ISOPERIMETRIC INEQUALITY FOR SOME EIGENVALUES OF AN INHOMOGENEOUS, FREE MEMBRANE

by

Catherine Bandle

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Abstract

Szegö's inequality concerning the second eigenvalue of a homogeneous, free membrane is extended to the case of an inhomogeneous free membrane. With the help of a variational principle and the conformal mapping technic upper bounds are constructed for the sum VjU_2 + V j ^ > where ^ and ^ denote the second and third eigenvalue. These bounds only depend on the total mass of the domain and on a simple expression involving the mass distribution and its logarithm.

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1. This paper is concerned with the eigenvalue problem

(A) $$A_z cp+ppcp = O \quad \text{in } G \quad [A_z = \frac{\partial^2}{\partial x^2} + \frac{a^2}{\partial y}j\, J$$

$$IS - O \quad \text{on } \Gamma ,$$

where G is a simply connected domain in the z-plane (z=x+iy), T is its boundary and n denotes the outer normal. p(z) > O is the mass distribution. We assume that there exists a countable set of eigenvalues 0 = /i, < fi2 <. ^3 <. ...

Szegö [5] proved for membranes with constant p the isoperimetric inequality

n $$JL. + i- N \frac{2M}{\pi p_1}$$
$$\mu_2 \quad \mu_3$$

where M = JJ p dxdy ±s^ the total mass, $J_1(r)$ denotes the
G
Bessel function of order 1 and p,. = 1.8412... is the smallest value such that $J_n'(p_.) = 0$. Equality holds if and only if G is a circle. In his proof, Szegö uses conformal transplantation. Weinberger [61 extended this theorem to an N-dimensional not

necessarily simply connected membrane with constant mass distribution. This result can be stated as follows:

For all N-dimensional domains G .of given total mass, the N-dimensional sphere yields the maximum of JLU •

The aim of this paper is to generalize the inequality of SzegS to the case of an inhomogeneous membrane.

2. Problem (A) is equivalent to the eigenvalue problem Lcp+ucp=O in G, -f^ = 0 on T, where L = A^/P is the Beltrami operator of an abstract surfact T given by G and the line element ds^2 = p(z)|dz|^2 . Here, v =VP~ n is tne unit normal with respect to the metric of £. By the Theorema Egregium it follows that the Gaussian curvature is K^G - (-Alnp)/2p. We shall assume, that the inequality (- A In p)/2p <. K holds in G U T and that 2v - K M > 0.

In addition to problem (A) we consider in the w-plane (w = u + iv) the auxiliary problem

(B) $\Delta_w \hat{\varphi} + \mu g(w)\hat{\varphi} = 0$ $\hat{G} = \{w; |w| < \hat{R}\}$

$\frac{\partial \hat{\varphi}}{\partial \nu} = 0$ on $\hat{T} = \{w; |w| = \hat{R}\}$

$$\text{where } g(w) = \begin{cases} \dfrac{4c^2}{(1 \pm r^2)^2} & \text{if } K_o = \pm c^{-2} \\ 1 & \text{if } K_o = 0 \end{cases}$$

[r, θ polar coordinates of the w-plane] and $\hat{\nu} = \sqrt{g^{-1}}\, n$ is the unit normal with respect to the metric $d\hat{s}^2 = g(w)|dw|^2$. $\hat{R}$ is determined such that $\iint_{\hat{G}} g(w)\, du\, dv = \iint_{G} p(z)\, dx\, dy = M$. An elementary calculation yields

$$\hat{R}^2 = \begin{cases} \dfrac{M}{4\pi c^2 \mp M} & \text{if } K_o = \pm c^{-2} \\ \dfrac{M}{\pi} & \text{if } K_o = 0 \end{cases}$$

Because of our assumptions, we have in the case $K_o \neq 0$ $\hat{R}^2 \leq 1$. $\hat{G}$ with the metric $d\hat{s}^2$ can be interpreted as a geodesic circle on a surface of constant Gaussian curvature K_o. The eigenfunctions of problem (B) have the form

(2) $$\hat{\varphi}_k(r,\theta) = R_m(\hat{\mu}_k ; r) \begin{cases} \sin m\theta \\ \cos m\theta \end{cases} \qquad m = 1, 2, \ldots,$$

or

(3) $\phi_k(r, \theta) = R_0(\mu_k ; r)$

$R_m(\mu_k ; r)$ is an eigenfunction of

$$(rR')' - \frac{m^2}{r} R + \mu r g(w) R = 0 \quad \text{in } (0, \hat{R})$$

(4)

$$R'(\hat{R}) = 0, \qquad R(0) < \infty$$

PROPOSITION. The second eigenvalue μ_2 of problem (B) is degenerated; the corresponding eigenfunctions are $\phi_2(r, \theta) = R_1(\mu_2; r) \sin\theta$ and $\hat{\phi}_2(r, \theta) = R_1(\mu_2; r) \cos\theta$.

Proof. We have to show that the first eigenvalue $\mu_1^{(1)}$ of (4) corresponding to $m = 1$ is smaller than the second eigenvalue $\mu_2^{(0)}$ of (4) with $m = 0$. If $K_0 = 0$, then $\mu_1^{(1)} = (1.85 \ldots / R)^2$ and $\mu_2^{(0)} = (3.8 \ldots / R)^2$. Let $K_0 = \pm c^{-2}$ be different from zero. We introduce the new variable $z = \frac{r^2 - 1}{1 + r^2}$ if $K_0 > 0$, or $z = \frac{r^2 + 1}{1 - r^2}$ if $K_0 < 0$. The interval $[0, \hat{R}]$ is then transformed into $[\mp 1, b_\pm]$ where $b_\pm = \frac{\hat{R}^2 \mp 1}{1 \pm \hat{R}^2}$, and the differential equation takes the form

$$(5) \qquad \pm\{(1-z^2)\tilde{R}'\}' - \frac{m^2}{1-z^2}\tilde{R} + \hat{\mu} c^2 \tilde{R} = 0$$

with the boundary conditions

$$\tilde{R}(\mp 1) < \infty , \qquad \tilde{R}'(b_\pm) = 0 .$$

If $m = 1$, then the solutions of (5) can be written as $\tilde{R}(z) = \sqrt{\pm(1-z^2)}\, p'(z)$ [2] where $p(z)$ is a solution of the Legendre equation

$$(6) \qquad \pm\{(1-z^2)p'\}' + \hat{\mu}^{(1)} c^2 p = 0 \quad \text{in} \quad (\mp 1, b_\pm)$$

with the boundary conditions

$$p(\mp 1) < \infty , \qquad p'(b_\pm) = \pm(\hat{\mu}^{(1)} c^2 b_\pm)\, p(b_\pm)$$

Assume that $p'(z) < 0$ in $(\mp 1, b_\pm)$. Because of the boundary condition at $b_\pm$ we have $p(b_\pm) > 0$, and thus, $p(z) > 0$ in the interval $(\mp 1, b_\pm)$. According to the definition, $\hat{\mu}_2^{(0)}$ is the second eigenvalue of $\pm\{(1-z^2)p'\}' + \hat{\mu}^{(0)} c^2 p = 0$ in $(\mp 1, b_\pm)$ with $p(\mp 1) < \infty$ and $p'(b_\pm) = 0$. The corresponding eigenfunction vanishes at some point in $(\mp 1, b_\pm)$. By the Sturm Comparison Theorem it follows that $\hat{\mu}_1^{(1)} < \hat{\mu}_2^{(0)}$.

3. THEOREM, if (-£ In p)/2p ≤ K_o i£ G U T and if $2\pi - K_0 M$ £ O, <u>then the following inequality holds between the eigenvalues of the problems</u> (A) <u>and</u> (B)

$$\frac{1}{\mu_2} + \frac{1}{\mu_3} \geq \frac{1}{\hat{\mu}_2} + \frac{1}{\hat{\mu}_3} \tag{7}$$

<u>Proof</u>. The proof is based on the variational principle

$$\frac{1}{\mu_2} + \frac{1}{\mu_3} = \max_{v_2, v_3} \sum_{i=1}^{2} \{R[v_i]\}^{-1} \tag{8}$$

$$R[f] = \frac{D(f)}{\iint_G f^2 p \, dxdy} \qquad [D(f) = \iint_G \operatorname{grad}^2 f \, dxdy]$$

is the Rayleigh quotient. v_2 and v_3 range over all piecewise continuously differentiable functions with

$$D(v_2, v_3) = \iint_G \operatorname{grad} v_2 \operatorname{grad} v_3 \, dxdy = 0$$

and

$$\iint_G v_i \, p \, dxdy = 0 \qquad i = 2,3 .$$

Let $z(w)$ be a conformal mapping from G onto $\hat{G}$, and let $U_2(z) = \hat{\varphi}_2(w(z))$ and $U_3(z) = \hat{\varphi}_3(w(z))$ [$w(z)$ inverse function of $z(w)$] be the transplanted eigenfunctions. In [5] it is shown, that there always exists a function $z(w)$ such that $\iint_G U_i \, p \, dxdy = 0$ for $i = 2,3$ (1). Because of the invariance of the Dirichlet integral under conformal mapping, we have $D_G(U_2,U_3) = D_{\hat G}(\hat\varphi_2,\hat\varphi_3) = 0$. Therefore, U_2 and U_3 are admissible functions for the variational characterization (8). Since $D_G(U_2) = D_G(U_3)$, we conclude from (8) that

$$(9) \qquad \frac{1}{j_2^2}+\frac{1}{j_3^2} \geq \frac{\iint_G \left(\sum_{i=2}^{3} U_i^2\right) p \, dxdy}{W} = \frac{\int_0^{2\pi}\int_0^{\hat R} R_1^2(M_2;r)\, p(z(w)) \left|\frac{dz}{dw}\right|^2 r \, dr\, d\theta}{D_{\hat G}(\hat\varphi_2)}$$

We shall use the notations $\frac{d}{dr} m(r) = \int_0^{2\pi} p(z(w)) \left|\frac{dz}{dw}\right|^2 r \, d\theta$,

$\hat\Gamma_t = \{w; |w| = t\}$, $\hat G_t = \{w; |w| < t\}$, $\Gamma_t = z(\hat\Gamma_t)$ and $G_t = z(\hat G_t)$. $m(t) = \int_0^t \frac{d}{dr} m(r)dr$ is the total mass of G_t.

Integration by part yields

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(1) In [5] the existence of such a function $z(w)$ has only been proved for $p = 1$, but it is easy to see that the proof remains the same if p is an arbitrary positive and continuous function.

$$(10) \qquad \int_0^{\hat R} R_1^2 \frac{d}{dr} m(r)\,dr = R_1^2\, m(r) \Big|_0^{\hat R} - 2\int_0^{\hat R} R_1 \left(\frac{d}{dr} R_1\right) m(r)\,dr$$

$$= R_1^2(\hat R) M - 2\int_0^{\hat R} R_1 \left(\frac{d}{dr} R_1\right) m(r)\,dr$$

The next step will be to estimate $m(r)$. Consider the modul $\mu(\Gamma, \Gamma_t) = \{D(h)\}^{-1}$. h is the solution of the Dirichlet problem $\Delta h = 0$ in $G \setminus G_\tau$, $h = 0$ on Γ and $h = 1$ on Γ_t. The modul is invariant under conformal transformation, therefore

$$\mu(\Gamma, \Gamma_t) = \mu(\hat\Gamma, \hat\Gamma_t) = \hat\mu_t \text{ in } \hat G .$$

$$\hat m(t) = \int_0^{2\pi}\int_0^t g(w) r\, dr\, d\theta = \begin{cases} 4\pi c^2 t^2 / (1 + t^2) & \text{if } K = + c^{-2} \\ \pi t^2 & \text{if } K_0 = 0 \end{cases}$$

denotes the total mass of $\hat G^\tau = \{w;\ |w| < t\}$ with respect to the mass distribution of the problem (B). By Corollary 2 [1] we have

$$(11) \qquad \mu(\Gamma, \Gamma_t) = \frac{1}{4\pi}\left\{ \ln \frac{M}{4\pi c^2 \pm M} - \ln \frac{\hat m(t)}{4\pi c^2 \pm \hat m(t)} \right\}$$

$$\leq \frac{1}{4\pi}\left\{ \ln \frac{M}{4\pi c^2 \pm M} - \ln \frac{m(t)}{4\pi c^2 \pm m(t)} \right\}$$

if $K_0 = \pm c^{-2}$

or else

(12) $$Mr, r_t > -fcmJ \frac{\Lambda}{4\pi} \ln f_{(t)}$$ (1)

Therefore we obtain the estimation

(13) $$m(t) \leq \hat{m}(t) \quad \text{for all } t .$$

If $r \in [0,\hat{R}]$, then R-jJr) ^ $R_1(r)$ ^ 0. This statement is equivalent with $R_x(z)$ $R'_1(z)$ ^0 in I = (Tl^b + J (cf. sec. 1(5)). It follows immediately from the next result.

LEMMA. *Let* f(z) *be the first eigenfunction of the eigen-value problem* {σ(z) f'(z)}' + (λ - e(z)) f(z) = 0 *in* (a,b), f(a) = 0 *and* f(b) = 0. *If* σ(z) > 0 *and if* e(z) *is a non-increasing function, then we have* f(z)f'(z) ≤ 0 *in* (a,b).

This lemma will be proved by contradiction. Since f(z) is the first eigenfunction, it has constant sign in (a,b). We may assume that f(z) > 0. Suppose that f'(z) < 0 in some interval. Because of the boundary conditions there exists a

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(1) (11) and (12) are generalizations of a theorem by T. Carleman, Math. Z. 1 (1918), pp. 208-212 for the capacity of a condenser. They hold only under the assumptions (-A ln p)/2p ≤ K_o and (4π - K_oM) > 0.

point $x_o \in (a,b]$ such that $f'(x_o) = 0$ and $f''(x_o) \geq 0$. By multiplying the differential equation with $f(z)$ and integrating,

we obtain $$-\int_a^{x_o} a(z) f'^2(z)dz + \int_a^{x_o} (\lambda - e(z)) f^2(z)dz = 0.$$

From this relation it follows that $\lambda_1 > \inf_{z \in (a,x_o)} e(z)$ and because of the monotonicity of $e(z)$

$$\lambda_1 > e(x_o) .$$

At x_o, $f(z)$ satisfies $a(x_o)f''(x_o) + (\lambda - e(x_o))f(x_o) = 0$.

Since $a(x_o)$ and $(\lambda - e(x_o))$ are positive, $f''(z)$ and $f(z)$ must vanish at x_o. By the uniqueness theorem the only solution for which $f(x_o) = 0$ and $f'(x_o) = 0$ is $f(z) = 0$. But this is no eigenfunction of the eigenvalue problem.

(13) and the monotonicity of $R_1(r)$ together with (10) yield

$$\int_0^{\hat{R}} R_1^2(r)\ f_2(r)dr \geq \int_0^{\hat{R}} R_1^2(r)\ \hat{f}_2(r)dr$$

and hence by (9)

$$\sum_{i=2}^{3} \mu_i^{-2} \geq \frac{\int_0^{\hat{R}} R_1^2(r) f(r) dr}{D_2(\phi_2)} \geq \frac{\int_0^{\hat{R}} R_1^2(r)\ \hat{f}(r)dr}{D_2(\hat{\phi}_2)} = \sum_{i=2} \hat{\mu}_i^{-1} .$$

4. The following corollaries are some immediate consequences of the theorem in the previous section.

COROLLARY 1. Consider the eigenvalues $\hat{\mu}_j(K_0)$ of problem (B) as a function of the Gaussian curvature K_0. Suppose that the total mass M is fixed. If $2\pi - K_0 M > 0$, then $\hat{\mu}_2^{-1}(K_0) + \hat{\mu}_3^{-1}(K_0)$ is a monoton decreasing function of K_0.

COROLLARY 2. If $2\pi - K_0 M \geq 0$, then

$$\frac{1}{\mu_2} + \frac{1}{\mu_3} \geq \frac{M}{2\pi} \tag{14}$$

Proof; From the theorem in section 3 and Corollary 1 we have $\frac{1}{\mu_2} + \frac{1}{\mu_3} \geq \frac{1}{\hat{\mu}_2(K_0)} + \frac{1}{\hat{\mu}_3(K_0)} \geq \frac{1}{\hat{\mu}_2(2\pi/M)} + \frac{1}{\hat{\mu}_3(2\pi/M)} = \frac{M}{2\pi}$. $\hat{\mu}_j(2\pi/M)$ correspond to the eigenvalues of the half-sphere with the radius $\sqrt{\frac{M}{2\pi}}$.

REMARK: Corollary 2 can also be obtained from the inequality $\frac{1}{\lambda_1} + \frac{1}{\mu_2} + \frac{1}{\mu_3} \geq \frac{3M}{4\pi}$ [3] where λ_1 is the first eigenvalue of the membrane $\Delta u + \lambda p u = 0$ in G, $u = 0$ on Γ. This result together with the inequality $\mu_2 \geq \lambda_1$ [1] leads to Corollary 2.

References

1. Bandle, C.,"Konstruktion isoperimetrischer Ungleichungen der mathematischen Physik aus solchen der Geometrie",(To appear in Comm. Math. Helv.)

2. Courant, R., and D. Hilbert, Methods of Mathematical Physics, Vol. 1, New York.

3. Hersch, J., "Quatre propriétés isopérimétriques de membranes sphériques homogènes",C. R. Acad. Sc. Paris, t. 270 (1970) pp. 1645-1648.

4. Ince, *K.*, Ordinary Differential Equations, Dover, 1956.

5. Szegő, G.,"Inequalities for Certain Eigenvalues of a Membrane of Given Area", J. Rat. Mech. Anal., 2 (1954), pp. 45-54.

6. Weinberger, H. F., "An Isoperimetric Inequality for the N-Dimensional Free Membrane Problem", J. Rat. Mech. Anal. 5, (1956) pp. 633-636.