = 0, $T_1 y, T_1 x$ are linearly independent and the rank T_1 = 2. Noting that T-orthogonality coincides with T^-orthogonality on the plane [x,y] it follows from the preceding

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proposition that F(£) = c(T£,£) for all £e t^{x}>y] where c
2

is independent of £. In particular F(Kx) = K F(x) for all K ^ 0. Let now zeE. Let z = Ax + *r*\ where x JL *r*\ and A is a real number. Then

$$F(z) = F(AX+TJ) = F(Ax) + F(7?) = A^2F(x). + F(77).$$

If (TTJ,TJ) = 0 then F(?7) = 0. If (TTJJT)) ^ 0 from the preceding it follows that F(?7) = c(Trj,rj) where c is snch that F(x) = c(Tx,x). Thus

$$F(z) = A^2F(x) + c(T77,T?) = c(T(Ax+rj), \lambda x+\eta).$$

This completes the proof of the theorem

Next we proceed to the case when T-orthogonality is symmetric and F is an odd functional. In this case if x ± *x,* then F is linear on [x]. Thus,if (Tx,x) = 0 for all x, we expect F to be a linear functional. However we provide an example to show that this need not be the case when every vector x in E is isotropic and rank T = 2.

Theorem 3. Let T : E~^E be a linear mapping such that T-orthogonality is symmetric and rank T ^≥ *2.* Then every odd continuous T-orthogonally additive real valued function on E is linear, if there is at least one nonisotropic vector.

Proof. Since there is a nonisotropic vector and T-orthogonality is symmetric, the linear mapping T is symmetric. Further we note that since F is an odd orthogonally additive function, F is linear on [x] if x is isotropic. We proceed to verify that F is linear on [x] even if x is nonisotropic. As already noted in the second paragraph of the proof of the preceding theorem there is a vector y x x such that (Ty,y) ^ 0. We may even assume that (Ty,y) = ;f(Tx,x) . if (Ty,y) = (Tx,x) then since x x y, K(x+y) JL K(x-y) for all real numbers K. Thus noting that F is an odd function it is verified that F(2Kx) = 2F(Kx) and F(2Ky) = 2F(Ky). Further since for any real number m, m(x+y) x (x-y) it is verified that F((m+l)x) + F((m-l)y) = F (mx) + F (x) + F(xny) - F (y) . Now by straightforward induction it is verified that for integers m, F(mx) = mF(x) and F(my) = mF(y). Since x,y could be replaced by rx,ry, r a real number, F(mrx) = mF(rx) for all real numbers r and integers m. Hence for rationals $\frac{m}{n}$ we have $F(\frac{m}{n}x) = \frac{m}{n}F(x)$. Since F is continuous F is linear on [x]. If (Tx,x) = -(Ty,y), since x x y , x + y , x - y are isotropic vectors. Thus for any real number A, F(Mx+y)) = MF(x)+F(y)) and F(A(x-y)) = A [F (x)-F (y)] . Hence F(Ax) + F(Ay) = A(F(x) + F (y)) and F(Ax) -F(Ay) =A[F(x) -F(y)]. Thus F(Ax) = AF(x). Hence F is linear on all 1-dimensional subspaces of E.

We proceed to show that F is indeed linear on E. Since F is linear on each line in E and orthogonally additive it is enough to show that in any two dimensional subspace [x,y] there are two linearly independent orghogonal vectors. Let x,y be two linearly independent vectors. If x x y we have two orthogonal vectors in $[x,y]$. If $x \perp y$, but $(Tx,x) \neq 0$ $((Ty,y) \neq 0)$ the pair x, $x + ay(y,y+ax)$ where $a = -\frac{(Tx,x)}{(Tx,y)}$ $\left(a = -\frac{(Ty,y)}{(Tx,y)}\right)$ is verified to be a pair of the required type in the subspace $[x,y]$. If $(Tx,x) = 0 = (Ty,y)$ then the pair $x + y$, $x - y$ is one such since T is symmetric. This completes the proof of linearity of F. Thus $F \in E^*$.

Before proceeding to the case when every vector is T-isotropic let us recall that according to lemma 2 if the rank of T is an odd integer ≥ 3 then there is at least one non-isotropic vector. We start with a preliminary result dealing with the case when rank $T = 4$.

Proposition 3. If $\dim E = 4$ and $T : E \longrightarrow E^*$ is a symmetric linear isomorphism and if every vector is isotropic, then every odd orthogonally additive continuous real valued function on E is linear.

Proof. Let $e_1 \in E \sim \{0\}$. Since $Te_1 \neq 0$, the subspace $M = Te_1^{-1}(0)$ is 3-dimensional. Let e_2 be a vector in $Te_1^{-1}(0)$ such that e_1, e_2 are linearly independent. Since Te_2 and Te_1 are linearly independent there is a vector e_3 such that $e_1 \perp e_3$ and $(Te_2, e_3) = 1$ and a vector e_4 such that $e_2 \perp e_4$ and $(Te_1, e_4) = 1$. It is verified that $\{e_1, e_2, e_3, e_4\}$ is a base for E and representing linear functionals f on E with vectors in E by the isomorphism $f \longleftrightarrow (f(e_1), f(e_2), f(e_3), f(e_4))$

it is verified from the properties that every vector is isotropic and orthogonality is symmetric that

$$Te_1 = e_4,\ Te_2 = e_3,\ Te_3 = -e_2 \text{ and } Te_4 = -e_1 .$$

Since for every $x \in E$, $x \perp x$ it follows that F is linear on $[x]$ for every $x \in E$. Thus if $x \perp y$ then F is linear on the subspace $[x,y]$. Since $e_1 \perp [e_1,e_2,e_3]$, $e_2 \perp [e_1,e_2,e_4]$, $e_3 \perp [e_1,e_3,e_4]$, $e_4 \perp [e_2,e_3,e_4]$ and $[e_2,e_3] \perp [e_1,e_4]$ it is enough to verify that F is linear on the subspaces $[e_2,e_3]$ and $[e_1,e_4]$. Consider a typical vector, say $\lambda e_2 + \mu e_3$ in $[e_2,e_3]$. It is verified that $e_1 + \lambda e_2 \perp \mu e_3 - \lambda\mu e_4$ and $e_1 - \lambda\mu e_4 \perp \lambda e_2 + \mu e_3$. Thus

$$F(e_1+\lambda e_2+\mu e_3-\lambda\mu e_4) = F(e_1+\lambda e_2) + F(\mu e_3-\lambda\mu e_4)$$

Since $e_1 \perp e_2$ and $e_3 \perp e_4$,

(1) $F(e_1-\lambda\mu e_4) + F(\lambda e_2+\mu e_3) = F(e_1) + F(\lambda e_2) + F(\mu e_3) - F(\lambda\mu e_4)$.

Once again since $e_1 + \lambda e_2 + \mu e_3 \perp \lambda e_2 + \lambda\mu e_4$ and $e_3 \perp e_1 - \lambda\mu e_4$ it follows that

$$\begin{aligned} F(e_1+\mu e_3-\lambda\mu e_4) &= F(\mu e_3)+F(e_1-\lambda\mu e_4) \\ &= F(e_1+\lambda e_2+\mu e_3)-F(\lambda e_2+\lambda\mu e_4) \\ &= F(e_1) + F(\lambda e_2+\mu e_3) - [F(\lambda e_2) + F(\lambda\mu e_4)] \end{aligned}$$

Thus

(2) $F(e_1-\lambda\mu e_4) - F(\lambda e_2+\mu e_3) = F(e_1) - F(\lambda\mu e_4) - F(\lambda e_2) - F(\mu e_3)$

From equations (1) and (2) and from the linearity of F on each line in E it follows that

$$F(\lambda e_2+\mu e_3) = F(\lambda e_2) + F(\mu e_3) = \lambda F(e_2) + \mu F(e_3) \quad \text{and}$$

$$F(e_1-\lambda\mu e_4) = F(e_1) - \lambda\mu F(e_4).$$

Thus F is verified to be linear on the subspaces $[e_2,e_3]$ and $[e_1,e_4]$. Hence F is a linear functional on E.

Theorem 4. Let E be an arbitrary topological vector space and T : E—>E be a linear mapping such that rank T ^> 3 and (Tx,x) = 0 for all xeE, and T-orthogonality is symmetric. If F is a continuous orthogonally additive functional on E, then F is linear.

Proof. Let e_1, e_4 be an arbitrary pair of linearly independent vectors. If e_1 x e_4 then since F is linear on [x] for each xeE, F is linear on the subspace $[e_1, e_4]$. Next let e_1 X e_4. Since e_1 ^ e_4* e_4 i e_4 and Te_1 / 0 ^ Te_4 it is verified that Te_1, Te_4 are linearly independent. Since x x x for all xeE and dim T ^ 3, it follows from the remarks preceding the proposition 3 that dim T ^ 4. Thus there exists a vector £, say £ = Ae_4 + h, where heTe_1^{-1}(0) such that T£ ∉ [Te_1,Te_4]. Now let h = Me_1 + e_2 where e_2 i e_1. Then it is verified that Te_2 ^ [Te_1, Te_4] and e_1 JL e_2, e_4 x e_2.

Now let e_3 be a vector in $Te_1^{-1}(0)$ ∩ $Te_4^{-1}(0)$ such that $e_2 \perp e_3$. It is verified that $Te_3 \not\in [Te_1, Te_2, Te_4]$. Further it is verified that the rank of $T_1 = T|E^4$ is 4, where $E^4 = [e_1, e_2, e_3, e_4]$ and the T-orthogonality restricted to E^4 coincides with T_1-orthogonality. Thus applying the preceding proposition, it is inferred that $F|E^4$ is linear. Hence F is linear on $[e_1, e_2]$ completing the proof of the theorem.

Before summarizing the results we discuss an example showing that preceding theorem cannot be improved.

Example. Consider $E = R^2$. Let $\{e_1, e_2\}$ be a base of E. Let T be the operator defined by $Te_1 = e_2$ and $Te_2 = -e_1$. Then it is verified that $(Tx,x) = 0$ for $x \in R^2$. Let $F : R^2 \longrightarrow R$ defined by, $F(ae_1 + be_2) = (a^3 + b^3)^{1/3}$. It is verified that F is a continuous T-orthogonally additive odd functional on R^2. Thus in the preceding theorem rank $T \geq 3$ cannot be replaced by rank $T \geq 2$.

Since every orthogonally additive functional F is the sum of an even and an odd orthogonally additive functional we can summarize the results of this section as follows.

<u>Theorem 5.</u> Let $T : E \longrightarrow E^*$ be a linear mapping such that $\dim E \geq 2$. If T-orthogonality is symmetric and if there is at least one non-isotropic vector, then a continuous function $F : E \longrightarrow R$ is orthogonally additive only if there are a real number c and a functional $\ell \in E^*$ such that $F(x) = c(Tx,x) + \ell(x)$ for all $x \in E$. If T is as above except

that every vector in E is isotropic^then if $\dim T$ ^ 3 every continuous orthogonally additive functional is linear.

In conclusion it might be remarked that if the quadratic form associated with the linear mapping T is not continuous on E, then $c = 0$ in Theorems 2 and 5.

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