NEMITSKY OPERATORS ON
SOBOLEV SPACES

by

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Introduction. The present paper deals with situations in which a function $g(\underset{\sim}{x}, t_1, \ldots, t_m)$ provides, via composition, a mapping ("Nemitsky operator") between Sobolev spaces. That is, we take g to be a function satisfying "Caratheodory conditions" and we analyze circumstances under which for every system of functions $u_1, \ldots, u_m \in W_{1,q}(\Omega)$, where $W_{1,q}(\Omega)$ is the class of L_q functions with L_q summable strong first derivatives on the domain $\Omega \subset R_n$, the composite function given by $v(\underset{\sim}{x}) = g(\underset{\sim}{x}, u_1(\underset{\sim}{x}), \ldots, u_m(\underset{\sim}{x}))$ belongs to $W_{1,p}(\Omega)$, with preassigned $1 \leq p < \infty$. We suppose in addition to the Caratheodory conditions that, roughly speaking, g is locally absolutely continuous on lines in R_m parallel to the axes and has a similar but weaker property for lines in Ω. This implies in some sense that the partial derivatives of g exist almost everywhere in $\Omega \times R_m$, and in our hypotheses one prescribes for each of these partial derivatives a function of an appropriate kind which dominates it almost everywhere. Then whenever $u_1, \ldots, u_m$ are such that their composites with the dominating functions lie in appropriate $L_r(\Omega)$ spaces, it is shown that v lies in $W_{1,p}(\Omega)$. (We remark that the analysis for the case $p = 1$ is considerably more complex than when $p > 1$.)

The above results are quite different from those usually studied with Nemitsky operators since such operators are generally examined only on spaces, such as the $L_r(\Omega)$ spaces, which are normal lattices of measurable functions ([10],[11]). These results should be of interest in the study of partial differential equations which involve nonlinear functions satisfying weak smoothness requirements.

In an earlier paper ([12]) we have likewise analyzed situations in which a function g provides a mapping between Sobolev spaces. The methods of that paper are quite different from those used here and are restricted to situations in which a chain rule is available for the partial derivatives of v. Moreover, there the chain rule was an essential ingredient in determining when a function g provides a mapping of the desired kind, while in the present paper a chain rule is generally not valid for the situations under study.

The approach we follow here relies heavily on a characterization of the spaces $W_{1,q}(\Omega)$ due to Gagliardo [6], Morrey [14] and Calkin [3]. It also utilizes a theorem of Hardy-Littlewood [8] on difference quotients under translation, as well as certain classical results of Tonelli [15, p. 123] on absolutely continuous curves.

The plan of the paper is as follows. Section 1 is devoted to preliminaries. Section 2 deals with the basic problem for the case $p > 1$. Section 3 extends these results to the case $p = 1$. Section 4 analyzes continuity properties of the Nemitsky operators for the case $p > 1$. Finally in Section 5, by restricting attention to the particular case of functions g which are independent of $\underset{\sim}{x} \in \Omega$ and by strengthening our previous conditions, we obtain a chain rule for the partial derivatives of v.

§1. Preliminaries

The following notations will be used in this paper.

A point in the Euclidean space R_n will hereafter be denoted by $x = (x_1, \ldots, x_n)$; the Euclidean norm will be denoted by $|\cdot|$. We shall denote by $\mathcal{L}_k$, k-dimensional Lebesgue measure and by $\mathcal{H}_k$, k-dimensional Hausdorff measure. We shall use the same symbol for an equivalence class of functions (relative to Lebesgue measure) as for a representative of that class. The meaning will be clear from context.

If f is a real function defined in a domain Ω in R_n, we denote by $\frac{\partial f}{\partial x_k}$ the classical partial derivative (with respect to x_k), wherever it exists; and we denote by $\partial_{x_k} f$ the distribution derivative (with respect to x_k) of f in Ω, whenever it is meaningful.

By $C^k(\Omega)$ we denote, as usual, the class of real functions which are continuous and possess continuous derivatives, up to order k, in Ω. The subspace of $C^k(\Omega)$ consisting of those functions whose support is a compact subset of Ω will be denoted by $C_o^k(\Omega)$. The class of real functions $\{f\}$ such that f is Lipschitz in Ω will be denoted by $\mathrm{Lip}(\Omega)$; the class of real functions $\{f\}$ such that f is Lipschitz in every compact subset of Ω will be denoted by $\mathrm{Lip}^{loc}(\Omega)$. Finally, we denote by $L_p(\Omega)$ the class of real functions $\{f\}$ which are Lebesgue p-summable in Ω. The class of functions $\{f\}$ which belong to $L_p(\Omega')$ for every bounded domain Ω' such that $\overline{\Omega'} \subset \Omega$, will be denoted by $L_p^{loc}(\Omega)$. The standard norm in $L_p(\Omega)$ will be denoted by $\|\cdot\|_{L_p(\Omega)}$.

A real function f defined on an open subset O of the real line R_1, is said to be locally absolutely continuous (or l.a.c.) on O, if it is absolutely continuous on every compact subinterval of O. Similarly, f is said to be locally of bounded variation (or l.b.v.) on O, if it is of bounded variation on every compact subinterval of O. If $O = \bigcup_{n=1}^{\infty} I_n$, where $\{I_n\}$ is a family of disjoint open intervals, we denote:

$$(1.1) \qquad \text{tot.var.}_{O}[f] = \sum_{1}^{\infty} \text{tot.var.}_{I_n}[f],$$

which may be finite or infinite.

In this paper, such notions, as "null set" and "almost everywhere", will always refer to the measure $\mathcal{L}_n$, except when another measure is specified.

We bring now a number of preliminary results that will be needed in the following sections.

If f is any real function defined on an interval I on the real line, it is known that the domain of existence of f' is an $\mathcal{L}_1$-measurable set and f' is an $\mathcal{L}_1$-measurable function on this domain. Moreover if f is Borel measurable, then f' is Borel measurable. (See Saks [15], p. 112-113.)

For functions in more than one variable we have the following two results:

<u>Lemma</u> 1.1. Let f be a real function defined in a domain Ω in R_n. If f is Borel measurable, then the domains of existence of each of the derivatives $\frac{\partial f}{\partial x_j}$, $(j = 1,\ldots,n)$, are Borel sets, and the derivatives are Borel functions on their respective domains.

For proof, see Marcus and Mizel [12, Lemma 4.1].

<u>Lemma</u> 1.2. Let f be a real measurable function in a domain Ω in R_n. Suppose that f is continuous on almost every line τ parallel to the x_i-axis (where i is a fixed index). Then the domain of existence of $\frac{\partial f}{\partial x_i}$ is measurable and $\frac{\partial f}{\partial x_i}$ is a measurable function on this domain.

This result is probably known, but we have not been able to locate any reference for it. Therefore, we present a proof below. If instead of considering the standard derivative $\frac{\partial f}{\partial x_i}$ one considers approximate partial derivatives, then results on their measurability may be found for instance in Saks [15] and Federer [5]. Actually, a result that is stronger than Lemma 1.2 (it assumes only that f is measurable) is stated in Haslam - Jones and Burkill [2], but their proof contains a serious gap.

Before we proceed with the proof of the lemma, we introduce a notation that will be useful in this proof and elsewhere.

For any function f we denote:

$$(1.2) \qquad \delta_h^i f(x) = \frac{1}{h}[f(x + he^i) - f(x)], \qquad (h \neq 0),$$

where $e^i = (\delta_{i,1},\ldots,\delta_{i,n})$.

<u>Proof</u> <u>of</u> <u>Lemma</u> 1.2. Let $\overline{D}_i f$ [resp. $\underline{D}_i f$] denote the upper [resp. the lower] derivative of f in the x_i-direction. It is sufficient to show that both of these extreme derivatives are

measurable. Indeed, if this is shown, then the domain of existence of $\frac{\partial f}{\partial x_i}$ is precisely the set where $\overline{D}_i f - \underline{D}_i f = 0$, which is measurable; on this set $\frac{\partial f}{\partial x_i}$ coincides with the measurable function $\overline{D}_i f$.

By definition $\overline{D}_i f(x) = \limsup_{h \to 0} \delta_h^i f(x)$. We also define:

$$(1.3) \qquad \overline{D}_i^{\text{rat.}} f(x) = \limsup_{r \to 0} \delta_r^i f(x),$$

where r is restricted to the rational numbers. Clearly, $\overline{D}_i^{\text{rat.}} f$, being the lim sup of a countable family of measurable functions, is itself a measurable function in Ω. If we similarly define $\underline{D}_i^{\text{rat.}} f$ (with lim sup replaced by lim inf), then by the same token, $\underline{D}_i^{\text{rat.}} f$ is measurable in Ω. Let A be the set where $\overline{D}_i^{\text{rat.}} f = \underline{D}_i^{\text{rat.}} f$. In A we define $\overline{D}_i^{\text{rat.}} f = D_i^{\text{rat.}} f$. Then A is a measurable set and $D_i^{\text{rat.}} f$ is measurable on A. If B denotes the domain of existence of $\frac{\partial f}{\partial x_i}$, then clearly $B \subseteq A$ and $\frac{\partial f}{\partial x_i} = D_i^{\text{rat.}} f$ in B. We shall show that $A - B$ is a null set, thereby proving the assertion of the lemma.

Let τ be a line parallel to the x_i-axis such that f is continuous on $\tau \cap \Omega$. We shall show that $A \cap \tau = B \cap \tau$.

Let $x \in A \cap \tau$ and let $\{h_\nu\}_1^\infty$ be any sequence of numbers tending to zero, such that $h_\nu \neq 0$, $(\nu = 1,2,\ldots)$. We may assume that the sequence of points $\{x + h_\nu e^i\}_1^\infty$ is contained in a compact subinterval of $\tau \cap \Omega$. Choose a sequence of rational numbers $\{r_\nu\}_1^\infty$ such that $\lim_{\nu \to \infty} r_\nu/h_\nu = 1$ and such that $|f(x + r_\nu e^i) - f(x + h_\nu e^i)| \leq h_\nu/\nu$. (Here we use the continuity of f on $\tau \cap \Omega$.) Then:

$$(1.4) \qquad \lim_{\nu \to \infty} \delta^i_{h_\nu} f(x) = \lim_{\nu \to \infty} \left[\frac{r_\nu}{h_\nu} \delta^i_{r_\nu} f(x) + \frac{f(x+h_\nu e^i) - f(x+r_\nu e^i)}{h_\nu}\right]$$

$$= \lim_{\nu \to \infty} \delta^i_{r_\nu} f(x) = D_i^{\text{rat.}} f(x).$$

It follows from (1.4) that $A \cap \tau = B \cap \tau$. Since this holds for almost every line τ parallel to the x_i-axis, it follows that $A - B$ is a null set, and the proof is complete.

<u>Corollary</u> 1.1. Let f have the properties stated in the lemma. If, for almost every line τ parallel to the x_i-axis, $\frac{\partial f}{\partial x_i}$ exists $\mathcal{L}_1$ - a.e. on $\tau \cap \Omega$, then $\frac{\partial f}{\partial x_i}$ exists a.e. in Ω and is measurable.

By Fubini's theorem, this is an immediate consequence of the lemma.

<u>Definition</u> 1.1. Let Ω be a domain in R_n. We denote by $A_i(\Omega)$ the class of functions $\{f\}$ such that:

(i) f is a real measurable function in Ω.

(ii) For almost every line τ parallel to the x_i-axis, f is l.a.c. on $\tau \cap \Omega$.

The intersection $\bigcap_{1}^{n} A_i(\Omega)$ will be denoted by $A(\Omega)$.

The class of functions $\{f\}$ such that f coincides a.e. in Ω with a function $\tilde{f}$ in $A_i(\Omega)$ [resp. $A(\Omega)$] will be denoted by $A'_i(\Omega)$ [resp. $A'(\Omega)$].

Finally, suppose that condition (ii) is replaced by:

(ii)' For almost every line τ parallel to the x_i-axis, f is continuous and l.b.v. on $\tau \cap \Omega$.

Then, the spaces corresponding to A_i, A'_i, A, A' will be denoted by B_i, B'_i, B, B' respectively. Note that $A_i \subset B_i$, $A'_i \subset B'_i$, etc.

<u>Remark</u> 1.1. By Corollary 1.1, if $f \in B_i(\Omega)$, then $\frac{\partial f}{\partial x_i}$ exists a.e. in Ω and is measurable.

<u>Definition</u> 1.2. Let $f \in B'_i(\Omega)$ and let $\tilde{f} \in B_i(\Omega)$ such that $\tilde{f} = f$ a.e. in Ω. We denote by $\partial'_{x_i} f$ the equivalence class of measurable functions in Ω which contains the function $\frac{\partial \tilde{f}}{\partial x_i}$. Then $\partial'_{x_i} f$ will be called the <u>strong approximate derivative</u> of f with respect to x_i.

We note that if $\tilde{f}_1, \tilde{f}_2 \in B_i(\Omega)$ and $\tilde{f}_1 = \tilde{f}_2$ a.e. in Ω, then $\frac{\partial \tilde{f}_1}{\partial x_i} = \frac{\partial \tilde{f}_2}{\partial x_i}$ a.e. in Ω.

<u>Lemma</u> 1.3. Suppose that $f \in L_1^{loc}(\Omega) \cap A'_i(\Omega)$. If $\partial'_{x_i} f \in L_1^{loc}(\Omega)$, then

(1.5) $$\partial'_{x_i} f = \partial_{x_i} f \quad \text{a.e. in } \Omega.$$

Proof. Let $\tilde{f} \in A_i(\Omega)$ such that $\tilde{f} = f$ a.e. in Ω. Let $\varphi \in C_o^\infty(\Omega)$. If τ is a line parallel to the x_i-axis, such that $\tilde{f}$ is l.a.c. on $\tau \cap \Omega$, we have:

$$\int_{\tau\cap\Omega} \frac{\partial \tilde{f}}{\partial x_i} \cdot \varphi dx_i = -\int_{\tau\cap\Omega} \tilde{f} \frac{\partial\varphi}{\partial x_i} dx_i.$$

Hence, by Fubini's theorem:

$$\int_\Omega \frac{\partial \tilde{f}}{\partial x_i} \varphi dx = -\int_\Omega \tilde{f} \frac{\partial\varphi}{\partial x_i} dx = -\int_\Omega f \frac{\partial\varphi}{\partial x_i} dx$$

which proves the assertion of the lemma.

We denote by $W_{k,p}(\Omega)$ (k a positive integer; $p \geq 1$), the Sobolev space of real functions $\{f\}$ such that f and its distribution derivatives up to order k, belong to $L_p(\Omega)$. This space is provided with the standard norm:

$$\|f\|_{W_{k,p}(\Omega)} = \sum_{|\alpha|\leq k} \|\partial_x^\alpha f\|_{L_p(\Omega)},$$

where $\alpha = (\alpha_1,\ldots,\alpha_n)$, $0 \leq \alpha_j$ is an integer $(j = 1,\ldots,n)$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$ and $\partial_x^\alpha f = \partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n} f$.

The space of functions $\{f\}$ such that $f \in W_{k,p}(\Omega')$ for every bounded domain Ω' such that $\overline{\Omega'} \subset \Omega$ is denoted by $W_{k,p}^{loc}(\Omega)$. The product space $W_{k,p_1}(\Omega) \times \ldots \times W_{k,p_m}(\Omega)$ will be denoted by $W_{k,\tilde{p}}(\Omega)$, where $\tilde{p} = (p_1,\ldots,p_m)$.

The following characterization of $W_{1,p}(\Omega)$, $(1 \leq p)$, is due to Gagliardo [6]. (Most of the essential features of this result (at least in local form) are also contained in the papers of Calkin [3] and Morrey [14].)

Lemma 1.4. Let $1 \leq p < \infty$. A function f, defined in Ω, belongs to $W_{1,p}(\Omega)$ if and only if:

(i) $f \in A'(\Omega)$;

(ii) $\partial'_{x_i} f \in L_p(\Omega)$, $(i = 1,\ldots,n)$;

(iii) $f \in L_p(\Omega)$.

Moreover, if $f \in W_{1,p}(\Omega)$ then $\partial'_{x_i} f = \partial_{x_i} f$ a.e. in Ω, $(i = 1,\ldots,n)$.

Finally if Ω is bounded and has the cone property, then condition (iii) may be omitted.

Remark 1.2. 1. The result was not stated in this form in [6], but is an immediate consequence of Sections 1 and 2 of that paper.

2. As a consequence of this lemma we have the following local result:

$f \in W^{loc}_{1,p}(\Omega)$ if and only if:

(i) $f \in A'(\Omega)$;

(ii)' $\partial'_{x_i} f \in L^{loc}_p(\Omega)$, $i = 1,\ldots,n$.

3. If $k > 1$, then $W_{k,p}(\Omega)$ may be characterized inductively

by:

(1.6) $$f \in W_{k,p}(\Omega) \Leftrightarrow \partial_{x_i} f \in W_{k-1,p}(\Omega).$$

The following two results on Sobolev spaces are well known. For proofs see for instance Agmon [1] (pp. 42-45). Combined, these results yield an alternative characterization of $W_{1,p}(\Omega)$, for $1 < p < \infty$.

If Ω is a domain in R_n and Ω' a set in R_n, the notation $\Omega' \subset\subset \Omega$ means that $\overline{\Omega'}$ is a compact subset of Ω. The boundary of Ω will be denoted by $\partial\Omega$.

<u>Lemma</u> 1.5. Let $f \in W_{1,p}(\Omega)$, where $1 \leq p < \infty$. If Ω' is an open set such that $\Omega' \subset\subset \Omega$ and if $0 < h < \text{dist.}(\Omega', \partial\Omega)$, then:

(1.7) $$\|\delta_h^i f\|_{L_p(\Omega')} \leq \|\partial_{x_i} f\|_{L_p(\Omega)}, \quad (i = 1,\ldots,n).$$

<u>Lemma</u> 1.6. Let $f \in L_p(\Omega)$, where $1 < p < \infty$. Suppose that there exists a number C such that, for every open set Ω' with $\Omega' \subset\subset \Omega$, and for sufficiently small $|h|$:

(1.8) $$\|\delta_h^i f\|_{L_p(\Omega')} \leq C.$$

Then $\partial_{x_i} f \in L_p(\Omega)$ and $\|\partial_{x_i} f\|_{L_p(\Omega)} \leq C$. In particular, if (1.8) holds for all i, $(i = 1,\ldots,n)$, then $f \in W_{1,p}(\Omega)$.

<u>Remark</u>. In [1] these results are stated for $p = 2$. But the proofs given there, with only minor modifications, yield also the more general results stated above. The special assumption on Ω, included in the statement of Theorem 3.15 of [1], was made for convenience only and is not really needed (see [1 , p. 11]).

The next lemma deals with the situation considered in Lemma 1.6, for the case $p = 1$, and f a function of one variable. It is due to Hardy and Littlewood [8].

<u>Lemma</u> 1.7. Let $f \in L_1(I)$, where I is an interval on the real line. Suppose that for every compact subinterval of I, say I', and for every sufficiently small $|h|$ we have:

$$(1.9) \qquad \int_{I'} |f(\sigma + h) - f(\sigma)|/|h| \, d\sigma \leq C,$$

where C is a constant independent of I' and h. Then f coincides $\mathfrak{L}_1$ - a.e. in I with a function $\tilde{f}$ of bounded variation such that:

$$(1.10) \qquad \text{tot.var.}_I [\tilde{f}] \leq C.$$

We shall need also the following two results due to Valée Poussin [19]. (Stronger versions of these results were obtained by Serrin and Varberg [17]; their proofs are much simpler than the original proofs of Valée Poussin.)

<u>Lemma</u> 1.8. Let s be an absolutely continuous real valued function on an interval I of the real line. Let N be an $\mathcal{L}_1$-null set on R_1 and set $M = s^{-1}(N) \cap I$. Then $s' = 0$ $\mathcal{L}_1$ - a.e. in M.

<u>Lemma</u> 1.9. Let w and s be absolutely continuous real valued functions on intervals J and I respectively. If $s(I) \subset J$ and if $w \; s$ is absolutely continuous in I, then:

$$(1.11) \qquad [w \circ s]' = [w' \circ s]s', \qquad \mathcal{L}_1 \text{ - a.e. in } I,$$

provided that we interpret the right side as zero whenever $s'(\sigma) = 0$ (even if $(w' \circ s)(\sigma)$ is undefined or infinite).

Conversely, if with the above convention $[w' \circ s]s'$ is summable on I, then $w \circ s$ is absolutely continuous on I and (1.11) holds.

The next result is due to Serrin [16] (unpublished). For a more general result see Marcus and Mizel [12] (Theorem 4.3).

<u>Lemma</u> 1.10. Let $g : R_1 \to R_1$ be an l.a.c. function. Suppose that $u \in W^{loc}_{1,1}(\Omega)$ and set $v = g \circ u$. Then $v \in W^{loc}_{1,1}(\Omega)$ if and only if the following condition holds:

$$(1.12) \qquad v_i = [g' \circ u]\partial_{x_i} u \in L^{loc}_1(\Omega), \quad (i = 1, \ldots, n),$$

the product being interpreted as zero whenever $\partial_{x_i} u = 0$.

Moreover, if (1.12) holds we have $v_i = \partial_{x_i} v$ a.e. in Ω, $(i = 1,\dots,n)$.

For the sake of convenience we give a proof of this lemma.

Proof. By Lemma 1.4, there exists $\tilde{u} \in A(\Omega)$ such that $\tilde{u} = u$ a.e. in Ω and $\frac{\partial \tilde{u}}{\partial x_i} = \partial_{x_i} u$ a.e. in Ω. Set $\tilde{v} = g \circ \tilde{u}$.

First we assume that (1.12) holds. Let τ be a line parallel to the x_i-axis, such that $\tilde{u}$ is l.a.c. on $\tau \cap \Omega$ and such that $\tilde{v}_i = [g' \circ \tilde{u}] \frac{\partial \tilde{u}}{\partial x_i} \in L_1^{loc}(\Omega)$. Then by Lemma 1.9, $\tilde{v}$ is l.a.c. on $\tau \cap \Omega$ and $\frac{\partial \tilde{v}}{\partial x_i} = \tilde{v}_i$ $\mathcal{L}_1$ - a.e. in $\tau \cap \Omega$. Since this holds for almost every line τ parallel to the x_i-axis $(i = 1,\dots,n)$ it follows that $\tilde{v} \in A(\Omega)$ and that $\frac{\partial \tilde{v}}{\partial x_i} = \tilde{v}_i$ a.e. in Ω. Hence by (1.12) and Lemma 1.4 (see also Remark 1.2(3)) it follows that $\tilde{v} \in W_{1,1}^{loc}(\Omega)$.

Now, suppose that $v \in W_{1,1}^{loc}(\Omega)$. Then $\tilde{v} = g \circ \tilde{u} \in A(\Omega)$. Indeed, $\tilde{v}$ is continuous on $\tau \cap \Omega$, for almost every line τ parallel to one of the axes and $\tilde{v} = v$ a.e. in Ω. Since $v \in A'(\Omega)$ (by Lemma 1.4) it follows that $\tilde{v} \in A(\Omega)$.

If τ is a line parallel to the x_i-axis such that both $\tilde{u}$ and $\tilde{v}$ are l.a.c. on $\tau \cap \Omega$, it follows from Lemma 1.9 that:

$$\frac{\partial \tilde{v}}{\partial x_i} = [g' \circ \tilde{u}] \frac{\partial \tilde{u}}{\partial x_i} .$$

Since $\tilde{v} \in W_{1,1}^{loc}(\Omega)$ this implies (1.12). This completes the proof of the lemma.

Corollary 1.2. If Ω is bounded and has the cone property, the statement of the lemma is valid also in the case the "$v \in W_{1,1}^{loc}(\Omega)$" is replaced by "$v \in W_{1,p}(\Omega)$" and "$v_i \in L_1^{loc}(\Omega)$" is replaced by "$v_i \in L_p(\Omega)$, $(1 \leq p < \infty)$".

This follows immediately by Lemma 1.4.

§2. On a Class of Nemitsky Operators

Let Ω be a domain in R_n and let $g = g(x, \underset{\sim}{t})$ be a real function defined in $\Omega \times R_m$. Here $x = (x_1, \ldots, x_n)$ denotes a point in R_n and $\underset{\sim}{t} = (t_1, \ldots, t_m)$ denotes a point in R_m.

Definition 2.1. A function g as above is called a Caratheodory function if:

(i) For a.e. $x \in \Omega$, $g(x, \cdot)$ is a continuous function on R_m.

(ii) For every fixed $\underset{\sim}{t} \in R_m$, $g(\cdot, \underset{\sim}{t})$ is a measurable function in Ω.

With a given Caratheodory function g, we associate an operator G defined by:

$$G\underset{\sim}{u}(x) = g(x, \underset{\sim}{u}(x)) = (g \circ \underset{\sim}{u})(x)$$

where $\underset{\sim}{u} = (u_1,\ldots,u_m)$ is measurable in Ω. Such an operator G is called a <u>Nemitsky operator</u>.

By a theorem due to Caratheodory [4], if $\underset{\sim}{u}$ is a measurable vector valued function as above, $G\underset{\sim}{u}$ is measurable in Ω. (For the proof see also Vainberg [18], p. 152.)

In this section we consider the following problem:

Given a vector valued function $\underset{\sim}{u} \in W_{1,\tilde{q}}(\Omega)$, where $\tilde{q} = (q_1,\ldots,q_m)$, state conditions on g such that $G\underset{\sim}{u} \in W_{1,p}(\Omega)$.

In the theorems that follow we describe a set of rather weak conditions on g which imply this property of the associated operator G.

First we introduce:

<u>Definition</u> 2.2. Let g be a real function defined on $\Omega \times R_m$. We shall say that g is an <u>l.a.c. Caratheodory function</u> if:

(i) There exists a null subset of Ω, to be denoted by $N = N_g$, such that for every fixed $x \in \Omega - N$:

(a) $g(x,\cdot)$ is continuous in R_m;

(b) For every line parallel to one of the axes in R_m, $g(x,\cdot)$ restricted to this line is locally absolutely continuous.

(ii) For every fixed $\underset{\sim}{t} \in R_m$, $g(\cdot,\underset{\sim}{t}) \in A'(\Omega)$.

Note that a function g as above is in particular a

Caratheodory function.

With this definition we have:

Theorem 2.1. Let Ω be a bounded domain in R_n, possessing the cone property. Let g be an l.a.c. Caratheodory function in $\Omega \times R_m$. Let $p > 1$ and $\tilde{q} = (q_1,\ldots,q_m)$ where $q_i \geq p$.

Given certain functions a, b, a_i, $b_{i,j}$ suppose that:

I. For every fixed $\underset{\sim}{t} \in R_m$:

$$|\partial'_{x_i} g(x,\underset{\sim}{t})| \leq a(x) + b(\underset{\sim}{t}), \quad \text{a.e. in } \Omega, \quad (i = 1,\ldots,n), \tag{2.1}$$

where $\partial'_{x_i} g$ denotes the strong approximate derivative of g, as in Definition 1.2.

II. The inequality:

$$\left|\frac{\partial g(x,\underset{\sim}{t})}{\partial t_k}\right| \leq a_k(x) + \sum_{j=1}^{m} b_{k,j}(t_j), \quad (k = 1,\ldots,m), \tag{2.2}$$

holds at every point $(x,\underset{\sim}{t}) \in (\Omega - N) \times R_m$ at which the estimated derivative exists in the classical sense. Here, $N = N_g$ is the set mentioned in Definition 2.2.

The assumptions on a, b, a_k, $b_{k,j}$ are as follows:

$$0 \leq a \in L_p(\Omega); \tag{2.3}$$

$$0 \leq b \text{ is continuous in } R_m; \tag{2.4}$$

(2.5) $\quad 0 \leq a_k \in L_{q_k'}(\Omega)$, where $\frac{1}{q_k'} + \frac{1}{q_k} = \frac{1}{p}$, $(k = 1,\ldots,m)$;

(2.6) $\quad 0 \leq b_{k,j}$ is an extended real valued Borel measurable function on R_1, $(k,j = 1,\ldots,m)$;

(2.7) $$b_{k,k} \in L_1^{loc}(R_1), \qquad (k = 1,\ldots,m).$$

Let $\underset{\sim}{u} = (u_1,\ldots,u_m) \in W_{1,\underset{\sim}{q}'}(\Omega)$ and suppose that:

(2.8) $$b \circ \underset{\sim}{u} \in L_p(\Omega);$$

(2.9) $$b_{k,j} \circ u_j \in L_{q_k'}(\Omega), \qquad (k,j = 1,\ldots,m; k \neq j);$$

(2.10) $\quad [b_{k,k} \quad u_k] \partial_{x_i} u_k \in L_p(\Omega)$, $(k = 1,\ldots,m; i = 1,\ldots,n)$, where the product is to be interpreted as zero whenever $\partial_{x_i} u_k = 0$. Then $v = g \circ \underset{\sim}{u} \in W_{1,p}(\Omega)$.

<u>Proof</u>. Let M be a countable dense set in R_m. For each fixed $\underset{\sim}{t}' \in R_m$ we may modify $g(\cdot,\underset{\sim}{t}')$ on a null subset of Ω, say $N_{t'}$, such that the modified function is in $A(\Omega)$. We make this modification for every $\underset{\sim}{t}' \in M$, leaving $g(x,\underset{\sim}{t})$ unchanged when $\underset{\sim}{t} \notin M$, and denote the modified function by g_o. Let $N_M = \bigcup_{t' \in M} N_{t'}$; then N_M is a null subset of Ω. Note that g_o may differ from g only on $N_M \times M$. Clearly g_o satisfies all the assumptions that have been made with respect to g, except that

in II $N = N_g$ may have to be replaced by a larger null set (e.g. $N \cup N_M$).

Since $g(x,\underset{\sim}{u}(x)) = g_o(x,\underset{\sim}{u}(x))$ a.e. in Ω it is sufficient to prove that $g_o \circ \underset{\sim}{u} \in W_{1,p}(\Omega)$.

In order to simplify the notation we shall assume that the given function g already satisfies the additional condition:

$$(2.11) \qquad g(\cdot,\underset{\sim}{t}') \in A(\Omega), \qquad \forall \underset{\sim}{t}' \in M,$$

with M as above. Then (by Definition 1.2), for every fixed $\underset{\sim}{t}' \in M$:

$$\partial'_{x_i} g(x,t') = \frac{\partial g}{\partial x_i}(x,t'), \quad \text{a.e. in } \Omega, \quad (i = 1,\dots,n).$$

Hence, taking into account that M is countable, there exists a null subset of Ω, say N_1, such that (by (2.1)):

$$(2.1)' \quad \left|\frac{\partial g}{\partial x_i}(x,\underset{\sim}{t})\right| \le a(x) + b(\underset{\sim}{t}), \quad \forall\ (x,\underset{\sim}{t}) \in (\Omega - N_1) \times M, \quad (i = 1,\dots,n).$$

Let $\underset{\sim}{t}^o$ be a fixed point in R^m. Then $g(\cdot,\underset{\sim}{t}^o) \in A'(\Omega)$ and by (2.1) $\partial'_{x_i} g(\cdot,\underset{\sim}{t}^o) \in L_p(\Omega)$, $(i = 1,\dots,n)$. Hence by Lemma 1.4 it follows that:

$$(2.12) \qquad g(\cdot,\underset{\sim}{t}^o) \in W_{1,p}(\Omega), \qquad \forall \underset{\sim}{t}^o \in R_m.$$

Denote:

$$(2.13)\qquad \beta_i(\rho) = \int_0^\rho b_{i,i}(\rho)\,d\rho, \quad -\infty < \rho < \infty, \quad i = 1,\ldots,n.$$

Then β_i is l.a.c. on R_1. By (2.10) and Corollary 1.2 it follows that

$$(2.14)\qquad \beta_i \circ u_i \in W_{1,p}(\Omega), \quad i = 1,\ldots,n.$$

(In deriving (2.12) and (2.14) we used the fact that Ω is bounded and has the cone property.)

We prove now that $v = g \circ \underset{\sim}{u} \in L_p(\Omega)$. First we remark that, by the theorem of Caratheodory [4] mentioned before, v is a measurable function in Ω.

If $x \in \Omega - N$, then by (2.2):

$$(2.15)\qquad |g(x,\underset{\sim}{t}) - g(x,\underset{\sim}{t}^o)| \leq \left|\int_{t_1^o}^{t_1} \frac{\partial g}{\partial t_1}(x,\sigma,t_2^o,\ldots,t_m^o)\,d\sigma\right| + \ldots$$

$$+ \left|\int_{t_m^o}^{t_m} \frac{\partial g}{\partial t_m}(x,t_1,\ldots,t_{m-1},\sigma)\,d\sigma\right| \leq$$

$$\leq \sum_{i=1}^{m} \{a_i(x)|t_i - t_i^o| + \sum_{j=1}^{i-1} b_{i,j}(t_j)|t_i - t_i^o|$$

$$+ \sum_{j=i+1}^{m} b_{i,j}(t_j^o)|t_i - t_i^o| + |\beta_i(t_i) - \beta_i(t_i^o)|\}.$$

To simplify the notation we denote:

$$c_i^{(1)}(\underset{\sim}{t}) = \sum_{j=1}^{i-1} b_{i,j}(t_j), \quad c_i^{(2)}(\underset{\sim}{t}) = \sum_{j=i+1}^{m} b_{i,j}(t_j),$$

(2.16)

$$c_i(\underset{\sim}{t}) = c_i^{(1)}(\underset{\sim}{t}) + c_i^{(2)}(\underset{\sim}{t}).$$

Then, for $x \in \Omega - N$, we have:

$$(2.17) \quad |g(x,\underset{\sim}{t}) - g(x,\underset{\sim}{t}^o)| \leq \sum_{i=1}^{m} [a_i(x) + c_i^{(1)}(\underset{\sim}{t}) + c_i^{(2)}(\underset{\sim}{t}^o)]|t_i - t_i^o|$$

$$+ \sum_{i=1}^{m} |\beta_i(t_i) - \beta_i(t_i^o)|.$$

Now, pick a point $\underset{\sim}{t}^o$ such that $b_{i,j}(t_j^o)$ is finite for $i,j = 1,\ldots,m$, $i \neq j$. (Clearly, by (2.9), such a point exists.) Then, from (2.5), (2.9), (2.14), (2.17) we get:

$$(2.18) \quad \|v - g(\cdot,t^o)\|_{L_p(\Omega)} \leq \sum_{i=1}^{m} [\|a_i\|_{L_{q_i'}(\Omega)} + \|c_i^{(1)} \circ \underset{\sim}{u}\|_{L_{q_i'}(\Omega)}]\|u_i - t_i^o\|_{L_{q_i}(\Omega)}$$

$$+ \sum_{i=1}^{m} c_i^{(2)}(\underset{\sim}{t}^o)\|u_i - t_i^o\|_{L_p(\Omega)}$$

$$+ \sum_{i=1}^{m} \|\beta_i \circ u_i - \beta_i(t_i^o)\|_{L_p(\Omega)}.$$

Note that since Ω is bounded and $u_i \in L_{q_i}(\Omega)$, with $q_i \geq p$, it follows that $u_i \in L_p(\Omega)$. Taking into account (2.12) we conclude that $v \in L_p(\Omega)$.

In order to show that $v \in W_{1,p}(\Omega)$ we have to discuss some additional properties of g.

First we observe that g satisfies the following condition:

For a.e. line τ in R_n, parallel to the x_i-axis:

$$(2.19) \qquad |g(x',\underset{\sim}{t})-g(x'',\underset{\sim}{t})| \le \left|\int_{x'_i}^{x''_i} a(x)\,dx_i\right| + b(\underset{\sim}{t})\,|x'_i-x''_i|,$$

for every $\underset{\sim}{t} \in R_m$ and every two points x',x'' lying in one interval of $\tau \cap \Omega$, such that $x',x'' \notin N$. (The index i will be kept fixed throughout the following part of the proof.)

Indeed, for a.e. line τ parallel to the x_i-axis the following three conditions are satisfied:

(α_1) $\tau \cap (N \cup N_1)$ is an $\mathcal{L}_1$-null set;

(α_2) "a" restricted to $\tau \cap \Omega$ is in L_p;

(α_3) $g(\cdot,\underset{\sim}{t})$ is l.a.c. on $\tau \cap \Omega$ for every $\underset{\sim}{t} \in M$.

This follows from (2.3), (2.11) and the fact that $N \cup N_1$ is a null set.

If τ is a line as above, satisfying conditions (α_1), (α_2), (α_3), and if $\underset{\sim}{t} \in M$, (2.19) follows immediately from (2.1)', for any two points x',x'' lying in a subinterval of $\tau \cap \Omega$. If in addition $x',x'' \notin N$, then by the continuity of g with respect to $\underset{\sim}{t}$ (Definition 2.2(i)) and by the continuity of b in R_m we conclude that (2.19) holds for every $\underset{\sim}{t} \in R_m$. Here we are using the fact that M is a dense subset of R_m.

Furthermore, for such a line τ and for every $\underset{\sim}{t} \in R_m$, we can modify $g(\cdot,\underset{\sim}{t})$ on $\tau \cap N$ in such a way that the modified function will be l.a.c. on $\tau \cap \Omega$ and will satisfy (2.19) for every two points x', x'' lying in a subinterval of $\tau \cap \Omega$. Indeed, for every fixed $\underset{\sim}{t} \in R_m$, $g(\cdot,\underset{\sim}{t})$ restricted to $(\tau \cap \Omega) - N$ is uniformly continuous (by (2.19)). Since $\tau \cap N$ is an $\mathcal{L}_1$-null set and hence has a dense complement, it follows (by a standard argument) that $g(\cdot,\underset{\sim}{t})$ may be redefined on $\tau \cap N$ in such a way that it will become uniformly continuous on $\tau \cap \Omega$. Obviously, the function thus modified will have the properties stated above.

Denote the function resulting from this modification of g on all lines τ as above, by g_i. Note that $g_i(x,\underset{\sim}{t}) = g(x,\underset{\sim}{t})$ for all $(x,\underset{\sim}{t}) \in (\Omega - N) \times R_m$. Hence g_i will also satisfy condition (i) of Definition 2.2 as well as inequalities (2.2) and (2.17).

Set $v_i = g_i \circ \underset{\sim}{u}$; clearly $v = v_i$ a.e. in Ω. Let τ be a line parallel to the x_i-axis satisfying conditions (α_1)-(α_3). Let $x^o \in (\tau \cap \Omega) - N$ and let $|h| \neq 0$ be a sufficiently small so that x^o and $x_h^o = x^o + he^i$ belong to one subinterval of $\tau \cap \Omega$. Then by (2.17) and (2.19):

(2.20) $$|\delta_h^i v_i(x^o)| \le \frac{1}{|h|}|g_i(x_h^o,\underset{\sim}{u}(x_h^o)) - g_i(x^o,\underset{\sim}{u}(x_h^o))|$$

$$+ \frac{1}{|h|}|g_i(x^o,\underset{\sim}{u}(x_h^o)) - g_i(x^o,\underset{\sim}{u}(x^o))|$$

$$\le \frac{1}{|h|}\int_o^{|h|} a(x^o + \xi e^i)d\xi + (b\circ \underset{\sim}{u})(x_h^o)$$

$$+ \sum_{k=1}^m [a_k(x^o) + (c_k^{(1)}\circ \underset{\sim}{u})(x_h^o) + (c_k^{(2)}\circ \underset{\sim}{u})(x^o)]|\delta_h^i u_k(x^o)|$$

$$+ \sum_{k=1}^m |\delta_h^i(\beta_k\circ u_k)(x^o)|.$$

Let Ω' be an open subset of Ω such that $\overline{\Omega'} \subset \Omega$ and let $h_o = \text{dist.}(\Omega',\partial\Omega)$. Since $u_k \in W_{1,q_k}(\Omega)$ and $\beta_k\circ u_k \in W_{1,p}(\Omega)$, it follows (by Lemma 1.5) that:

(2.21) $$\begin{cases} \|\delta_h^i u_k\|_{L_{q_k}(\Omega')} \le \|\partial_{x_i} u_k\|_{L_{q_k}(\Omega)}, \\ \|\delta_h^i[\beta_k\circ u_k]\|_{L_p(\Omega)} \le \|\partial_{x_i}[\beta_k\circ u_k]\|_{L_p(\Omega)}, \quad (0 < |h| < h_o). \end{cases}$$

By (2.20) and (2.21):

(2.22) $$\|\delta_h^i v_i\|_{L_p(\Omega')} \le \|a\|_{L_p(\Omega)} + \|b\circ \underset{\sim}{u}\|_{L_p(\Omega)}$$

$$+ \sum_{k=1}^m [\|a_k\|_{L_{q_k'}(\Omega)} + \|c_k\circ \underset{\sim}{u}\|_{L_{q_k'}(\Omega)}]\|\partial_{x_i} u_k\|_{L_{q_k}(\Omega)}$$

$$+ \sum_{k=1}^m \|\partial_{k_i}(\beta_k\circ u_k)\|_{L_p(\Omega)}, \quad (0 < |h| < h_o).$$

Here we used Minkowski's inequality and Hölder's inequality. For reference we shall denote the right side of (22) by K_i. We note that K_i does not depend on Ω' or h, for sufficiently small $|h|$.

Since $v_i \in L_p(\Omega)$, $(1 < p < \infty)$, it follows from (2.22) (by Lemma 1.6) that $\partial_{x_i} v_i \in L_p(\Omega)$ and that:

(2.23) $$\|\partial_{x_i} v_i\|_{L_p(\Omega)} \leq K_i.$$

We remark that this is the only place in our arguments where the assumption $1 < p$ was needed.

Now, $v_i = v$ a.e. in Ω; hence, it follows that $\partial_{x_i} v \in L_p(\Omega)$ and that:

(2.24) $$\|\partial_{x_i} v\|_{L_p(\Omega)} \leq K_i.$$

Finally, since this holds for every i, $(i = 1,\ldots,n)$ we conclude that $v \in W_{1,p}(\Omega)$.

<u>Corollary</u> 2.1. In addition to the assumptions of the theorem, suppose that $b_{i,j}(0)$ is finite for $i,j = 1,\ldots,m$ $(i \neq j)$. Then, without loss of generality we may assume also that $b_{i,j}(0) = 0$ for i,j as above. In this case, v satisfies the following inequality:

$$(2.25)\quad \|v\|_{W_{1,p}(\Omega)} \le \|g(\cdot,\underset{\sim}{0})\|_{L_p(\Omega)} + \|a\|_{L_p(\Omega)} + \|b\circ \underset{\sim}{u}\|_{L_p(\Omega)}$$

$$+ \sum_{k=1}^{m} [\|a_k\|_{L_{q'_k}(\Omega)} + \|c_k \circ \underset{\sim}{u}\|_{L_{q'_k}(\Omega)}]\|u_k\|_{W_{1,q_k}(\Omega)}$$

$$+ \sum_{k=1}^{m} \|\beta_k \circ u_k\|_{W_{1,p}(\Omega)}.$$

This inequality follows immediately from (2.18) with $\underset{\sim}{t}^{o} = (0,\ldots,0)$ and (2.24).

<u>Corollary</u> 2.2. If Ω is any domain in R_n (possibly unbounded), the theorem will still be valid if we make the following additional assumptions:

$$(2.26)\quad \begin{cases} b_{i,j}(0) = 0, & (i,j = 1,\ldots,m; i \neq j), \\ g(\cdot,\underset{\sim}{0}) \in L_p(\Omega), & \\ \beta_i \circ u_i \in L_p(\Omega), & (i = 1,\ldots,n), \end{cases}$$

with β_i as defined in the proof of the theorem.

The assumption that Ω is bounded and has the cone property has been used only in the proof of (2.12), (2.14) and (2.18). Since it is enough, for our purposes, to obtain (2.18) with $\underset{\sim}{t}^{o} = (0,\ldots,0)$, the conditions (2.26) make it possible to dispense with this special assumption on Ω.

Corollary 2.3. Under the assumptions of the theorem we have:

$$(2.27)\quad |\partial_{x_i} v(x)| \le a(x) + (b\circ \underset{\sim}{u})(x) + \sum_{k=1}^{m} [a_k(x) + \sum_{j=1}^{m} (b_{k,j}\circ u_j)(x)]|\partial_{x_i} u_k(x)|,$$

a.e. in Ω, $(i = 1,\ldots,n)$, the products on the right being interpreted as zero whenever $\partial_{x_i} u_k = 0$.

Proof. Let $\tilde{u}_k$, $\tilde{v}$ be functions in $A(\Omega)$ such that $v = \tilde{v}$ and $u_k = \tilde{u}_k$ $(k = 1,\ldots,m)$, a.e. in Ω. Such functions exist by Lemma 1.4. We denote $v^* = g(x,\underset{\sim}{\tilde{u}})$.

By (2.14), $\beta_i\circ \tilde{u}_i \in W_{1,p}(\Omega)$ so that (by Lemma 1.4) $\beta_i\circ\tilde{u}_i\in A'(\Omega)$. But $\beta_i\circ\tilde{u}_i$ is continuous on every segment where $\tilde{u}_i$ is continuous. Therefore $\beta_i\circ\tilde{u}_i\in A(\Omega)$, $(i = 1,\ldots,n)$.

Let τ be a line parallel to the x_i-axis such that τ satisfies conditions (α_1)-(α_3) and in addition:

$$(2.28)\quad \begin{cases} \underset{\sim}{\tilde{u}},\ \tilde{v},\ \beta_k\circ u_k\ (k = 1,\ldots,m) \text{ are l.a.c. on } \tau\cap\Omega; \\ b_{k,j}\circ\tilde{u}_j \text{ restricted to } \tau\cap\Omega \text{ is locally summable} \\ \qquad (k,j = 1,\ldots,m; k\neq j); \\ \tilde{v} = v^* \quad \mathcal{L}_1 \text{ - a.e. on } \tau\cap\Omega. \end{cases}$$

These conditions are satisfied by a.e. line τ parallel to the x_i-axis.

Further, let x^o be a point in $(\tau\cap\Omega) - N$ such that:

$$(2.29)\quad \begin{cases} x^o \text{ is a Lebesgue point for each of the functions } a \\ \quad \text{and } b_{k,j} \circ \tilde{u}_j \ (k,j = 1,\ldots,m; k \neq j) \text{ restricted} \\ \quad \text{to } \tau \cap \Omega; \\ \dfrac{\partial \tilde{v}}{\partial x_i},\ \dfrac{\partial \tilde{\underset{\sim}{u}}}{\partial x_i} \text{ and } \dfrac{\partial(\beta_k \circ \tilde{u}_k)}{\partial x_i} \text{ exist at } x^o; \\ \tilde{v}(x^o) = v^*(x^o). \end{cases}$$

These conditions are satisfied by $\mathfrak{L}_1$ - almost every point on $\tau \cap \Omega$.

Let $\{h_\nu\}_1^\infty$ be a sequence of non-zero numbers such that $h_\nu \to 0$ and such that the points $x^\nu = x^o + h_\nu e^i$ $(\nu = 1,2,\ldots)$ lie in the subinterval of $\tau \cap \Omega$ which contains x^o. Moreover, choose h_ν in such a manner that $x^\nu \notin N$, $\tilde{v}(x^\nu) = v^*(x^\nu)$, $(\nu=1,2,\ldots)$ and finally $b_{k,j} \circ \tilde{u}_j(x^\nu) \to b_{k,j} \circ \tilde{u}_j(x^o)$, $(k,j = 1,\ldots,m; k \neq j)$. This is possible since by condition (2.29) x^o is actually a point of approximate continuity of the functions $b_{k,j} \circ \tilde{u}_j$ restricted to $\tau \cap \Omega$. (For the definition of approximate continuity see Saks [15], p. 132.)

By (2.17) and (2.19) we obtain:

$$|\delta^i_{h_\nu}\tilde{v}(x^o)| = |\delta^i_{h_\nu} v^*(x^o)| \leq \frac{1}{|h|}\int_o^{|h|} a(x^o + \xi e^i)d\xi + (b \circ \tilde{\underset{\sim}{u}})(x^o_h)$$

$$+ \sum_{k=1}^{m} [a_k(x^o) + (c_k^{(1)} \circ \tilde{\underset{\sim}{u}})(x^o_h) + (c_k^{(2)} \circ \tilde{\underset{\sim}{u}})(x^o)]|\delta^i_{h_\nu}\tilde{u}_k(x^o)|$$

$$+ \sum_{k=1}^{m} |\delta^i_{h_\nu}(\beta_k \circ \tilde{u}_k)(x^o)|.$$

Letting $\nu \to \infty$ we get:

$$(2.30) \quad |\frac{\partial \tilde{v}}{\partial x_i}(x^o)| \leq a(x^o) + (b \circ \tilde{\underset{\sim}{u}})(x^o) + \sum_{k=1}^{m} [a_k(x^o) + (c_k \circ \tilde{\underset{\sim}{u}})(x^o)]|\frac{\partial \tilde{u}_k}{\partial x_i}(x^o)| + \sum_{k=1}^{m} |\frac{\partial(\beta_k \circ \tilde{u}_k)}{\partial x_i}(x^o)|.$$

This inequality holds $\mathcal{L}_1$ - a.e. on $\tau \cap \Omega$ for almost every line τ parallel to the x_i-axis. Since both sides of the inequality are measurable functions, it follows (by Fubini's theorem) that the inequality holds a.e. in Ω. Finally we note that by Lemma 1.10 :

$$(2.31) \qquad \frac{\partial(\beta_k \circ \tilde{u}_k)}{\partial x_i} = [b_{k,k} \circ \tilde{u}_k] \cdot \frac{\partial \tilde{u}_k}{\partial x_i}, \quad \text{a.e. in } \Omega,$$

the product being interpreted as zero whenever $\frac{\partial \tilde{u}_k}{\partial x_i} = 0$. Combining (2.30) and (2.31) and taking into account that $\frac{\partial \tilde{u}_k}{\partial x_i} = \partial_{x_i}\tilde{u}_k$ a.e. in Ω, $(k = 1,\dots,m; i = 1,\dots,n)$ we obtain (2.27).

§3. On a Class of Nemitsky Operators (Con't)

In the previous section we considered Nemitsky operators associated with l.a.c. Caratheodory functions, which map an R_m-valued function $\underset{\sim}{u} \in W_{1,\tilde{q}}(\Omega)$ to a function in $W_{1,p}(\Omega)$ with $p > 1$. The case $p = 1$ requires a different treatment. In the present section we deal with this special case.

Theorem 3.1. Let $p = 1$ and $\tilde{q} = (q_1,\ldots,q_m)$ with $q_1 = \ldots = q_m = 1$. Then, under the assumptions of Theorem 2.1 with p and $\tilde{q}$ as above we have $v = g \circ \underset{\sim}{u} \in W_{1,1}(\Omega)$.

Proof. As in the proof of Theorem 2.1, we may and shall assume that g satisfies condition (2.11) and inequality (2.1)', where M is a dense countable subset of R_m. We also use the various notations introduced in the proof of that theorem.

By Lemma 1.4, there exist functions $\tilde{u}_k \in A(\Omega)$ such that

$$\tilde{u}_k = u_k \quad \text{and} \quad \frac{\partial \tilde{u}_k}{\partial x_i} = \partial_{x_i} u_k \quad \text{a.e. in } \Omega, \ (k = 1,\ldots,m; i = 1,\ldots,n).$$

Denote by N_2 the set of points in Ω where at least one of the following relations does not hold:

$$(3.1) \quad \underset{\sim}{u}(x) = \underset{\sim}{\tilde{u}}(x); \quad a_k(x) \le \|a_k\|_{L_\infty(\Omega)}; (b_{k,j} \circ u_j)(x) \le \|b_{k,j} \circ u_j\|_{L_\infty(\Omega)},$$

$(k,j = 1,\ldots,m; k \neq j)$. By (2.5) and (2.9) these relations hold on a subset of Ω of full measure, so that N_2 is a null set.

Let τ be a line parallel to the x_i-axis such that conditions (α_1)-(α_3) (described in the proof of Theorem 2.1) are satisfied and such that:

(α_4) $\tilde{\underset{\sim}{u}}$ and $\beta_k \circ \tilde{u}_k$ $(k = 1,\dots,m)$ are l.a.c. on $\tau \cap \Omega$

(α_5) $N_2 \cap \tau$ is an $\mathcal{L}_1$ - null set.

Clearly, almost every line τ parallel to the x_i-axis satisfies these conditions.

Let then τ be a line as above and let I be a compact subinterval of $\tau \cap \Omega$. Denote $v^* = g \circ \tilde{\underset{\sim}{u}}$ and let $x', x'' \in I - N$. Then by (2.17) and (2.19) we have:

$$(3.2)\quad |v^*(x') - v^*(x'')| \le \left|\int_{x'_i}^{x''_i} a(x)\,dx_i\right| + (b \circ \tilde{\underset{\sim}{u}})(x')\,|x'_i - x''_i|$$
$$+ \sum_{k=1}^{m} [a_k(x'') + (c_k^{(1)} \circ \tilde{\underset{\sim}{u}})(x')$$
$$+ (c_k^{(2)} \circ \tilde{\underset{\sim}{u}})(x'')]\,|\tilde{u}_k(x') - \tilde{u}_k(x'')|$$
$$+ \sum_{k=1}^{m} |(\beta_k \circ \tilde{u}_k)(x') - (\beta_k \circ \tilde{u}_k)(x'')|.$$

In particular, if $x', x'' \in I - (N \cup N_2)$ we have:

$$(3.3)\quad |v^*(x')-v^*(x'')| \leq |\int_{x_i'}^{x_i''} a(x)\,dx_i| + (b\circ\tilde{u})(x')\,|x_i'-x_i''|$$

$$+ \sum_{k=1}^{m} [\|a_k\|_{L_\infty(\Omega)} + \|c_k\circ\underset{\sim}{\tilde{u}}\|_{L_\infty(\Omega)}]\cdot|\tilde{u}_k(x')-\tilde{u}_k(x'')|$$

$$+ \sum_{k=1}^{m} |(\beta_k\circ\tilde{u}_k)(x') - (\beta_k\circ\tilde{u}_k)(x'')|.$$

By (3.3), v^* restricted to $\tau \cap \Omega$, is uniformly continuous on $I - (N \cup N_2)$. Hence, by a standard argument, v^* can be redefined on $I \cap (N \cup N_2)$ in such a manner that the modified function will be continuous in I and will satisfy (3.3) at every point in I.

Modifying v^* in this manner on all the compact subintervals of $\tau \cap \Omega$ and for all lines τ parallel to the x_i-axis as above, we denote the modified function by v_i^*. Note that this modification involves only points x in $N \cup N_2$ so that $v_i^* = v^*$ a.e. in Ω. By (3.3), $v_i^* \epsilon A_i(\Omega)$. Using inequality (3.2) together with the above remarks concerning v_i^* we obtain (as in the proof of Corollary 2.3):

$$(3.4)\quad |\frac{\partial v_i^*}{\partial x_i}(x)| \leq a(x) + (b\circ\underset{\sim}{\tilde{u}})(x) + \sum_{k=1}^{m} [a_k(x)+(c_k\circ\underset{\sim}{\tilde{u}})(x)]\,|\frac{\partial \tilde{u}_k}{\partial x_i}(x)|$$

$$+ \sum_{k=1}^{m} |\frac{\partial(\beta_k\circ\tilde{u}_k)(x)}{\partial x_i}|, \quad \text{a.e. in } \Omega.$$

Hence, $\frac{\partial v_i^*}{\partial x_i} \in L_1(\Omega)$ and:

$$(3.5) \quad \left\|\frac{\partial v_i^*}{\partial x_i}\right\|_{L_1(\Omega)} \leq \|a\|_{L_1(\Omega)} + \|b \circ \underset{\sim}{\tilde{u}}\|_{L_1(\Omega)}$$

$$+ \sum_{k=1}^{m} [\|a_k\|_{L_\infty(\Omega)} + \|c_k \circ \underset{\sim}{\tilde{u}}\|_{L_p(\Omega)}] \left\|\frac{\partial \tilde{u}_k}{\partial x_i}\right\|_{L_1(\Omega)}$$

$$+ \sum_{k=1}^{m} \left\|\frac{\partial(\beta_k \circ \tilde{u}_k)}{\partial x_i}\right\|_{L_1(\Omega)} .$$

As in the proof of Theorem 2.1, $v \in L_1(\Omega)$ and satisfies (2.18) (with $p = 1$, $q_i = 1$, $q_i' = \infty$). Since $v = v_i^*$ a.e. in Ω it follows (by Lemma 1.3) that:

$$(3.6) \qquad \partial_{x_i} v = \frac{\partial v_i^*}{\partial x_i} \quad \text{a.e. in } \Omega,$$

so that $\partial_{x_i} v \in L_1(\Omega)$. This result holds for every i, $(i = 1,\ldots,n)$. Therefore $v \in W_{1,1}(\Omega)$ and the proof is complete.

Corollary 3.1. The statements of Corollaries 2.1, 2.2 and 2.3 are valid also in the case $p = 1$, $\tilde{q} = (1,\ldots,1)$.

The proofs are the same as those of the above mentioned corollaries. Actually inequality (2.27) has already been obtained in the proof of the theorem (see (3.4) and (3.6)).

Corollary 3.2. Let $p = 1$ and $\tilde{q} = (q_1, \ldots, q_m)$ with $q_i \geq 1$. Let Ω, g and $\underset{\sim}{u}$ satisfy the condition of Theorem 2.1 with p and $\tilde{q}$ as above. In addition suppose that:

$$(3.7) \qquad a_k \in L_\infty^{loc}(\Omega), \quad b_{k,j} \circ u_j \in L_\infty^{loc}(\Omega), \quad (k,j = 1,\ldots,m; k \neq j).$$

Then $v = g \circ \underset{\sim}{u} \in W_{1,1}(\Omega)$.

Proof. Since $W_{1,q_k}(\Omega) \subset W_{1,1}(\Omega)$, all the assumptions of the theorem are satisfied in compact subdomains of Ω (by (3.7)). Hence $v \in W_{1,1}^{loc}(\Omega)$.

By Corollary 3.1, inequality 2.17 is valid. Therefore $\partial_{x_i} v \in L_1(\Omega)$, $(i = 1,\ldots,n)$. In fact we have:

$$(3.8) \quad \|\partial_{x_i} v\|_{L_1(\Omega)} \leq \|a\|_{L_1(\Omega)} + \|b \circ \underset{\sim}{u}\|_{L_1(\Omega)} + \sum_{k=1}^{m} [\|a_k\|_{L'_{q_k}(\Omega)} + \|c_k \circ \underset{\sim}{u}\|_{L_{q'_k}(\Omega)}] \|\partial_{x_i} u_k\|_{L_{q_k}(\Omega)} + \sum_{k=1}^{m} \|(b_{k,k} \circ \underset{\sim}{u}) \partial_{x_i} u_k\|_{L_1(\Omega)}, \quad (i = 1,\ldots,n).$$

As in the proof of Theorem 2.1, $v \in L_1(\Omega)$ and satisfies (2.18) (with $p = 1$). Hence $v \in W_{1,1}(\Omega)$.

Remark. Note that the statements of Corollaries 2.1, 2.2, 2.3 are valid also under the assumptions of the above corollary. The validity of Corollary 2.3 in this case follows from its validity under the assumptions of Theorem 3.1 (and has been used in the proof above). For the other two, the proof is the same as before.

The following theorem deals with the same case as Corollary 3.2, but without the additional assumption (3.7).

<u>Theorem</u> 3.2. Let $p = 1$ and $\tilde{q} = (q_1,\ldots,q_m)$ with $q_i \geq 1$. Let Ω, g and $\underset{\sim}{u}$ satisfy the conditions of Theorem 2.1 with p and $\tilde{q}$ as above. In addition suppose that g is continuous in $\Omega \times R_m$. Then $v = g \circ \underset{\sim}{u} \in B'(\Omega)$ and the strong approximate derivatives $\partial'_{x_i} v$, $(i = 1,\ldots,n)$, belong to $L_1(\Omega)$.

<u>Proof</u>. Let $\tilde{u}_k$ be a function in $A(\Omega)$ such that $\tilde{u}_k = u_k$ a.e. in Ω, $(k = 1,\ldots,m)$. Denote $v^* = g \circ \underset{\sim}{\tilde{u}}$; then $v^* = v$ a.e. in Ω.

As in the proof of Theorem 2.1 we have:

(3.9) $\qquad v \in L_1(\Omega)$ and $\beta_k \circ u_k \in W_{1,1}(\Omega)$, $\quad (k = 1,\ldots,m)$.

Furthermore, $\beta_k \circ \tilde{u}_k \in A(\Omega)$.

Let τ be a line parallel to the x_i-axis such that τ satisfies conditions (α_1)-(α_4) (which are stated in the proofs of Theorem 2.1 and Theorem 3.1) and in addition:

$$(3.10)\quad\begin{cases} \tilde{u}_k|_{\tau\cap\Omega} \in W_{1,q_k}(\tau\cap\Omega), & (k=1,\ldots,m); \\ \beta_k \circ \tilde{u}_k|_{\tau\cap\Omega} \in W_{1,1}(\tau\cap\Omega), & (k=1,\ldots,m); \\ a_k|_{\tau\cap\Omega} \in L_{q_k'}(\tau\cap\Omega), & (k=1,\ldots,m); \\ b_{k,j} \circ \tilde{u}_j|_{\tau\cap\Omega} \in L_{q_k'}(\tau\cap\Omega), & (k,j=1,\ldots,m; k\neq j); \\ v^*|_{\tau\cap\Omega} \in L_1(\tau\cap\Omega); & \\ b \circ \tilde{\underset{\sim}{u}}|_{\tau\cap\Omega} \in L_1(\tau\cap\Omega); & \end{cases}$$

We observe that, since $\tilde{u}_k \in W_{1,q_r}(\Omega) \cap A(\Omega)$, the first condition in (3.10) is satisfied by a.e. line τ parallel to the x_i-axis. The same remark applies to the second condition in (3.10). It is clear that also the other conditions in (3.10) as well as (α_1)-(α_4) are satisfied by a.e. line τ as above.

Let I be an open interval contained in $\tau \cap \Omega$ let I' be a compact subinterval of I. Denote by h_o the distance between I' and the boundary of I.

If x', x'' are two points in $I - N$, the difference $|v^*(x') - v^*(x'')|$ may be estimated as in (3.2). In particular, if $h \neq 0$ is a fixed number such that $|h| < h_o$, we have:

$$(3.11)\quad |\delta_h^i v^*(x)| \leq \frac{1}{|h|}\int_o^{|h|} a(x+\xi e^i)d\xi + (b\circ\tilde{u})(x+he^i)$$

$$+ \sum_{k=1}^{m} [a_k(x) + (c_k^{(1)}\circ \underset{\sim}{\tilde{u}})(x)$$

$$+ (c_k^{(2)}\circ \underset{\sim}{\tilde{u}})(x+he^i)]\,|\delta_h^i\tilde{u}_k(x)|$$

$$+ \sum_{k=1}^{m} |\delta_h^i(\beta_k\circ\tilde{u}_k)(x)|,$$

for $\mathcal{L}_1$ - a.e. point $x\in I'$.

Integrating over I' and using the one-dimensional version of Lemma 1.5 we obtain:

$$(3.12)\quad \|\delta_h^i v^*\|_{L_1(I')} \leq \|a\|_{L_1(I)} + \|b\circ\underset{\sim}{\tilde{u}}\|_{L_1(I)}$$

$$+ \sum_{k=1}^{m} [\|a_k\|_{L_{q_k'}(I)} + \|c_k\circ\underset{\sim}{\tilde{u}}\|_{L_{q_k'}(I)}]\,\|\frac{\partial\tilde{u}_k}{\partial x_i}\|_{L_{q_k}(I)}$$

$$+ \sum_{k=1}^{m} \|\frac{\partial(\beta_k\circ\tilde{u}_k)}{\partial x_i}\|_{L_1(I)},$$

for all $0 < |h| < h_o$.

By Lemma 1.7, v^* coincides $\mathcal{L}_1$ - a.e. in I, with a function of bounded variation on I. Since this result holds for every subinterval of $\tau\cap\Omega$, v^* coincides $\mathcal{L}_1$ - a.e. in $\tau\cap\Omega$ with a function v_τ^* which is locally of bounded variation on $\tau\cap\Omega$. Moreover (by Lemma 1.7 and (3.12)):

$$(3.13)\quad \text{tot.var.}_{\tau\cap\Omega}[v^*_\tau] \le \|a\|_{L_1(\tau\cap\Omega)} + \|b\circ \underset{\sim}{\tilde{u}}\|_{L_1(\tau\cap\Omega)}$$

$$+ \sum_{k=1}^{m} [\|a_k\|_{L_{q'_k}(\tau\cap\Omega)} + \|c_k\circ \underset{\sim}{\tilde{u}}\|_{L_{q'_k}(\tau\cap\Omega)}] \|\frac{\partial \tilde{u}_k}{\partial x_i}\|_{L_{q_k}(\tau\cap\Omega)}$$

$$+ \sum_{k=1}^{m} \|\frac{\partial(\beta_k\circ \tilde{u}_k)}{\partial x_i}\|_{L_1(\tau\cap\Omega)}.$$

Up to this point we have not made use of the continuity of g in $\Omega \times R_m$. We observe now, that by this assumption, $v^*|_{\tau\cap\Omega}$ is continuous (because $\tilde{u}|_{\tau\cap\Omega}$ is continuous). Since $v^* = v^*_\tau$ $\mathfrak{L}_1$ - a.e. on $\tau \cap \Omega$, it follows that v^* is also l.b.v. on $\tau \cap \Omega$. (This is easily verified directly from the definition.) Furthermore, $v^*_\tau = v^*|_{\tau\cap\Omega} + s$, where s is a saltus function on $\tau \cap \Omega$, such that $s = 0$ everywhere, except on a countable set of points, on $\tau \cap \Omega$, (see Saks [15], p. 97-98). Clearly, $\frac{ds}{dx_i} = 0$ wherever it exists, so that $\frac{\partial v^*_\tau}{\partial x_i} = \frac{\partial v^*}{\partial x_i}$ $\mathfrak{L}_1$ - a.e. on $\tau \cap \Omega$. Finally, by a well-known property of functions of bounded variation (see Saks [15], pp. 119 and 121):

$$(3.14)\qquad \int_{\tau\cap\Omega} |\frac{\partial v^*}{\partial x_i}|\,dx_i = \int_{\tau\cap\Omega} |\frac{\partial v^*_\tau}{\partial x_i}|\,dx_i \le \text{tot.var.}_{\tau\cap\Omega}[v^*_\tau].$$

Note that these results hold for a.e. line τ parallel to the x_i-axis, $(i = 1,\dots,n)$. Hence, in particular, $v^* \in B(\Omega)$ and the derivatives $\frac{\partial v^*}{\partial x_i}$, $(i = 1,\dots,n)$, exist a.e. in Ω and are measurable there. Combining (3.13) and (3.14) and using Fubini's theorem we get:

$$(3.15)\quad \left\|\frac{\partial v^*}{\partial x_i}\right\|_{L_1(\Omega)} \leq \|a\|_{L_1(\Omega)} + \|b \circ \tilde{\underset{\sim}{u}}\|_{L_1(\Omega)}$$

$$+ \sum_{k=1}^{m} [\|a_k\|_{L_{q_k'}(\Omega)} + \|c_k \circ \tilde{\underset{\sim}{u}}\|_{L_{q_k'}(\Omega)}] \left\|\frac{\partial \tilde{u}_k}{\partial x_i}\right\|_{L_{q_k}(\Omega)}$$

$$+ \sum_{k=1}^{m} \left\|\frac{\partial(\beta_k \circ \tilde{u}_k)}{\partial x_i}\right\|_{L_1(\Omega)}, \quad (i = 1,\ldots,n).$$

Since $v = v^*$ a.e. in Ω, the assertion of the theorem is proved.

For the next result we introduce:

Definition 3.1. Let g be a real function defined in $\Omega \times R_m$ and let w_k be a function in $A(\Omega)$, $(k = 1,\ldots,m)$. We shall say that g has the (N^*) property with respect to $\underset{\sim}{w} = (w_1,\ldots,w_m)$, if for almost every line τ parallel to one of the axes in R_n, the function $g \circ \underset{\sim}{w}$, restricted to $\tau \cap \Omega$, has the (N) property (i.e. it takes $\mathcal{L}_1$ - null subsets of $\tau \cap \Omega$ into null sets).

The following lemma provides sufficient conditions for a function g to have the (N^*) property.

Lemma 3.1. Let g be an l.a.c. Caratheodory function in $\Omega \times R_m$. Suppose that g satisfies conditions I and II of Theorem 2.1. We assume also that the null set $N = N_g$, mentioned in Definition 2.2 (and also in Condition II), satisfies $\mathcal{H}_{n-1}(N) = 0$.

The assumptions on a, b, a_k, $b_{k,j}$ are as follows:

(3.16) $$0 \le a \in L_1^{loc}(\Omega);$$

(3.17) $0 \le b$ is continuous in R_m;

(3.18) a_k, $(k = 1,\ldots,m)$ are non-negative measurable functions, finite everywhere in Ω;

(3.19) $$\begin{cases} b_{k,j},\ (k,j = 1,\ldots,m) \text{ are non-negative Borel functions, finite everywhere in } R_1; \\ 0 \le b_{k,k} \in L_1^{loc}(R_1),\ (k = 1,\ldots,m). \end{cases}$$

Let $w_k \in A(\Omega)$, $(k = 1,\ldots,m)$, and set $\underset{\sim}{w} = (w_1,\ldots,w_m)$. Suppose that:

(3.20) $$(b_{k,k} \circ w_k) \frac{\partial w_k}{\partial x_i} \in L_1^{loc}(\Omega), \quad (k = 1,\ldots,m; i = 1,\ldots,n).$$

Then g has the (N^*) property with respect to $\underset{\sim}{w}$.

Proof. Let M be a countable dense subset of R_m. As in the proof of Theorem 2.1 we obtain inequality (2.1)' for $(x,\underset{\sim}{t}) \in (\Omega - N_1) \times R_m$, where N_1 is a null subset of Ω.

Defining β_k as in Theorem 2.1, we note that by (3.20) and Lemma 1.9, $\beta_k \circ w_k \in A(\Omega)$.

Let τ be a line parallel to the x_i-axis (the index i will be kept fixed throughout the proof) such that:

$$(3.21) \quad \begin{cases} \tau \cap N \text{ is empty;} \\ \tau \cap N_1 \text{ is an } \mathcal{L}_1\text{-null set;} \\ g(\cdot,\underset{\sim}{t}) \text{ is l.a.c. on } \tau \cap \Omega \text{ for every } \underset{\sim}{t} \in M; \\ a|_{\tau\cap\Omega} \in L_1^{loc}(\tau \cap \Omega); \\ w_k \text{ and } \beta_k \circ w_k, \ (k = 1,\ldots,m) \text{ are l.a.c. on } \tau \cap \Omega; \end{cases}$$

We remark that each of the conditions in (3.21) is satisfied by a.e. line τ parallel to the x_i-axis. With respect to the first condition, this follows from the fact that $\mathcal{H}_{n-1}(N) = 0$, since this implies that the projection of N on the hyperplane $x_i = 0$ is an $\mathcal{L}_{n-1}$-null set.

Let I be an open subinterval of $\tau \cap \Omega$ such that $\bar{I} \subset \tau \cap \Omega$. It is sufficient to show that $\mu = g \circ \underset{\sim}{w}$ has the (N) property on I.

By (2.17) and (2.19) we obtain:

$$(3.22)\quad |\mu(x')-\mu(x'')| \le \left|\int_{x_i'}^{x_i''} a(x)\,dx_i\right| + (b \circ \underset{\sim}{w})(x')\,|x_i'-x_i''|$$

$$+ \sum_{k=1}^{m} [a_k(x'') + (c_k^{(1)} \circ \underset{\sim}{w})(x')$$

$$+ (c_k^{(2)} \circ \underset{\sim}{w})(x'')]\,|w_k(x')-w_k(x'')|$$

$$+ \sum_{k=1}^{m} |(\beta_k \circ w_k)(x') - (\beta_k \circ w_k)(x'')|,$$

for every $x', x'' \in I$. (Here we use the fact that I does not intersect N.)

Let Q be an $\mathcal{L}_1$-null subset of I. We have to show that $\mu(Q)$ is a null set.

Let:

$$(3.23)\quad J_\ell = \{x \in I \mid a_k(x) \le \ell,\ (b_{k,j} \circ w_j)(x) \le \ell \quad \text{for} \quad k,j = 1,\dots,m;\ k \ne j\},$$

where $\ell = 1,2,\dots$ Then $I = \bigcup_1^\infty J_\ell$ and we set $Q_\ell = Q \cap J_\ell$.

In the next part of the proof we shall keep ℓ fixed. It is sufficient to prove that $\mu(Q_\ell)$ is a null set.

Given $\varepsilon > 0$, let O be an open subset of I containing Q_ℓ such that $\mathcal{L}_1(O) < \varepsilon$. The set O may be written as a countable union of disjoint open intervals $\{I_\nu\}_{\nu=1}^\infty$. We may assume that every interval I_ν contains at least one point of

Q_ℓ; we pick such a point in I_ν and denote it by x^ν.

Let $x' \in I_\nu \cap Q_\ell$. Then by (3.22) and (3.23):

$$(3.24) \qquad |\mu(x') - \mu(x^\nu)| \le \int_{I_\nu} a(x)\,dx_i + \mathcal{L}_1(I_\nu) \sup_I b \circ \underset{\sim}{w}$$

$$+ c_1 \ell \sum_{k=1}^{m} |w_k(x') - w_k(x^\nu)|$$

$$+ \sum_{k=1}^{m} |\beta_k \circ w_k(x') - \beta_k \circ w_k(x^\nu)|,$$

where c_1 is a constant depending only on m. Note that $b \circ \underset{\sim}{w}$ is continuous on $\tau \cap \Omega$ so that $\sup_I b \circ \underset{\sim}{w} < \infty$.

Denote $s_\nu = \sup_{x' \in I_\nu \cap Q_\ell} |\mu(x') - \mu(x^\nu)|$. Then, by (3.24):

$$(3.25) \qquad \sum_1^\infty s_\nu \le \int_O a(x)\,dx_i + \varepsilon \cdot \sup_I (b \circ \underset{\sim}{w}) + c_1 \ell \sum_{k=1}^{m} \int_O \left|\frac{\partial w_k}{\partial x_i}\right| dx_i$$

$$+ \sum_{k=1}^{m} \int_O \left|\frac{\partial(\beta_k \circ w_k)}{\partial x_i}\right| dx_i .$$

All the integrals in (3.25) are integrals of functions belonging to $L_1(I)$. Hence, they tend to zero when $\varepsilon \to 0$.

Obviously $\mu(Q_\ell)$ is covered by a countable family of closed intervals whose total length is $2 \sum_1^\infty s_\nu$. Hence, the outer measure of $\mu(Q_\ell)$ is bounded by twice the right side of (3.25), which tends to zero when $\varepsilon \to 0$. Hence $\mu(Q_\ell)$ is a null set. This completes the proof of the lemma.

<u>Theorem</u> 3.3. Let $p = 1$ and $\tilde{q} = (q_1,\ldots,q_m)$ with $q_i \geq 1$. Suppose that Ω, g and $\underset{\sim}{u}$ are as in Theorem 2.1 with p and $\tilde{q}$ as above. In addition suppose that g is continuous in $\Omega \times R_m$, that $\mathcal{H}_{n-1}(N) = 0$ (where $N = N_g$) and that the functions a_k, $b_{k,j}$, $(k,j = 1,\ldots,m; k \neq j)$ are finite everywhere in Ω and R_1 respectively.

Then $v = g \circ \underset{\sim}{u} \in W_{1,1}(\Omega)$.

<u>Proof</u>. By Theorem 3.2 $v \in B'(\Omega)$. Let $\tilde{u}_k$ be a function in $A(\Omega)$ such that $\tilde{u}_k = u_k$ a.e. in Ω. If $v^* = g \circ \underset{\sim}{\tilde{u}}$, then $v^* = v$ a.e. in Ω. Furthermore v^* is continuous on $\tau \cap \Omega$, for a.e. line τ parallel to one of the axes in R_n. Therefore, $v^* \in B(\Omega)$, (see Definition 1.1).

By Lemma 3.1, g has the (N^*) property with respect to $\underset{\sim}{\tilde{u}}$. Hence, for a.e. line τ parallel to one of the axes in R_n, $v^*|_{\tau\cap\Omega}$ has the following properties: it is continuous, it has the (N) property and it is locally of bounded variation. Therefore, by a theorem of Banach-Zarecki (see Saks [15], p.227) $v^*|_{\tau\cap\Omega}$ is l.a.c. on $\tau \cap \Omega$. Hence $v^* \in A(\Omega)$.

Again by Theorem 3.2, $\frac{\partial v^*}{\partial x_i} \in L_1(\Omega)$, $(i = 1,\ldots,n)$. Hence, by Lemma 1.4, $v \in W_{1,1}(\Omega)$. This completes the proof of the theorem.

§4. Demicontinuity of $G : W_{1,\tilde{q}}(\Omega) \rightarrow W_{1,p}(\Omega)$

In this section we describe first a general class of conditions on g which ensure that G is a demicontinuous (i.e. "strong $\rightarrow$ weak" continuous) mapping from $W_{1,\tilde{q}}(\Omega)$ to $W_{1,p}(\Omega)$. Then in order to illustrate the scope of our theorem we give a relatively explicit result in which g is assumed to satisfy power type growth conditions. The methods are based on those used in proving Theorem 2.1.

Our main result follows. Recall that a bounded transformation between normed spaces is one which maps bounded sets to bounded sets.

Theorem 4.1. Let Ω be a bounded domain in R_n possessing the cone property and let $p > 1$, $\tilde{q} = (q_1,\ldots,q_m)$, $q_i \geq p$, be given. Suppose that g is an l.a.c. Caratheodory function in $\Omega \times R_m$ which fulfills the estimates of Theorem 2.1 with functions a, b, a_k, $b_{k,j}$ which satisfy (2.3)-(2.7), a_k and $b_{k,j}$ being everywhere finite. Suppose also that $b_{k,k}$ is continuous, $k = 1,\ldots,m$, and that under functional composition the functions b, $b_{k,j}$, $k,j = 1,\ldots,m$, satisfy:

$$(4.1)\quad \begin{cases} b_{k,j} \text{ defines a mapping from } L_{q_j^*}(\Omega) \text{ to } L_{q'_k}(\Omega), \\ q_j^* = \dfrac{nq_j}{n-q_j}\,; \\ b \text{ defines a mapping from } L_{q_1^*}(\Omega) \times\ldots\times L_{q_m^*}(\Omega) \text{ to } L_p(\Omega). \end{cases}$$

Then G maps $W_{1,\tilde{q}}(\Omega)$ into $W_{1,p}(\Omega)$ and is demicontinuous. Moreover G is continuous as a mapping from $W_{1,\tilde{q}}(\Omega)$ to $L_p(\Omega)$.

Remark 4.1. It is implied by (4.1) that the following estimates hold ([7], Theorem 1 (slightly modified)):

$$(4.2)\quad \begin{cases} |b_{k,j}(\sigma)| \le c_{k,j} + d_{k,j}|\sigma|^{q'_k/q_j^*}, \quad k,j = 1,\ldots,m. \\ |b(\sigma_1,\ldots,\sigma_m)| \le c(1 + |\sigma_1|^{q_1^*} +\ldots+ |\sigma_m|^{q_m^*})^{1/p} \end{cases}$$

Thus all the composition mappings in (4.1) are bounded. In addition, the mappings associated with $b_{k,k}$ $(k = 1,\ldots,m)$ and with b are continuous ([9],[11],[13]).

Proof. In order to prove demicontinuity we must show that for any sequence $\underset{\sim}{u}^\nu = (u_1^\nu,\ldots,u_m^\nu)$, $\nu \ge 1$, convergent to a limit $\underset{\sim}{u}^o = (u_1^o,\ldots,u_m^o)$ in $W_{1,\tilde{q}}(\Omega)$, the functions $v^\nu = G\underset{\sim}{u}^\nu$ converge weakly to $v^o = G\underset{\sim}{u}^o$ in $W_{1,p}(\Omega)$.

We show first that the sequence v^ν converges to v^o in $L_p(\Omega)$. According to (2.17) we have for all $x \in \Omega - N$,

$$(4.3) \quad |g(x,\underset{\sim}{u}^\nu(x)) - g(x,\underset{\sim}{u}^o(x))| \leq \sum_{k=1}^{m} [a_k(x) + c_k^{(1)}(\underset{\sim}{u}^\nu(x)) + c_k^{(2)}(\underset{\sim}{u}^o(x))] |u_k^\nu(x) - u_k^o(x)| + \sum_{k=1}^{m} |\beta_k(u_k^\nu(x)) - \beta_k(u_k^o(x))|$$

where the β_k are defined as in (2.13). Now by Sobolev's imbedding theorem, $u_k^\nu \in L_{q_k^*}(\Omega)$, $\nu \geq 0$, and the convergence of $\underset{\sim}{u}^\nu$ to $\underset{\sim}{u}^o$ in $W_{1,\tilde{q}}(\Omega)$ implies that

$$u_k^\nu \to u_k^o \quad \text{in} \quad L_{q_k^*}(\Omega), \qquad k = 1,\ldots,m.$$

Thus (4.1_1) and Corollary 1.2 give

$$(4.4) \qquad \partial_{x_i}(\beta_k \circ u_k^\nu) = (b_{k,k} \circ u_k^\nu)\partial_{x_i} u_k^\nu, \qquad \nu = 0,1,\ldots.$$

Moreover by using Remark 4.1 we also deduce that

$$b_{k,k} \circ u_k^\nu \to b_{k,k} \circ u_k^o \quad \text{in} \quad L_{q_k'}(\Omega), \quad k = 1,\ldots,m.$$

Applying this result to (4.4) we obtain:

$$(4.5)\quad \|\partial_{x_i}(\beta_k \circ u_k^\nu) - \partial_{x_i}(\beta_k \circ u_k^o)\|_{L_p(\Omega)} \le \|b_{k,k} \circ u_k^\nu - b_{k,k} \circ u_k^o\|_{L_{q_k'}(\Omega)} \times$$

$$\times \|\partial_{x_i} u_k^\nu\|_{L_{q_k}(\Omega)}$$

$$+ \|b_{k,k} \circ u_k^o\|_{L_{q_k'}(\Omega)} \|\partial_{x_i} u_k^\nu - \partial_{x_i} u_k^o\|_{L_{q_k}(\Omega)}$$

$$\to 0.$$

Moreover, we obtain from (4.2_1) the following estimate

$$(4.6)\quad |\beta_k(\sigma') - \beta_k(\sigma'')| = \left|\int_{\sigma'}^{\sigma''} b_{k,k}(\sigma)\,d\sigma\right|$$

$$\le [c_k + d_k(|\sigma''|^{q_k^*/q_k'} + |\sigma'|^{q_k^*/q_k'})]\,|\sigma'' - \sigma'|.$$

This ensures that the sequence $\beta_k \circ u_k^\nu$ converges in $L_p(\Omega)$:

$$\|\beta_k \circ u_k^\nu - \beta_k \circ u_k^o\|_{L_p(\Omega)} \le \|c_k + d_k(|u_k^\nu|^{q_k^*/q_k'} + |u_k^o|^{q_k^*/q_k'})\|_{L_{r_k}(\Omega)} \|u_k^\nu - u_k^o\|_{L_{q_k^*}(\Omega)}$$

$$\le \mathcal{L}_n(\Omega)^{1/n}[c_k \mathcal{L}_n(\Omega)^{1/q_k'} +$$

$$+ d_k(\|u_k^\nu\|_{L_{q_k^*}(\Omega)}^{q_k^*/q_k'} + \|u_k^o\|_{L_{q_k^*}(\Omega)}^{q_k^*/q_k'})]\,\|u_k^\nu - u_k^o\|_{L_{q_k^*}(\Omega)}$$

$$\to 0,$$

where $\dfrac{1}{r_k} = \dfrac{1}{p} - \dfrac{1}{q_k^*} = \dfrac{1}{q_k'} - \dfrac{1}{n}$.

Utilizing this fact in conjunction with (4.1_1) we obtain from (4.3) the estimate

$$(4.7)\quad \|v^\nu - v^o\|_{L_p(\Omega)} \le \sum_{k=1}^{m} [\|a_k + c_k^{(1)} \circ \underset{\sim}{u}^\nu + c_k^{(2)} \circ \underset{\sim}{u}^o\|_{L_{q_k'}(\Omega)} \|u_k^\nu - u_k^o\|_{L_{q_k}(\Omega)}]$$

$$+ \sum_{k=1}^{m} \|\beta_k \circ u_k^\nu - \beta_k \circ u_k^o\|_{L_p(\Omega)}$$

$$\longrightarrow 0.$$

In order to complete the proof it suffices, by reflexivity of $L_p(\Omega)$, to show that for each i the functions $\partial_{x_i} v^\nu$ form a bounded set in $L_p(\Omega)$. For it then follows, by a standard argument, that $\partial_{x_i} v^\nu \longrightarrow \partial_{x_i} v^o$ weakly in $L_p(\Omega)$ and hence $v^\nu \longrightarrow v^o$ weakly in $W_{1,p}(\Omega)$. However, by (2.22) we have for any open subset Ω' of Ω such that $\overline{\Omega'} \subset \Omega$ the estimate

$$(4.8)\quad \|\delta_h^i v^\nu\|_{L_p(\Omega')} \le \|a\|_{L_p(\Omega)} + \|b \circ \underset{\sim}{u}^\nu\|_{L_p(\Omega)}$$

$$+ \sum_{k=1}^{m} [\|a_k\|_{L_{q_k'}(\Omega)} + \|c_k \circ \underset{\sim}{u}^\nu\|_{L_{q_k'}(\Omega)}] \|\partial_{x_i} u_k^\nu\|_{L_{q_k}(\Omega)}$$

$$+ \sum_{k=1}^{m} \|\partial_{x_i}(\beta_k \circ u_k^\nu)\|_{L_p(\Omega)}, \quad (0 < |h| < h_o), \quad i=1,\ldots,h.$$

By (4.2) and (4.5) the right side of (4.8) is bounded uniformly in ν, and hence by Lemma 1.6 we have the requisite boundedness for $\|\partial_{x_i} v^{\nu}\|_{L_p(\Omega)}$, $i = 1,\dots,n$, $\nu \geq 0$.

The next result can be regarded as a corollary to Theorem 4.1. As mentioned above it is stated primarily for illustrative purposes.

<u>Theorem</u> 4.2. Let Ω be a bounded domain in R_n possessing the cone property and let $p > 1$, $\tilde{q} = (q_1,\dots,q_m)$ with $q_i \geq p$, be given. Suppose that g is an l.a.c. Caratheodory function in $\Omega \times R_m$ which satisfies the following estimates for certain functions $a \in L_p(\Omega)$, $a_k \in L_{q_k'}(\Omega)$, $k = 1,\dots,m$:

(I) $|\partial'_{x_i} g(x,\underset{\sim}{t})| \leq a(x) + b \sum_{j=1}^{m} |t_j|^{\nu_j}$ a.e. in Ω, where

$\nu_j = q_j/p$, $(i = 1,\dots,n)$;

(II) $\left|\frac{\partial g(x,\underset{\sim}{t})}{\partial t_k}\right| \leq a_k(x) + b_k \sum_{j=1}^{m} |t_j|^{\nu_{k,j}}$ for all $(x,\underset{\sim}{t}) \in (\Omega - N) \times R_m$

at which the left side exists, where $\nu_{k,j} = q_j^*/q_k'$ $(k = 1,\dots,m)$.

Here $N = N_g$ is the null set mentioned in Definition 2.1.

Then G maps $W_{1,\tilde{q}}(\Omega)$ into $W_{1,p}(\Omega)$ and is demicontinuous. Moreover G is continuous as a mapping from $W_{1,\tilde{q}}(\Omega)$ to $L_p(\Omega)$.

§5. Chain Rules

In the present section we describe certain results in which the mapping G has the additional feature that a chain rule holds. That is, one has, for each $i = 1,\dots,n$, equality between the derivative $\partial_{x_i}(g \circ \underset{\sim}{u})$ and the (properly interpreted) combination $(\partial_{x_i} g) \circ \underset{\sim}{u} + [(\nabla_{\underset{\sim}{u}} g) \circ \underset{\sim}{u}]\partial_{x_i}\underset{\sim}{u}$. Associated with the existence of such a chain rule are, as is clear, stronger continuity properties than in previous sections.

Our first result applies to functions g which are independent of x.

Lemma 5.1. Let I be an open interval in R_1. Let $g = g(t_1,\dots,t_m)$ be an l.a.c. Caratheodory function (i.e. g is continuous in R_m and the restriction of g to any line parallel to one of the axes in R_m is an l.a.c. function).

Suppose that the inequality:

$$(5.1) \qquad \left|\frac{\partial g(\underset{\sim}{t})}{\partial t_k}\right| \leq a_k + \sum_{j=1}^{m} b_{k,j}(t_j), \qquad (k = 1,\dots,m),$$

holds at every point in R_m at which the estimated derivative exists, where a_k are constants and the functions $b_{k,j}$ satisfy the conditions described below.

(5.2) $0 \leq b_{k,j}$ is a real valued Borel function on R_1,

$(k,j = 1,\dots,m; k \neq j)$.

(5.3) $$0 \leq b_{k,k} \in L_1^{loc}(R_1), \quad (k = 1,\dots,m).$$

Let $w_k : I \to R_1$ be an l.a.c. function such that $w'_k \in L_q^{loc}(I)$, $(k = 1,\dots,m)$, for some q, $1 \leq q < \infty$. Denote $\underset{\sim}{w} = (w_1,\dots,w_m)$.

Suppose that g has a differential at each point of the set $T_w = \underset{\sim}{w}(I) \subset R_m$, except for an $\mathcal{H}_1$-null set.

Finally suppose that:

(5.4) $$b_{k,j} \circ w_j \in L_{q'}^{loc}(I), \quad (k,j = 1,\dots,m; k \neq j), \text{ where } \frac{1}{q'} + \frac{1}{q} = 1,$$

(5.5) $$(b_{k,k} \circ w_k) w'_k \in L_1^{loc}(I), \quad (k = 1,\dots,m).$$

Then $\mu = g \circ \underset{\sim}{w}$ is l.a.c. on I and the chain rule holds, i.e.

(5.6) $$\mu' = \sum_{k=1}^{m} \left(\frac{\partial g}{\partial t_k} \circ \underset{\sim}{w}\right) w'_k, \qquad \mathcal{L}_1 \text{ - a.e. on } I,$$

the products on the right being interpreted as zero whenever their second factor is zero.

<u>Proof</u>. By Theorem 3.3 (for the case $n = 1$), $\mu \in W_{1,1}^{loc}(I)$. But μ is continuous on I; hence μ is l.a.c. on I. (Recall that every function in $W_{1,1}^{loc}(I)$ coincides a.e. in I with an l.a.c. function.)

Denote by σ a general point in R_1. Let $J = [a,b]$ be a compact subinterval of I. Let s be the arc length of the absolutely continuous curve $\underset{\sim}{w} = \underset{\sim}{w}(\sigma)$, $a \leq \sigma \leq b$, with $s(a) = 0$.

Then s is a monotonic increasing, absolutely continuous function in J and $s' = |\underset{\sim}{w}'|$ a.e. in J. Denote by J^* the interval $0 \leq s \leq s(b)$.

There exists a unique function $\underset{\sim}{w}^* = \underset{\sim}{w}^*(s)$, $(s \in J^*)$, such that $\underset{\sim}{w}^*(s(\sigma)) = \underset{\sim}{w}(\sigma)$, $(\sigma \in J)$. Indeed, if σ_1, σ_2 are two distinct points in J such that $s(\sigma_1) = s(\sigma_2)$, it follows (by monotonicity) that $s(\sigma) =$ const. in the interval between σ_1 and σ_2 and hence $s'(\sigma) = |\underset{\sim}{w}'(\sigma)| = 0$ in this interval. But an absolutely continuous function whose derivative is zero a.e. in an interval is necessarily a constant in that interval. Hence $\underset{\sim}{w}(\sigma_1) = \underset{\sim}{w}(\sigma_2)$.

Furthermore $|\underset{\sim}{w}^*(s_1) - \underset{\sim}{w}^*(s_2)| \leq |s_1 - s_2|$, $\forall\ s_1, s_2 \in J^*$, (see Saks [13], p. 123). Hence $\underset{\sim}{w}^*$ is absolutely continuous and in fact Lipschitz in J^*. Therefore, by Lemma 1.9, we have:

$$(5.7) \qquad \frac{d\underset{\sim}{w}(\sigma)}{d\sigma} = \frac{d\underset{\sim}{w}^*}{ds}(s(\sigma)) \cdot s'(\sigma), \quad \text{a.e. in } J.$$

Noting that by (5.4) $(b_{k,j} \circ w_j)|\underset{\sim}{w}'(\sigma)| \in L_1(J)$, we have (by a known theorem on change of variables; see for instance Federer [5], p. 245):

$$(5.8) \qquad \int_0^{s(b)} (b_{k,j} \circ w_j^*)(s)\,ds = \int_a^b (b_{k,j} \circ w_j)(\sigma)\, s'(\sigma)\, d\sigma,$$

so that $b_{k,j} \circ w_j^* \in L_1(J^*)$.

By Lemma 1.8, $w_k' = [\frac{dw_k^*}{ds} \circ s]s'$, $\mathcal{L}_1$ - a.e. in J. Hence, by (5.5), $(b_{k,k} \circ w_k)[\frac{dw_k^*}{ds} \circ s]s' \in L_1(J)$. Therefore, by the same theorem on change of variables we have:

$$(5.9) \quad \int_0^{s(b)} (b_{k,k} \circ w_k^*)(s) \frac{dw_k^*(s)}{ds} ds = \int_a^b (b_{k,k} \circ w_k)(\sigma) \frac{dw_k^*(s(\sigma))}{ds} s'(\sigma) d\sigma$$

so that $(b_{k,k} \circ w_k^*) \frac{dw_k^*}{ds} \in L_1(J)$.

Finally w_k^* is Lipschitz in J and $|\frac{dw_k^*}{ds}| \leq 1$ $\mathcal{L}_1$- a.e. in J, so that $w_k^* \in W_{1,1}(J)$, $(k = 1,\ldots,m)$.

Hence g and $\underset{\sim}{w}^*$ satisfy all the assumptions of Theorem 3.3 in J so that $g \circ \underset{\sim}{w}^* = \mu^*$ is l.a.c. in J^*.

Now, if σ_o is a point in J such that $\frac{dw}{d\sigma}$ exists at σ_o and the differential of g exists at $\underset{\sim}{y} = \underset{\sim}{w}(\sigma_o)$, then (5.6) holds, as can be verified by an elementary computation. Let M be the set of points in J where at least one of these conditions does not hold. Then M consists of a null set, plus a set M' such that $\underset{\sim}{w}(M')$ is an $\mathcal{H}_1$-null set. Hence, by Lemma 1.8, $\frac{d\underset{\sim}{w}}{d\sigma} = 0$ $\mathcal{L}_1$ - a.e. in M. It follows that $s' = 0$ $\mathcal{L}_1$ - a.e. in M, so that $s(M)$ is a null set (Saks [15], p. 227). This implies that $\mu(M) = \mu^*(s(M))$ is a null set. Here we use the fact that μ^* is l.a.c. on J^*. Appealing once more to Lemma 1.8, and using the fact that μ is a.c. in J, we conclude that $\frac{d\mu}{d\sigma} = 0$ $\mathcal{L}_1$ - a.e. in M. Therefore (5.6) holds $\mathcal{L}_1$ - a.e. in M and everywhere in J-M. Since J was an arbitrary compact subinterval of I, the proof is completed.

Theorem 5.1. Let Ω be a domain in R_n. Let $g = g(t_1,\ldots,t_m)$ be an l.a.c. Caratheodory function which satisfies all the assumptions of Lemma 5.1.

Let $u_k \in W^{loc}_{1,q}(\Omega)$, for some q, $1 \leq q < \infty$, $(k = 1,\ldots,m)$. Denote $\underset{\sim}{u} = (u_1,\ldots,u_m)$.

Letting S_g denote the set of points in R_m where g does not possess a differential, suppose that S_g intersects every absolutely continuous curve in R_m, on an $\mathcal{H}_1$-null set.

Finally suppose that:

$$(5.10) \qquad b_{k,j} \circ u_j \in L^{loc}_{q'}(\Omega), \quad (k,j = 1,\ldots,m; k \neq j), \text{ where } \frac{1}{q'} + \frac{1}{q} = 1,$$

$$(5.11) \qquad (b_{k,k} \circ u_k) \frac{\partial u_k}{\partial x_i} \in L^{loc}_1(\Omega), \quad (k = 1,\ldots,m).$$

Then $v = g \circ \underset{\sim}{u}$ is in $W^{loc}_{1,1}(\Omega)$ and the chain rule holds:

$$(5.12) \qquad \partial_{x_i} v = \sum_{k=1}^{m} \left(\frac{\partial g}{\partial t_k} \circ \underset{\sim}{u}\right) \partial_{x_i} u_k, \qquad \text{a.e. in } \Omega,$$

the products on the right being interpreted as zero whenever their second factor is zero.

Proof. By Theorem 3.3, $v \in W^{loc}_{1,1}(\Omega)$. If $\tilde{u}_k$ is a function in $A(\Omega)$ such that $\tilde{u}_k = u_k$ a.e. in Ω then $v^* = g \circ \underset{\sim}{\tilde{u}}$ is continuous on $\tau \cap \Omega$ for a.e. line τ parallel to one of the axes in R_n. Furthermore $v^* = v$ a.e. in Ω. Therefore, by Lemma 1.4, $v^* \in A(\Omega)$.

Let τ be a line parallel to the x_i-axis such that $\underset{\sim}{\tilde{u}}$ is l.a.c. on $\tau \cap \Omega$ and such that:

$$(5.13)\quad \begin{cases} \left.\dfrac{\partial \tilde{u}_k}{\partial x_i}\right|_{\tau\cap\Omega} \in L_q^{loc}(\tau \cap \Omega), & (k=1,\dots,m); \\ b_{k,j} \circ \tilde{u}_j\big|_{\tau\cap\Omega} \in L_{q'}^{loc}(\tau \cap \Omega), & (k,j=1,\dots,m; k\neq j); \\ \left.\left[(b_{k,k} \circ \tilde{u}_k)\dfrac{\partial \tilde{u}_k}{\partial x_i}\right]\right|_{\tau\cap\Omega} \in L_1^{loc}(\tau \cap \Omega), & (k=1,\dots,m). \end{cases}$$

Since $\dfrac{\partial \tilde{u}_k}{\partial x_i} = \partial_{x_i} u_k$ a.e. in Ω, $(k = 1,\dots,m)$, it is clear that these conditions are satisfied by a.e. line τ parallel to the x_i-axis.

By Lemma 1.5, v^* is l.a.c. on $\tau \cap \Omega$ and:

$$(5.14)\qquad \frac{\partial v^*}{\partial x_i} = \sum_{k=1}^{m} \left(\frac{\partial g}{\partial t_k} \circ \underset{\sim}{\tilde{u}}\right) \frac{\partial \tilde{u}_k}{\partial x_i}\,, \qquad \mathcal{L}_1 \text{ - a.e. on } \tau \cap \Omega.$$

As g is continuous on every line parallel to one of the axes in R_m, g is a Borel function (see Caratheodory [4]). Hence $\dfrac{\partial g}{\partial t_k}$ is a Borel function (Marcus and Mizel [12], Lemma 4.1). Therefore, the right side of (5.14) is a measurable function in Ω, and of course, the same is true of the left side of (5.14) (by Lemma 1.2). Since (5.14) holds $\mathcal{L}_1$ - a.e. on $\tau \cap \Omega$ for a.e. line τ parallel to the x_i-axis, it follows that it holds a.e. in Ω.

Finally, since $\frac{\partial v^*}{\partial x_i} = \partial_{x_i} v$ a.e. in Ω (Lemma 1.4), (5.12) follows from (5.14). This completes the proof of the theorem.

Remark. Other cases in which the chain rule holds, for composition of functions of the form discussed above, are presented in Marcus and Mizel [12]. As noted in the Introduction, the methods used there are quite different from those of the present paper. We mention in particular that if g depends not only on $\underset{\sim}{t}$ but also on x and if it is a locally Lipschitz function in $\Omega \times R_m$, then under a hypothesis on S_g as above, the chain rule holds for $v = g \circ \underset{\sim}{u}$ with $u_k \in W_{1,1}^{loc}(\Omega)$, $(k = 1,\ldots,m)$, ([12], Theorem 2.1).

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