A MATHEMATICAL THEORY

OF LATERAL SENSORY INHIBITION

by

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Report 71-27

February 1971

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February 1971

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To be published in the Archive for Rational Mechanics and Analysis.

Introduction

It is a fundamental tenet of neurophysiology that sensory receptors of a given type do not operate independently; the ultimate response associated with a point is affected by the stimulation of other, nearby, points.

For example, in vision the magnitude of the signal reported to the central nervous system by the photoreceptors at a point is not completely determined (even in a steady state) by the intensity of the light received at that point, but is instead influenced by the signals from other points. The psychophysical experiments of Ernst Mach [1865, 1] [1866, 1] [1868, 1][#] suggested to him that this is true in human vision.

[#] English translations of these and related papers of Mach are given in the treatise of Ratliff [1965, 2].

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Employing the methods of modern electrophysiology,[##] Hartline [1949, 1]

[##] Cf. Hartline & Graham [1932, 1], and Hartline, Wagner, & MacNichol [1952, 1].

demonstrated the phenomenon for the ommatidia of the compound eye of the horseshoe crab *Limulus*, and Hartline & Ratliff [1954, 1] [1957, 3] [1958, 2][###]

[###] See also Hartline, Wagner, & Ratliff [1956, 1], Ratliff & Hartline [1959, 5], Kirschfeld & Reichardt [1964, 3], and Ratliff [1965, 2].

gave a quantitative description of the steady state interactions of these ommatidia. The individual ganglion cells of vertebrate retinae often exhibit a specialized sensitivity to temporal changes in illumination.[#]

[#] Cf. Hartline [1938, 1]. For a detailed psychophysical investigation of temporal properties of human vision, with the emphasis on spatially uniform fields, see Kelly [1961, 2] [1969, 1].

These transient responses can be attributed to the integration of lateral excitatory and inhibitory influences,[##] and detailed electrophysiological

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[##] The experiments of Ratliff, Hartline, & Miller [1963, 1] on <u>Limulus</u> illustrate that lateral inhibition can cause pronounced sensitivity to the onset of illumination, even for invertebrate compound eyes. See also Ratliff, Knight, Toyoda, & Hartline [1967, 1] and the discussion of Cornsweet [1970, 1, Chapter 14].

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measurements have been made of such lateral interactions for the retinae of goldfish,[###] frogs,[####] cats,[#####] and many other vertebrates. In

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[###] Cf. Wagner, MacNichol, & Wolbarsht [1963, 2].

[####] Cf. Hartline [1939, 1], Barlow [1953, 1], Lettvin, Maturana, McCulloch, & Pitts [1959, 4], and Lipetz [1961, 3] [1962, 1].

[#####] Cf. Kuffler [1953, 2], Barlow, FitzHugh, & Kuffler [1957, 1], and Baumgartner [1961, 1].

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mammals the retinal ganglion cells are third-order neurons. A typical ganglion cell appears to receive signals from many receptors; some of these signals may, perhaps, be inhibitory. This so-called receptive field of a ganglion cell is usually large, and there is considerable overlap of neighboring fields. Furthermore, several types of horizontal cells and amacrine cells appear to help spread the influence of receptors over large lateral areas. Of course, for most mammals, including the primates, the eyes are in continual motion, even when the animal attempts steady fixation. Thus, when a human observer views a temporally constant, but spatially non-uniform, field of illumination, although he "sees" the field as temporally constant, the individual ganglion cells which form the observer's optic nerves are not firing at a constant frequency, but are instead exhibiting a transient behavior. In the presence of time-dependent "self-inhibition", retinal image motion can give rise to an apparent lateral interaction; for, in a spatially non-uniform field of illumination, the average degree of adaption to light for an individual ganglion cell is not determined by the intensity of light at one point in the visual field, but is instead influenced by the intensity distribution in the region ℓ swept across the receptive field of that ganglion cell. In an apparent steady state, ℓ could be large even if the receptive field of the ganglion cell contained but one photoreceptor. It is clear from this that observed lateral interactions in steady state psychophysical experiments, of the type performed by Mach, result from combined effects of retinal motion and the anatomical overlap (and/or horizontal interconnection) of the receptive fields of the ganglion cells.

In the sense of hearing in mammals, as Békésy [1928, 1]# pointed out, mutual interaction of nerve fibers serving adjacent regions of the basilar membrane of the cochlea can account for the sharpness of pitch discrimination in the presence of a high degree of damping.##

#See also [1955, 1] [1959, 1].

##Inhibitory interaction in auditory pathways has been observed by Galambos & Davies [1944, 1]. See also Katsuki, Watanabe, & Suga [1959, 2].

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Lateral interactions of the nerves carrying impulses from the sensory cells of the skin play a role in the localization of the sense of touch.### There is also some evidence that lateral interaction of receptors of fixed type may occur in the senses of smell and taste#### and enhance the ability of an animal to use chemoreceptors to determine the direction of the source of an odor.#####

###Cf. Békésy [1955, 1] [1958, 1] [1959, 1], and Mountcastle & Powell [1959, 3].

####Cf. Békésy [1964, 1,2].

#####[1964, 1].

In this essay I attempt to construct a phenomenological theory of the lateral interactions of sensory receptors. The mathematical framework employed appears appropriate to those senses, such as vision and touch, for which the "sensory inputs" can be described by real-valued functions on a continuum.

The emphasis here is laid on the sense of vision. Now, in vision, particularly monocular vision, the sensory domain, often called the "field of vision", is a part of a spherical surface $\mathcal{S}$. The center of curvature for $\mathcal{S}$ is (in the case of simple vertebrate eyes) the nodal point P of the ocular lens; this "optical center" P lies between the cornea and the retina; (in humans P is about 7 mm behind the foremost point of the cornea).## The sensory input to the eye is usually considered

##Cf. [1970, 1, pp. 10 & 447].

a function i on $\mathcal{S}$, with $i(\underset{\sim}{x})$ the intensity of the light received by the eye from all sources lying on the line from P which intersects $\mathcal{S}$ at $\underset{\sim}{x}$. Here, in the interest of simplicity, I shall suppose that $\mathcal{S}$ can be represented approximately by a subset of a flat plane E.

Under a much more severe approximation, one may also describe the sensory input to cutaneous pressure and temperature sensors by employing functions on a two-dimensional flat space; thus some of the results obtained here, albeit presented in a terminology appropriate to the sense of vision, may suggest psychophysical experiments for the sense of touch.

The present work is concerned with spatial interactions only, and throughout my discussion of vision I have in mind achromatic or monochromatic fields. Thus, neither time nor wavelength occur explicitly. The results obtained may be applied to either steady-state experiments or to experiments in which the stimulus (light) is briefly flashed. I assume that in application to steady-state experiments on human vision, the natural retinal tremor will be present; as mentioned above, it is <u>not</u> intended that "steady-state" should imply a "constant rate of firing" for each neuron of the optic nerve.

In accord with the experiments of Hartline & Ratliff [1957, 3] [1958, 2] on <u>Limulus</u> and the general type of circuitry suggested by the microscopic anatomy of primate retinae, I assume that the lateral interactions "fall off gradually with distance"[#] and are, furthermore,

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[#] This expression is rendered precise in my article [1971, 1] on "retardation theorems". Definitions and results given there will be employed here.

"recurrent"[##] in the sense that the response at a point is determined by

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[##] Cf. [1965, 2, p. 108].

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the excitation at that point and the response (rather than the excitation) at other points. As will become apparent when the present theory is laid out, "recurrent systems" have the property that the "excitation field" ϵ is mathematically determined when the "response field" ψ is specified, but not always vice versa. In the asymptotic limit of "spread-out fields", if one is given ϵ on a large region $\mathcal{R}$ in E, calculation of ψ requires the solution of a field equation which need not have a unique solution on $\mathcal{R}$ unless certain data for ψ are specified at the boundary of $\mathcal{R}$. This suggests that the present theory may supply a mathematical framework for the discussion of certain primitive optical illusions.

2. Basic Assumptions

Let the "field of vision" be represented by a flat plane E, i.e. a two-dimensional Euclidean point-space, and let V be the translation space of E. Thus the difference $\underset{\sim}{x}-\underset{\sim}{y}$ of two points $\underset{\sim}{x}, \underset{\sim}{y}$ in E is an element of the vector space V, and V has a positive definite inner product "$\cdot$". A *field* ϕ on E is here a Lebesgue-measurable function mapping E into the real numbers R.

Two fields of particular importance are the *intensity field* i and the *response field* ψ; $i(\underset{\sim}{x})$ is the intensity of light at the point $\underset{\sim}{x}$, and $\psi(\underset{\sim}{x})$ is the animal's visual response, at the point $\underset{\sim}{x}$, to the field i; i.e. $\psi(\underset{\sim}{x})$ is the "intensity seen at $\underset{\sim}{x}$" when the "actual" intensity distribution is given by i. One may interpret $\psi(\underset{\sim}{x})$ as the number of impulses per second in the neuron or neurons of the optic nerve associated with the point $\underset{\sim}{x}$ in E. This interpretation is not necessary to the theory, however. In humans, $\psi(\underset{\sim}{x})$ can be identified with the "apparent brightness" an observer sees at $\underset{\sim}{x}$. Various matching tests[#] and threshold experiments[##] have been designed to enable a person to communicate to others quantitative

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[#] Cf., e.g., Davidson [1968, 2].

[##] Cf., e.g., Schade [1956, 2], Campbell & Green [1965, 1], Campbell & Robson [1968, 1].

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information about the apparent brightness he sees at each point in a given intensity field.# In the theory to be developed here, the field i

#A survey is given in Chapters 11-13 of [1970, 1].

is determined (by a relation not yet stated) when the field ψ is given on E, albeit specification of i may not yield a unique ψ.

It is assumed that for each visual system there is a strictly increasing function $\mathfrak{f}$ which gives the intensity i° corresponding to a given <u>constant</u> response field ψ°:

$$i^\circ = \mathfrak{f}(\psi^\circ). \tag{2.1}$$

Whether or not ψ and i are constant on E, one may state that knowledge of the field i is equivalent to knowledge of the <u>excitation</u> <u>field</u> ϵ defined by the equation

$$\epsilon(\underset{\sim}{x}) \;\underline{\underline{\mathrm{def}}}\; \mathfrak{f}^{-1}(i(\underset{\sim}{x})), \tag{2.2}$$

which holds for each $\underset{\sim}{x}$ in E; ϵ is, in a certain sense, a "normalized form" of i. If i is given, then, for each $\underset{\sim}{x}$, the number $\epsilon(\underset{\sim}{x})$ is the value of the (unique)## constant response field ψ° corresponding to the constant

##Although i need not always determine ψ, it follows from (2.1) and the strict monotonicity of $\mathfrak{f}$ that, when a constant intensity field is specified, there is one and only one <u>constant</u> response field corresponding to it.

intensity field i° whose value is the number $i(\underset{\sim}{x})$. As $\mathfrak{f}^{-1}$ is single-valued and strictly increasing, each intensity field i determines a unique excitation field ϵ and vice-versa. Thus, $\epsilon(\underset{\sim}{x})$ is a measure of the intensity at $\underset{\sim}{x}$ in the units employed for ψ, and it is meaningful to discuss the "difference" between $\psi(\underset{\sim}{x})$ and $\epsilon(\underset{\sim}{x})$: $\psi(\underset{\sim}{x}) - \epsilon(\underset{\sim}{x})$ measures the effect, on the response at $\underset{\sim}{x}$, of spatial variations in response.

Throughout this article I shall take as the basic fields the excitation ϵ and the response ψ. Because of its importance to applications and because of differences between this and other definitions of "excitation", I should like to emphasize a consequence of the definition (2.2): If $\psi(\underset{\sim}{x})$ is interpreted as the rate of firing of certain neurons of the visual pathway, then $\epsilon(\underset{\sim}{x})$ must be identified with the rate of activity which would be manifested by the same neurons if ψ and the intensity of light were spatially constant and the intensity had the value it presently has at $\underset{\sim}{x}$.

Let $\mathfrak{S}$ be the set of all real-valued Lebesgue-measurable functions on V. The elements ϕ_* of $\mathfrak{S}$ are called centered fields. If ϕ is a field on E and $\underset{\sim}{x}$ a point in E, then ϕ may be "centered on $\underset{\sim}{x}$" by defining a function $\phi_{\underset{\sim}{x}}$ in $\mathfrak{S}$ by

$$\phi_{\underset{\sim}{x}}(\underset{\sim}{v}) = \phi(\underset{\sim}{x} + \underset{\sim}{v}), \qquad \underset{\sim}{v} \in V. \tag{2.3}$$

The main postulate of the present theory is that for each visual system and each $c > 0$, there is a real-valued function $\mathfrak{F}(\cdot;c)$ which is defined on a subset $\mathfrak{D}_c$ of $\mathfrak{S}$, and which relates the response field ψ to the excitation field ϵ through the following <u>basic</u> <u>constitutive</u> <u>equation</u>:

$$\psi(\underset{\sim}{x}) = \epsilon(\underset{\sim}{x}) + \mathfrak{F}(\psi_{\underset{\sim}{x}};\psi(\underset{\sim}{x})). \tag{2.4}$$

This equation is assumed to hold at each point $\underset{\sim}{x}$ in E.

Although it is not important for some of the results to be obtained here, for simplicity I assume that the functional $\mathfrak{F}$ in (2.4), as well as the function $\mathfrak{f}$ in (2.1), is independent of $\underset{\sim}{x}$, i.e. that the visual system under consideration is intrinsically homogeneous.#

#As Cornsweet [1970, 1, p. 328, 329] remarks: Although the optical parts of human eyes are usually homogeneous near the optic axis, retinae and visual pathways do show anatomical inhomogeneities. For example, for humans the densities of rods and cones vary with position.

To describe the domain of $\mathfrak{F}(\cdot;c)$, let p be in $[1,\infty]$, and let h be a positive continuous function on $[0,\infty)$ such that, for some numbers $r > 0$ and $A > 0$,

$$s^r h(s) \text{ is monotone decreasing for } s > A. \tag{2.5}$$

By Remark 3 of [1971], $h(|\underset{\sim}{v}|)$, as a function of $\underset{\sim}{v}\in V$, is then an "influence function of order r" for V. For each centered field ϕ_*, put

$$\left.\begin{aligned} \|\phi_*\|_{h,p} &= \begin{cases} \sqrt[p]{\displaystyle\int_V \left(|\phi_*(\underset{\sim}{v})|h(|\underset{\sim}{v}|)\right)^p d\underset{\sim}{v}}\,, & \text{if } 1 \le p < \infty, \\ \operatorname*{ess\,sup}_V |\phi_*(\underset{\sim}{v})|h(|\underset{\sim}{v}|), & \text{if } p = \infty; \end{cases} \\ \|\phi_*\|'_h &= \sup|\phi_*(\underset{\sim}{v})|h(|\underset{\sim}{v}|). \end{aligned}\right\} \tag{2.6}$$

The integration and essential supremum shown above are taken with respect to Lebesgue measure on V. Let $\mathcal{L}_{h,p}$ be the set of all ϕ_* in $\mathfrak{S}$ for which $\|\phi\|_{h,p}$ is finite, and let $\mathcal{C}_h$ be the set of continuous functions ϕ_* in $\mathfrak{S}$ with $\|\phi_*\|'_h$ finite. Clearly, $\mathcal{C}_h$ is a Banach space with norm $\|\cdot\|'_h$. If we agree to consider the same two elements of $\mathcal{L}_{h,p}$ which differ on no more than a set of zero Lebesgue-measure, then, for each p in $[1,\infty]$, $\mathcal{L}_{h,p}$ is a Banach space with norm $\|\cdot\|_{h,p}$.

For each number c, let $c^\dagger$ be the constant function with value c, i.e.

$$c^\dagger(\underset{\sim}{v}) = c, \quad \text{for all } \underset{\sim}{v} \in V. \tag{2.7}$$

The present theory holds under <u>either one</u> of the following two (distinct) hypotheses about $\mathfrak{J}$:

(I) Let it be assumed that there exists p in $[1,\infty]$, an integer $m \geq 0$, and a positive continuous function h obeying (2.5) with

$$r > m + \frac{2}{p} \qquad \left(\frac{2}{p} = 0 \quad \text{if} \quad p = \infty\right), \tag{2.8}$$

such that for each $c > 0$, there is a $\delta_c > 0$ for which

$$\mathfrak{D}_c = \left\{\phi_* \mid \phi_* \in \mathcal{L}_{h,p}, \; \|\phi_* - c^\dagger\|_{h,p} < \delta_c\right\}, \tag{2.9}$$

i.e. such that the domain $\mathfrak{D}_c$ of $\mathfrak{J}(\cdot;c)$ is a spherical neighborhood of the positive constant function $c^\dagger$ in $\mathcal{L}_{h,p}$.

(II) Let it be assumed that there exists an integer $m \geq 0$ and a positive continuous function obeying (2.5) with

$$r > m, \tag{2.8$'$}$$

such that for each $c > 0$ there is a $\delta_c > 0$ for which

$$\mathfrak{D}_c = \left\{\phi_* \mid \phi_* \in \mathcal{C}_h, \ \ \|\phi_* - c^\dagger\|_h' < \delta_c\right\}. \tag{2.9$'$}$$

The value required for the integer m in (2.8) [or (2.8)′] will be given in the formulation of subsequent theorems. The larger m, the stronger the hypothesis on h. Whether one uses the relations (2.8) and (2.9) of (I) or the relations (2.8)′ and (2.9)′ of (II), none of the theorems to be given here require that m be greater than 4.

For compatibility with the definition (2.2) of ϵ, it is assumed that for each $c > 0$,

$$\mathfrak{I}(c^\dagger; c) = 0. \tag{2.10}$$

This relation, when combined with (2.4), states that, if the response ψ is constant on E, then the excitation ϵ must be a constant equal to ψ.

In the terminology employed in [1971,1], $\mathfrak{I}$ is said to be a <u>tame function of type</u> n if: (1) the relation (2.8) [or, under the hypothesis (II), the relation (2.8)′] holds with $m = n$, and (2) for each $c > 0$, $\mathfrak{I}$ is n-times Fréchet-differentiable at the constant function $c^\dagger$ in $\mathfrak{D}_c$. The differentiability assumption means that for each $c > 0$ there are on $\mathcal{L}_{h,p}$ [or on $\mathcal{C}_h$] bounded homogeneous polynomials $\delta^k\mathfrak{J}_c[\cdot]$ of degree $k = 1,\ldots,n$ such that for every ϕ_* with $c^\dagger + \phi_*$ in $\mathfrak{D}_c$,

$$\mathfrak{J}(c^\dagger+\phi_*;c) = \sum_{k=1}^{n} \frac{1}{k!}\,\delta^k\mathfrak{J}_c[\phi_*] + o\left(\|\phi_*\|^n\right), \tag{2.11}$$

where $\|\cdot\|$ stands for $\|\cdot\|_{h,p}$ if (2.9) holds and for $\|\cdot\|_h'$ if (2.9)$'$ holds. In writing (2.11) use is made of (2.10). The polynomial $\delta^k\mathfrak{J}_c[\cdot]$, or its polar form (i.e. the symmetric bounded k-linear form $\delta^k\mathfrak{J}_c(\cdots)$ defined by $\delta^k\mathfrak{J}_c(\phi_*,\ldots,\phi_*) = \delta^k\mathfrak{J}_c[\phi_*]$) is called the k^{th} Fréchet-derivative of $\mathfrak{J}$ at $c^\dagger$. For each ϕ_*, the function $c \mapsto \delta^k\mathfrak{J}_c[\phi_*]$, as a real-valued function on $(0,\infty)$, is assumed to be (n-k)-times differentiable, and $c \mapsto \mathfrak{J}_c[\phi_*;c]$ is assumed to be n-times differentiable.

Throughout this paper, unless the contrary is stated, it will be assumed that $\mathfrak{J}$ is a tame function of type 4; i.e. that (2.8) [or (2.8)$'$] holds with $m = 4$, and (2.11) holds with $n = 4$. Occasionally theorems will be stated employing a weaker hypothesis about $\mathfrak{J}$.

Let $\mathcal{O}$ be the orthogonal group on V; that is, the group of all linear transformations G of V into V for which $|G\underset{\sim}{v}| = |\underset{\sim}{v}|$ for every $\underset{\sim}{v}$ in V. If $\underset{\sim}{G}$ is in $\mathcal{O}$ and if ϕ_* is a function in $\mathfrak{S}$, one writes $\psi\circ\underset{\sim}{G}$ for the function defined by

$$(\psi\circ\underset{\sim}{G})(\underset{\sim}{v}) = \psi(\underset{\sim}{G}(\underset{\sim}{v})), \text{ for all } \underset{\sim}{v}\in V. \tag{2.12}$$

The idea that there are no "built-in preferred directions in the visual system" is rendered precise by assuming that $\mathfrak{J}$ is an isotropic functional; that is, that $\mathfrak{J}$ has $\mathcal{O}$ for its symmetry group.# Thus, I assume: For

#See [1971, 1, §6].

each $\underset{\sim}{G}$ in $\mathcal{O}$, $\mathfrak{J}$ obeys identity

$$\mathfrak{J}(\phi_* \circ \underset{\sim}{G}; c) = \mathfrak{J}(\phi_*; c), \tag{2.13}$$

for every $c > 0$ and for all ϕ_* in $\mathfrak{D}_c$.##

##It appears, from the psychophysical experiments of Campbell, Kulikowski, & Levinson [1966, 1] (see also Cornsweet [1970, 1, pp. 329, 330]), that, even after the optics of the eyes are corrected for possible astigmatism, the human visual system is not perfectly isotropic. At high spatial frequencies, measured thresholds of contrast for vertical and horizontal gratings are lower than for oblique gratings. This observed anisotropy is not very large for gratings with spatial frequencies less than 10 cycles per degree.

3. Retardation Theorems

Let n be either zero or a positive integer. A centered field ϕ_* in $\mathcal{L}_{h,p}$ is said to be n-times differentiable at $\underset{\sim}{v} = \underset{\sim}{0}$, if, after suitable alteration of ϕ_* on a set in V of measure zero,

$$\phi_*(\underset{\sim}{v}) = \phi_*^\circ + \sum_{k=1}^{n} \frac{1}{k!} \nabla^k \phi_*(\underset{\sim}{v},\ldots,\underset{\sim}{v}) + o(|\underset{\sim}{v}|^n), \tag{3.1}$$

where

$$\phi_*^\circ = \lim_{|\underset{\sim}{v}| \to 0} \phi_*(\underset{\sim}{v}) \tag{3.2}$$

is sometimes called the zeroth derivative of ϕ_* at $\underset{\sim}{v} = \underset{\sim}{0}$, and $\nabla^k \phi_*$, called the k^{th} gradient of ϕ_* at $\underset{\sim}{v} = \underset{\sim}{0}$, is a symmetric k-linear form on V. The existence, in the classical sense, of the ordinary gradient of order n at $\underset{\sim}{v} = \underset{\sim}{0}$ implies that (3.1) holds without any alteration of ϕ_*. If ϕ_* is continuous at $\underset{\sim}{0}$, then ϕ_* is "zero-times differentiable" at $\underset{\sim}{0}$, and ϕ_*°, defined in (3.2), is just $\phi_*(\underset{\sim}{0})$, the value of ϕ_* at $\underset{\sim}{0}$. If ϕ_* is in $\mathcal{C}_h$, then ϕ_* is automatically continuous, and one says that ϕ_* is n-times differentiable if (3.1) holds without any alteration of ϕ_*.

Let $\mathfrak{D}_c^n$ be the set of all ϕ_* in $\mathfrak{D}_c$ that are n-times differentiable. When $\nabla^2\phi_*$ exists, it is a second-order tensor on V, and $\nabla^1\phi_*$, written $\nabla\phi_*$, is represented by a vector in V. Of particular importance here are the trace of $\nabla^2\phi_*$, written $\Delta\phi_*$, and the magnitude of $\nabla\phi_*$, written $|\nabla\phi_*|$. If ϕ is a field on E, then at each point $\underset{\sim}{x}$ where $\phi_{\underset{\sim}{x}}$ (i.e. ϕ centered on $\underset{\sim}{x}$)

has first and second gradients, $\nabla\phi_{\underset{\sim}{x}}$, $\nabla^2\phi_{\underset{\sim}{x}}$, one uses the notation

$$\left.\begin{aligned} \Delta\phi(\underset{\sim}{x}) &\overset{\text{def}}{=\!=} \Delta\phi_{\underset{\sim}{x}} = \operatorname{trace} \nabla^2\phi_{\underset{\sim}{x}}, \\ (\nabla\phi(\underset{\sim}{x}))^2 &\overset{\text{def}}{=\!=} |\nabla\phi_{\underset{\sim}{x}}|^2 = (\nabla\phi_{\underset{\sim}{x}})\cdot(\nabla\phi_{\underset{\sim}{x}}). \end{aligned}\right\} \tag{3.3}$$

$\Delta\phi(\underset{\sim}{x})$ is the Laplacian of ϕ at $\underset{\sim}{x}$. Of course, with respect to a Cartesian coordinate system (x,y) on E,

$$\left.\begin{aligned} \Delta\phi(\underset{\sim}{x}) &= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}, \\ (\nabla\phi(\underset{\sim}{x}))^2 &= \left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2. \end{aligned}\right\} \tag{3.4}$$

Given a centered field ϕ_* and a number α in $(0,1]$, one may define a new centered field $\phi_*^{(\alpha)}$ by

$$\phi_*^{(\alpha)}(\underset{\sim}{v}) = \phi_*(\alpha\underset{\sim}{v}), \qquad \underset{\sim}{v}\in V. \tag{3.5}$$

$\phi_*^{(\alpha)}$ is called the α-retardation of ϕ_*. Roughly speaking, retardation replaces a centered field by one which is similar but more "spread out". If, for some $n \geq 0$, ϕ_* is in $\mathfrak{D}_c^n$, then, for sufficiently small $\alpha > 0$, so also is $\phi_*^{(\alpha)}$.# It follows from (3.5) and (3.1) that

#Cf. [1971, 1, Theorem 2, and Remarks ~~3~~ 5 & ~~4~~ 6].

$$\nabla^n\phi_*^{(\alpha)} = \alpha^n\nabla^n\phi_*, \tag{3.6}$$

and hence that

$$|\nabla\phi_*^{(\alpha)}|^2 = \alpha^2|\nabla\phi_*|^2, \qquad \Delta\phi_*^{(\alpha)} = \alpha^2\Delta\phi_*. \tag{3.7}$$

In view of the normalization and isotropy conditions, (2.10), and (2.13), one may read off, from Theorem 7 of [1971,1] the following asymptotic approximation for $\mathfrak{J}(\phi_{\underset{\sim}{x}}^{(\alpha)},\phi(\underset{\sim}{x}))$.

Theorem 1 (*retardation theorem for smooth fields*). *Suppose* $\mathfrak{J}$ *is a tame function of type* n, *with* $n \geq 2$. *For each centered* ϕ_* *field in* $\mathfrak{D}_c^n$ *with* $\phi_*^\circ = c$, *one has, for small* $\alpha > 0$,

$$\mathfrak{J}(\phi_*^{(\alpha)};c) = \beta(c)|\nabla\phi_*^{(\alpha)}|^2 + \gamma(c)\Delta\phi_*^{(\alpha)} + \kappa(\alpha), \tag{3.8}$$

where

$$\kappa(\alpha) = \begin{cases} o(\alpha^2) \text{ if } n = 2, \\ o(\alpha^3) \text{ if } n = 3, \\ O(\alpha^4) \text{ if } n \geq 4; \end{cases} \tag{3.9}$$

$\beta(\cdot)$ *and* $\gamma(\cdot)$ *are continuous real-valued functions on* $(0,\infty)$ *and are uniquely determined by* $\mathfrak{J}$; $\beta(\cdot)$ *is* (n-2)- *and* $\gamma(\cdot)$ *is* (n-1)-*times differentiable*.

To have a concise way to describe the manner in which $\mathfrak{J}$ determines $\beta(\cdot)$ and $\gamma(\cdot)$, let $\underset{\sim}{a}$ be an arbitrary element of V, let $\underset{\sim}{A}$ be an arbitrary symmetric linear transformation of V into V, and employ the symbols $\underset{\sim}{a}[\cdot]$ and $\underset{\sim}{A}[\cdot]$ for the homogeneous polynomials on V defined by

$$\underset{\sim}{a}[\underset{\sim}{v}] \stackrel{\text{def}}{=} \underset{\sim}{a}\cdot\underset{\sim}{v}, \qquad \underset{\sim}{A}[\underset{\sim}{v}] \stackrel{\text{def}}{=} \underset{\sim}{v}\cdot\underset{\sim}{A}\underset{\sim}{v}, \qquad \underset{\sim}{v}\in V. \tag{3.10}$$

Now, it can be shown that the real-valued functions $\underset{\sim}{a}[\cdot]$ and $\underset{\sim}{A}[\cdot]$ belong to $\mathcal{L}_{h,p}$ [or $\mathcal{C}_h$ if (2.9)′ is used] and that the isotropy of $\mathfrak{J}$ implies that $\delta^2\mathfrak{J}_c\big[\underset{\sim}{a}[\cdot]\big]$ and $\delta\mathfrak{J}_c\big[\underset{\sim}{A}[\cdot]\big]$ have the forms#

#See [1971, 1, Lemma 1, and eqs. (5.25), (7.6)].

$$\left.\begin{aligned} \delta^2\mathfrak{J}_c\big[\underset{\sim}{a}[\cdot]\big] &= 2\beta(c)\underset{\sim}{a}\cdot\underset{\sim}{a} \\ \delta\mathfrak{J}_c\big[\underset{\sim}{A}[\cdot]\big] &= 2\gamma(c)\,\mathrm{trace}\,\underset{\sim}{A}. \end{aligned}\right\} \tag{3.11}$$

That is, the restriction of $\delta^2\mathfrak{J}_c$ to linear functions of the type $(3.10)_1$ and the restriction of $\delta\mathfrak{J}_c$ to quadratic forms of the type $(3.11)_2$ are determined by certain numbers $\beta(c)$ and $\gamma(c)$ as shown in (3.11), and these numbers $\beta(c)$, $\gamma(c)$, as functions of c, give the functions $\beta(\cdot)$, $\gamma(\cdot)$ in the retardation theorem.

It follows from (3.7) that the terms shown explicitly on the right in (3.8), i.e. $\beta(c)|\nabla\phi_*^{(\alpha)}|^2$ and $\gamma(c)\Delta\phi_*^{(\alpha)}$, are of order $O(\alpha^2)$. By (3.9), the remainder term $\kappa(\alpha)$ is of higher order in α. Even if ϕ_* is only twice differentiable at $\underset{\sim}{0}$, this remainder is $o(\alpha^2)$. If ϕ_* is four-times, or more, differentiable at $\underset{\sim}{0}$, then## $\kappa(\alpha)$ is $O(\alpha^4)$. Hence,

##It is assumed that $\mathfrak{J}$ is a tame function of type 4.

for a centered field ϕ_* that is "smooth and spread out" in the sense that

ϕ_* is several-times differentiable at $\underset{\sim}{0}$ and does not vary rapidly on V, $\mathfrak{I}(\phi_*;\phi_*^\circ)$ is given, to an apparently good approximation, by the value of the functional $\mathfrak{G}$ obeying

$$\mathfrak{G}(\phi_*) = \beta(\phi_*^\circ)|\nabla\phi_*|^2 + \gamma(\phi_*^\circ)\Delta\phi_*. \tag{3.12}$$

If $\phi_* = \psi_{\underset{\sim}{x}}$ with ψ a smooth response field on ϵ, then $\phi_*^\circ = \psi(\underset{\sim}{x})$, $|\nabla\phi_*|^2 = (\nabla\psi(\underset{\sim}{x}))^2$, $\Delta\phi_* = \Delta\psi(\underset{\sim}{x})$, and (3.12) becomes

$$\mathfrak{G}(\psi_{\underset{\sim}{x}}) = \beta(\psi(\underset{\sim}{x}))(\nabla\psi(\underset{\sim}{x}))^2 + \gamma(\psi(\underset{\sim}{x}))\Delta\psi(\underset{\sim}{x}). \tag{3.13}$$

Thus, it follows from (2.4) and the retardation theorem for smooth fields that, when there is reason to believe that the response field ψ is at least twice differentiable and does not vary rapidly from point to point in E, as a first correction to the "zeroth-order relation"

$$\psi(\underset{\sim}{x}) = \epsilon(\underset{\sim}{x}), \tag{3.14}$$

we have

$$\psi(\underset{\sim}{x}) = \epsilon(\underset{\sim}{x}) + \beta(\psi(\underset{\sim}{x}))(\nabla\psi(\underset{\sim}{x}))^2 + \gamma(\psi(\underset{\sim}{x}))\Delta\psi(\underset{\sim}{x}). \tag{3.15}$$

If the field ψ is several-times differentiable, this last equation should serve as a good approximation to the basic constitutive equation (2.4). Indeed, if ψ is four-times, or more, differentiable, the error will be of order four in "the scale of distance".

In Cartesian coordinates (3.15) takes the form

$$\psi = \epsilon + \beta(\psi)\left[\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2\right] + \gamma(\psi)\left[\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right]. \tag{3.16}$$

Throughout the remainder of this section it will be assumed that ψ varies only in one direction, i.e. that ψ is independent of y. Then

$$\psi = \psi(x), \qquad \epsilon = \epsilon(x), \tag{3.17}$$

and (3.16) reduces to

$$\psi = \epsilon + \beta(\psi)\psi'^2 + \gamma(\psi)\psi'', \tag{3.18}$$

where $\psi' = d\psi/dx$, $\psi'' = d^2\psi/dx^2$. It is interesting to see how this equation must be modified at points where there are jumps in the derivatives of ψ, i.e. at points where the left- and right-hand derivatives of ψ exist but are unequal.

Let n be an integer equal to or greater than zero, let c be positive, and let $\underset{\sim}{e}_\chi$, $\underset{\sim}{e}_\eta$ be an orthonormal basis in V, so that each $\underset{\sim}{v}$ in V can be written $\underset{\sim}{v} = \chi\underset{\sim}{e}_\chi + \eta\underset{\sim}{e}_\eta$. Let $\hat{\mathfrak{D}}_c^n$ be the set of centered fields ϕ_* in $\mathfrak{D}_c$ that are constant in the direction $\underset{\sim}{e}_\eta$ and are n-<u>fold regular at</u> $\chi = 0$, in the sense that when ϕ_* is regarded as a function of χ, one has

$$\lim_{\chi\to 0+} \phi_*(\chi) = \lim_{\chi\to 0-} \phi_*(\chi) = \phi_*^\circ, \tag{3.19}$$

and the right-hand derivatives, ϕ_{*+}^k, and the left-hand derivatives, ϕ_{*-}^k, defined by

$$\left.\begin{aligned}
\phi'_{*+} &= \lim_{\chi\to 0+} \frac{1}{\chi}\left(\phi_*(\chi) - \phi^\circ_*\right),\\
\phi''_{*+} &= \lim_{\chi\to 0+} \frac{2}{\chi^2}\left(\phi_*(\chi) - \phi^\circ_* - \chi\phi_{*+}\right),\\
&\vdots\\
\phi^k_{*+} &= \lim_{\chi\to 0+} \frac{k!}{\chi^k}\left(\phi_*(\chi) - \phi^\circ_* - \sum_{j=1}^{k-1} \frac{\chi^j}{j!}\,\phi^{(j)}_{*+}\right),\\
\phi'_{*-} &= \lim_{\chi\to 0-} \frac{1}{\chi}\left(\phi_*(\chi) - \phi^\circ_*\right),\\
&\vdots\\
\phi^k_{*-} &= \lim_{\chi\to 0-} \frac{k!}{\chi^k}\left(\phi_*(\chi) - \phi^\circ_* - \sum_{j=1}^{k-1} \frac{\chi^j}{j!}\,\phi^j_{*-}\right),
\end{aligned}\right\} \qquad (3.20)$$

exist for $k = 1,\dots,n$. Of course, if $\mathfrak{D}_c \subset \mathcal{C}_h$, then (3.19) holds automatically for each ϕ_* in $\mathfrak{D}_c$; if $\mathfrak{D}_c \subset \mathcal{L}_{h,p}$, then we should say that a function ϕ_* in $\mathfrak{D}_c$ is also in $\hat{\mathfrak{D}}^n_c$ if and only if (3.19) and (3.20) hold after suitable alteration of ϕ_* on a set of measure zero. For the α-retardation of ϕ_*, there hold the relations

$$\phi^{(\alpha)'}_{*+} = \alpha\phi'_{*+}, \qquad \phi^{(\alpha)'}_{*-} = \alpha\phi'_{*-}, \qquad \phi^{(\alpha)''}_{*} = \alpha^2\phi''_{*}, \text{ etc.} \qquad (3.21)$$

It follows from Theorems 10 and 11 of [1971,1] that, as a consequence of (2.10) and (2.13), we have here the following supplement to Theorem 1.

Theorem 2 (*retardation theorem for fields with one-sided derivatives*). *If* $\mathfrak{J}$ *is a tame function of type* n, *with* $n \geq 2$, *then for each* ϕ_* *in* $\hat{\mathfrak{D}}_c^n$ *with* $\phi_*^\circ = c$,

$$\mathfrak{J}(\phi_*^{(\alpha)};c) = \beta(c)\phi_{*+}^{(\alpha)\prime}\phi_{*-}^{(\alpha)\prime} + \frac{1}{2}\gamma(c)\left(\phi_{*+}^{(\alpha)\prime\prime}+\phi_{*-}^{(\alpha)\prime\prime}\right) + \theta(c)\left(\phi_{*+}^{(\alpha)\prime}-\phi_{*-}^{(\alpha)\prime}\right)$$
$$+ \pi(c)\left(\phi_{*+}^{(\alpha)\prime}-\phi_{*-}^{(\alpha)\prime}\right)^2 + \omega(\alpha), \tag{3.22}$$

where

$$\omega(\alpha) = \begin{cases} o(\alpha^2) \text{ if } n = 2, \\ O(\alpha^3) \text{ if } n \geq 3; \end{cases} \tag{3.23}$$

$\beta(\cdot)$, $\gamma(\cdot)$ *are as in Theorem* 1; $\theta(\cdot)$ *and* $\pi(\cdot)$ *are also continuous functions determined by* $\mathfrak{J}$; $\theta(\cdot)$ *is* (n-1)- *and* $\pi(\cdot)$ *is* (n-2)-*times differentiable*.

Let $\underset{\sim}{e}_\chi$, $\underset{\sim}{e}_\eta$ be again a fixed orthonormal basis in V, and let λ_+ be the following function on V:

$$\lambda_+(\chi\underset{\sim}{e}_\chi + \eta\underset{\sim}{e}_\eta) = \begin{cases} \chi, \text{ if } \chi\in[0,\infty), \\ 0, \text{ if } \chi\in(-\infty,0). \end{cases} \tag{3.24}$$

The new visual functions occurring in (3.22), $\theta(\cdot)$ and $\pi(\cdot)$, are determined by $\mathfrak{J}$ through the relations

$$\left.\begin{aligned} \theta(c) &= \delta\mathfrak{J}_c[\lambda_+], \\ \pi(c) &= \frac{1}{2}\delta^2\mathfrak{J}_c[\lambda_+], \end{aligned}\right\} \tag{3.25}$$

which, like (3.11), hold for each $c > 0$. By the isotropy of $\mathfrak{J}$ (i.e. the assumption that (2.13) holds for all orthogonal tensors $\underset{\sim}{G}$), the right-hand sides in (3.25) are independent of the choice of $\underset{\sim}{e}_\chi$, $\underset{\sim}{e}_\eta$.

By (3.21), the term $\theta(c)\left(\phi_{*+}^{(\alpha)} - \phi_{*-}^{(\alpha)}\right)$ in (3.22) is of order $O(\alpha)$; the other terms shown explicitly on the right in (3.22) are $O(\alpha^2)$. Even if ϕ_* is only 2-fold regular at $x = 0$, the remainder $\omega(\alpha)$ is $o(\alpha^2)$; if ϕ_* is 3-fold regular and $\mathfrak{J}$ is a tame function of type 3, then $\omega(\alpha)$ is $O(\alpha^3)$. Hence, if the response field ψ does not vary rapidly from point to point and is known to have one-sided, rather than classical, derivatives of orders 1 and 2, the equation,

$$\psi = \epsilon + \beta(\psi)\overline{\psi'}^2 + \gamma(\psi)\langle\psi''\rangle + \theta(\psi)[\![\psi']\!] + \pi(\psi)[\![\psi']\!]^2, \qquad (3.26)$$

generalizes (3.18) and holds as an approximation to the basic constitutive equation (2.4); here $[\![\psi']\!]$ is the "jump" in the first derivative of ψ, $\langle\psi''\rangle$ is the "mean value" of the second derivative, and $\overline{\psi'}^2$ is the square of the "geometric mean" of the first derivative; i.e.

$$\left.\begin{aligned} [\![\psi']\!] &\stackrel{\text{def}}{=} \psi'_+ - \psi'_-, \\ \langle\psi''\rangle &\stackrel{\text{def}}{=} \tfrac{1}{2}(\psi''_+ + \psi''_-), \\ \overline{\psi'}^2 &\stackrel{\text{def}}{=} \psi'_+\psi'_-, \end{aligned}\right\} \qquad (3.27)$$

with $\begin{Bmatrix}\psi'_+ \\ \psi'_-\end{Bmatrix}$ the $\begin{Bmatrix}\text{right-hand} \\ \text{left-hand}\end{Bmatrix}$ first derivative, and $\begin{Bmatrix}\psi''_+ \\ \psi''_-\end{Bmatrix}$ the $\begin{Bmatrix}\text{right-hand} \\ \text{left-hand}\end{Bmatrix}$ second derivative of ψ:

$$\left.\begin{aligned} \psi'_+(x) &= \lim_{h\to 0+} \frac{1}{h}\Big[\psi(x+h) - \psi(x)\Big], \\ \psi''_+(x) &= \lim_{h\to 0+} \frac{2}{h^2}\Big[\psi(x+h) - \psi(x) - h\psi'_+(x)\Big], \end{aligned}\right\} \qquad (3.28)$$

etc. By Theorem 2, if ψ is such that, at each $\underset{\sim}{x}$ in E, $\psi_{\underset{\sim}{x}}$ is in $\hat{\mathfrak{D}}_3$, then (3.26) gives an approximation to (2.4) that holds to within an error of order three "in the scale of distance".

Both electrophysiological measurements on the compound eye of <u>Limulus</u>[#] and psychophysical studies of the visual system of humans[##],

[#]Particularly the studies of Hartline & Ratliff [1954, 1] [1957, 3] [1958, 2]; see also [1959, 5].

[##]Starting with the work of Mach [1865, 1] [1866, 1] [1868, 1] and continuing to modern measurements of spatial transfer functions and thresholds of contrast for periodic patterns; cf. Schade [1956, 2], Campbell & Green [1965, 1], Davidson [1968, 2], and the monographs [1965, 2] [1970, 1].

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suggest that the primary effect of lateral interaction in vision is inhibition rather than excitation, more precisely, that for each c, the linear functional $\delta\mathfrak{J}_c$ is negative semi-definite in the sense that, for each ϕ_* in $\mathcal{L}_{h,p}$ (or $\mathcal{C}_h$),

$$\phi_*(\underset{\sim}{v}) \geq 0 \text{ for all } \underset{\sim}{v} \text{ in } V \implies \delta\mathfrak{J}_c[\phi_*] \leq 0. \tag{3.29}$$

It further appears that for humans, and also for <u>Limulus</u> (when c is not too small), (3.29) should be strengthened to

$$\phi_*(\underset{\sim}{v}) \geq 0 \text{ for all } \underset{\sim}{v} \text{ in } V,\ \ \|\phi_*\| \neq 0,\ \implies \delta\mathfrak{J}_c(\phi_*) < 0, \tag{3.30}$$

where $\|\cdot\|$ stands for $\|\cdot\|_{h,p}$ or $\|\cdot\|'_{h}$, as appropriate. In view of $(3.10)_2$, $(3.11)_2$, (3.24), and $(3.25)_1$, the assertion (3.30) yields

$$\gamma(c) < 0, \qquad \theta(c) < 0. \tag{3.31}$$

The inequalities (3.31) are clearly compatible with the phenomena called "Mach bands". For example, from (3.26) and (3.31) one may conclude that <u>for each</u> $c > 0$ <u>there exists a positive number</u> $\delta = \delta(c)$ <u>such that when</u> ϵ <u>and</u> ψ <u>are continuous at</u> x <u>with</u> $\psi(x) = c$, <u>with</u> $|[\![\psi']\!]| < \delta(c)$ (<u>at</u> x), pt <u>and with either</u> $\psi'_+(x) = 0$ <u>or</u> $\psi'_-(x) = 0$, <u>then at</u> x <u>the following implications</u> pt <u>hold</u>:

$$\left.\begin{array}{l}\langle\psi''\rangle > 0, \; [\![\psi']\!] \geq 0 \implies \psi < \epsilon, \\ \langle\psi''\rangle \geq 0, \; [\![\psi']\!] > 0 \implies \psi < \epsilon, \\ \langle\psi''\rangle < 0, \; [\![\psi']\!] \leq 0 \implies \psi > \epsilon, \\ \langle\psi''\rangle \leq 0, \; [\![\psi']\!] < 0 \implies \psi > \epsilon;\end{array}\right\} \tag{3.32}$$

indeed, one may take $\delta(c) = \left|\frac{\theta(c)}{\pi(c)}\right|$ $\big(\delta(c) = \infty$ if $\pi(c) = 0\big)$. I shall discuss the theory of Mach bands in detail in a future article.

The electrophysiological data of Hartline & Ratliff [1957, 3] [1958, 2] indicate that, for the compound eye of <u>Limulus</u>, $\delta^2\mathfrak{J}_c$ is negative semi-definite, i.e. that for each $c > 0$ and for each ϕ_* in the domain of $\delta^2\mathfrak{J}_c$,

$$\delta^2\mathfrak{J}_c[\phi_*] \leq 0. \tag{3.33}$$

It follows from $(3.11)_1$ and $(3.25)_2$ that the relation (3.33) implies

$$\beta(c) \leq 0, \qquad \pi(c) \leq 0. \tag{3.34}$$

I know of no conclusive evidence, however, indicating that (3.33) holds in general in human vision. In fact, some data obtained by pt Fiorentini & Radici [1957, 2]# for the subjective brightness experienced pt by human observers of Mach patterns, suggest that $\pi(c)$ is positive, at least for some observers under, perhaps, special conditions; these data are, however, compatible with (3.31). Of course, neither the inequality (3.33) nor its consequences (3.34) are essential to the present theory.

#See Figure 2.12 of Ratliff's survey [1965, 2, p. 58].

On Anisotropic Visual Systems

Theorem 2, which justifies the asymptotic forms (3.18), (3.26) of the constitutive equation (2.4), applies, of course, to fields which vary in only the x-direction. To prove Theorem 2, one does not need to assume that $\mathfrak{J}$ is an isotropic functional. For the validity of (3.22) and its consequences (3.18), (3.26), it suffices that the symmetry group of $\mathfrak{J}$

contain the central inversion, i.e. that the identity (2.13) hold when $\underset{\sim}{G} = \underset{\sim}{Q}$, where

$$\underset{\sim}{Q}\underset{\sim}{v} = -\underset{\sim}{v}, \quad \text{for all } \underset{\sim}{v} \in V. \tag{3.35}$$

However, under this assumption (which is weaker than the assertion that (2.13) hold for all $\underset{\sim}{G}$ in $\mathcal{O}$), the coefficients $\beta(\psi)$, $\gamma(\psi)$, $\theta(\psi)$, $\pi(\psi)$ in (3.26) and (3.18) depend on the inclination of the x-axis. To give definite formulae for these coefficients, let $\underset{\sim}{e}_\chi$, $\underset{\sim}{e}_\eta$ be an orthonormal basis with $\underset{\sim}{e}_\chi$ pointing in the x-direction, let λ_+ be as in (3.24), and put

$$\lambda_-(\underset{\sim}{v}) = -\lambda_+(-\underset{\sim}{v}), \qquad \underset{\sim}{v} \in V. \tag{3.36}$$

If (2.13) holds only for $\underset{\sim}{G} = \underset{\sim}{Q}$, the relations (3.11) must be replaced by#

#Cf. Theorem 11 of [1971, 1].

$$\left.\begin{aligned} \beta(c) &= \tfrac{1}{2}\,\delta^2\mathfrak{J}_c[\lambda_+ + \lambda_-], \\ \gamma(c) &= \tfrac{1}{2}\,\delta\mathfrak{J}_c[\lambda_+^2 + \lambda_-^2] = \delta\mathfrak{J}_c[\lambda_+^2], \end{aligned}\right\} \tag{3.37}$$

θ and π are given again by the formulae (3.25), and there is no argument to show that the right-hand sides of (3.37) and (3.25) are independent of $\underset{\sim}{e}_\chi$.

4. Solutions of the Approximating Field Equation

If the response field ψ is twice differentiable, the basic constitutive equation (2.4) is approximated by the second-order partial differential equation (3.15), which, in turn, reduces to the ordinary differential equation (3.18) when ψ varies only in the x-direction.

It is clear that, in the theory of (3.15)-(3.18), specification of the excitation field alone in a region does not determine the response field in that region. In particular, when ϵ is given as a function of x on an interval of the form $I = (0,X)$, $0 < X \leq \infty$, and it is assumed that ψ is also a function of x for x in I, then (3.15) reduces to (3.18) on I, and, clearly, one cannot expect ψ to be determined on this interval unless some information is given about the behavior of ψ near the boundary of I. To determine ψ on I, one may, for example, specify $\psi(0+)$ and $\psi'(0+)$, the limiting values of ψ and ψ' as x approaches zero from the right.[#]

[#] If, in addition to $\psi(0+)$, one attempts to specify $\psi(X-)$, instead of $\psi'(0+)$, then the problem of determining ψ on I, given ϵ on I, becomes a non-linear "two-point problem" which does not always have a unique solution.

In this section I should like to discuss, in a preliminary way, the theory of equation (3.18), assuming that neither ϵ nor ψ varies greatly on I, so that β and γ may be treated as constants. Thus, here we are concerned with the second-order non-linear differential equation

$$\psi = \epsilon + \beta\psi'^2 + \gamma\psi'', \qquad \psi = \psi(x), \qquad x\epsilon(0,X). \tag{4.1}$$

So as to have a non-trivial theory, let us assume that neither β nor γ is zero, and, in accord with the expectation that the lateral interaction is an "inhibition of response", let us choose γ negative [see (3.29)-(3.31)]. Thus,

$$\beta \neq 0, \qquad \gamma < 0. \tag{4.2}$$

By inverse methods one easily obtains some special solutions of (4.1) corresponding to particular choices for the field ϵ. A few of these special solutions may be worth mentioning, for they appear to provide methods of determining β and γ.

First, it is clear that if ϵ is a linear function, i.e.,

$$\epsilon(x) = A + Bx, \tag{4.3}$$

then (4.1) has a linear solution for ψ which has the same slope as ϵ but is shifted $\begin{Bmatrix}\text{downward}\\ \text{upward}\end{Bmatrix}$ by the amount $|\beta|B^2$ if β is $\begin{Bmatrix}\text{negative}\\ \text{positive}\end{Bmatrix}$; that is

$$\psi(x) = A + \beta B^2 + Bx. \tag{4.4}$$

Of course, the solution (4.4) is not the only solution of (4.1) corresponding to (4.3), but it is the only such <u>linear</u> solution. Hence, on comparing the "perceived brightness" ψ of several linear intensity patterns with different slopes B, if one finds that ψ varies linearly with x, then, at each fixed x, ψ should be a linear function of B^2, and the slope of a plot of ψ versus B^2, with A in (4.3) held fixed, equals the parameter β. [More generally, it follows from (4.1) and (4.2) that, for $\left\{\begin{matrix}\text{positive}\\\text{negative}\end{matrix}\right\}$ β, whenever ψ'' is zero at a point x, $\psi(x)$ lies $\left\{\begin{matrix}\text{above}\\\text{below}\end{matrix}\right\}$ $\epsilon(x)$ by the amount $|\beta|\psi'(x)^2$.]

Second, one may ask which excitation fields ϵ give response fields ψ that vary sinusoidally with x. It is readily verified that to have

$$\psi(x) = a + b \sin \omega x, \tag{4.5}$$

one must have

$$\epsilon(x) = A + B \sin \omega x + C \cos 2\omega x, \tag{4.6}$$

with

$$\left.\begin{aligned} A &= a - \tfrac{1}{2}\beta b^2\omega^2, \\ B &= b(1+\gamma\omega^2), \\ C &= -\tfrac{1}{2}\beta b^2\omega^2. \end{aligned}\right\} \tag{4.7}$$

Furthermore, whenever ϵ has the form (4.6) with

$$C = -\frac{\beta\omega^2 B^2}{2(1+\gamma\omega^2)^2}, \tag{4.8}$$

among the solutions of (4.1) is one of the form (4.5) with

$$\left.\begin{aligned} a &= A + C, \\ b &= \frac{B}{1+\gamma\omega^2}\ . \end{aligned}\right\} \tag{4.9}$$

It follows from (4.6) and (4.7) that observation of a response field ψ of the simple sinusoidal form (4.5) implies that the excitation field ϵ is approximately of the same form, if and only if $|C| \ll |B|$ in (4.6), that is, if and only if

$$\left|\frac{\beta b\omega^2}{2}\right| \ll |1+\gamma\omega^2|. \tag{4.10}$$

It is clear that (4.10) holds, in particular, in the limit of small ω, and, in view of Theorem 1, one expects (4.1) to hold with good accuracy in this limit.

It appears from this that for small ω one may identify b/B with the type of transfer function[#] F measured for human subjects by Davidson

[#]See [1970, 1, Chapter 12] where the term "modulation transfer function" is used.

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[1968, 2]; hence, for such a transfer function,

$$F(\omega) = \frac{1}{1+\gamma\omega^2} + O(\omega^4) = 1 - \gamma\omega^2 + O(\omega^4). \tag{4.11}$$

Of course, that we here have $F(0) = 1$ is a consequence of the definition (2.2) for ϵ; this definition requires that $\epsilon = \psi$ when ψ is spatially constant. Davidson uses different units for the excitation,# and in his

#His use of intensity fields of the form $i = i_0 \exp[c \sin \omega x]$ is, however, consistent with excitation fields ϵ of the form $A + B \sin \omega x$, if (as seems in accord with experience) the response to nearly uniform fields varies as the logarithm of the intensity i.

units $F(0)$ is much less than 1. The relation (4.11) states that, for small ω, $F(\omega)$ should increase as ω^2 with the limiting slope of a plot of F versus ω^2 equal to $-\gamma$. Thus, in principle, it should be possible to determine γ experimentally. [Davidson found that the behavior of F at low spatial frequencies varied with the subject tested; one subject showed an (approximately) linear dependence of F on ω. Clearly, more experimental work is called for.]

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If one takes the equations (4.6)-(4.9) seriously for values of ω of the order $\sqrt{-1/\gamma}$, then one is led to the conclusion that the threshold of contrast## at which a person perceives the presence of a periodic pattern

##Michelson [1927, 1] defined the contrast of a grating to be the difference between the maximum and minimum luminance divided by twice the mean luminance. Schade [1956, 2], Campbell & Green [1965, 1], and Campbell & Robson [1968, 1], have devised methods of experimentally determining Schade's contrast sensitivity function, which gives the reciprocal of the threshold of contrast for perceiving the presence of a periodic grating pattern as a function of the spatial frequency of the pattern.

should be a minimum at a spatial frequency near $\sqrt{-1/\gamma}$. These results would also indicate that for spatial frequencies ω near the "critical frequency", $\sqrt{-1/\gamma}$, it should be difficult for an observer to judge, within a factor of two, the "true frequency" of a periodic excitation pt field, even if the field describes a pattern of high contrast: for such pt frequencies, C/B is large, and the response field (4.5) differs in shape from the excitation field (4.6). It is, however, dangerous to draw such conclusions from formulae which rest on Theorem 1 and are, therefore, valid only for low spatial frequencies.

The equation (3.18) implies that if, for $x \in (-\infty,\infty)$, the response field ψ has the form

$$\psi \quad = \quad A + \frac{1}{2}\,\Phi_2\,, \tag{4.12}$$

with Φ_2 the second derivative of the error integral, Φ_c, i.e.

$$\Phi_1(x) \;=\; \frac{2}{\sqrt{\pi}}\, e^{-x^2}, \qquad \Phi_2(x) \;=\; \frac{-4x}{\sqrt{\pi}}\, e^{-x^2}, \qquad \Phi_n(x) \;=\; \frac{d^n\Phi_1}{dx^n}, \tag{4.13}$$

then ϵ must have the form

$$\epsilon \quad = \quad \psi - \beta\psi'^2 - \gamma\psi'' = \; A + \frac{1}{2}\left(\Phi_2 - \beta\Phi_3^2 - \gamma\Phi_4\right). \tag{4.14}$$

Assuming, as we have been doing in this section, that γ and β are constant, and choosing the values $\gamma = -10^{-1}$ and $\beta = -10^{-2}$ for these visual parameters,#

#If x is measured in degrees, then γ has the dimension of degrees^{-2}. If $-\gamma = 10^{-1}$, then the quantity $x/\sqrt{-\gamma}$ equals 1 when x is approximately 19' (i.e., 0.316°) or when x equals the visual angle subtended by a marker of length 0.32 cm viewed at a distance of 58 cm. Of course, the magnitude of β depends on the units used for ψ (and ϵ).

one may easily draw graphs of response field (4.12) and the corresponding excitation field (4.14).## Such graphs (with A put equal to 1) are shown

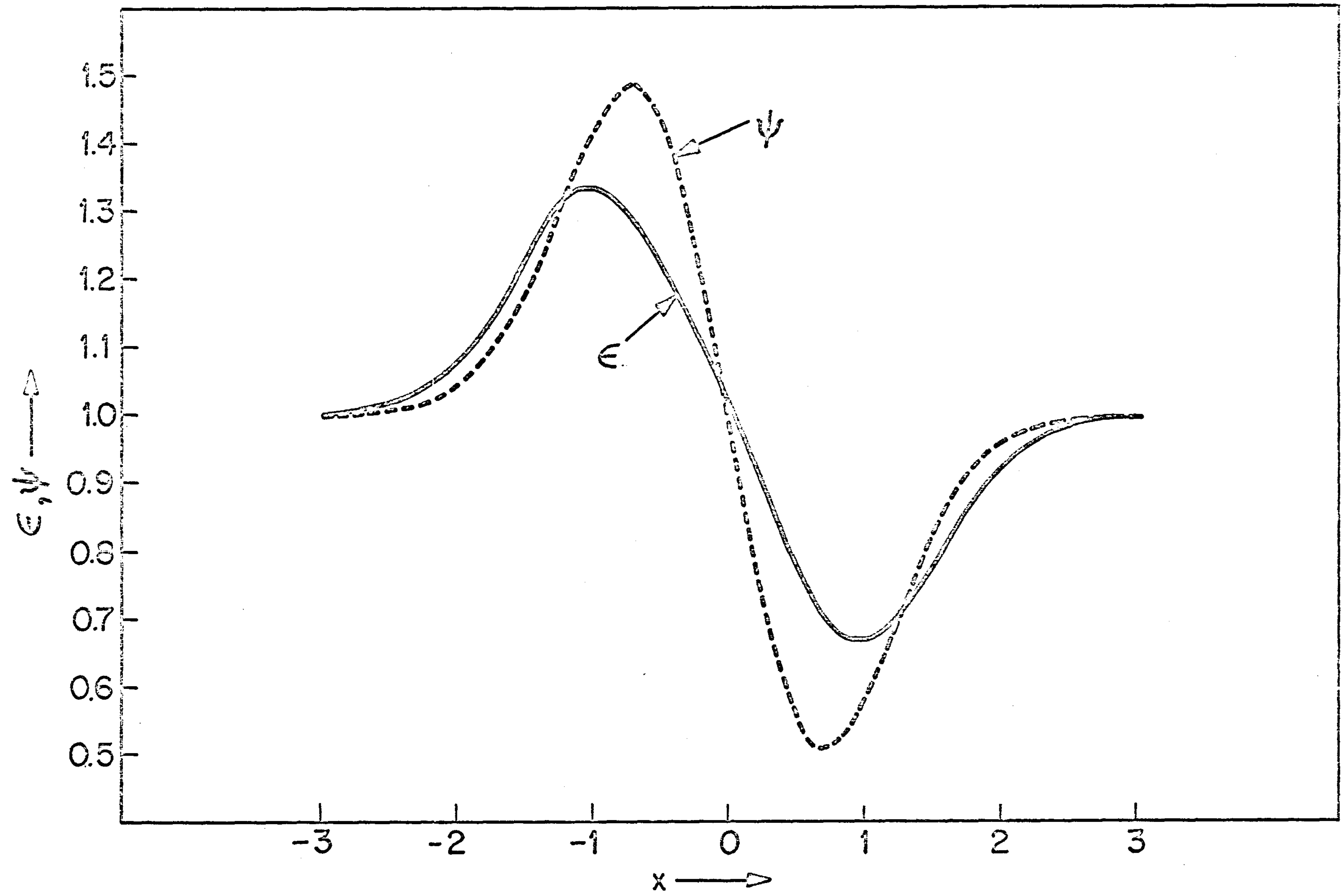
##The tables of Jahnke and Emde [1945, 1] facilitate the calculations.

in Figure 1. Note that here the primary effect of lateral inhibition is to produce, as expected, an intensification of contrast; that is, the response field shows a higher maximum and lower minimum than the excitation field. However, the maximum and the minimum in ψ are displaced slightly from the corresponding extrema in ϵ.

The extrema in response are here closer together than those in excitation. Perhaps an experimentor will find it possible to construct "optical illusions" which make use of such theoretically predicted shifts in the locations of extrema upon passage from luminance to subjective brightness.###

###In a subsequent article I shall discuss this point more fully and give solutions of (3.15) which bear on the demonstrations of O'Brien [1958, 3] and Cornsweet [1970, 1, page 274].

Figure 1. Dashed curve: graph of the response field (4.12) with A = 1. Solid curve: graph of the corresponding excitation field (4.14), assuming $\gamma = -10^{-1}$ and $\beta = -10^{-2}$.



1.5
1.4
1.3
1.2
1.1
1.0
0.9
0.8
0.7
0.6
0.5
ϵ, ψ
ψ
ϵ
-3
-2
-1
0
1
2
3
x

For each number D, let $\mathfrak{C}_D$ be the class of excitation fields ϵ for which there exist response fields ψ such that

$$\epsilon - \psi = D, \tag{4.15}$$

i.e. for which (4.1) has solutions ψ which differ from ϵ by a constant shift downward, equal to D. It is clear that, if βD is not positive, linear fields (4.3) are in $\mathfrak{C}_D$, but $\mathfrak{C}_D$ contains less trivial fields.

It follows from (4.1) and (4.15) that the elements ϵ of $\mathfrak{C}_D$ obey the equation

$$-\beta\epsilon'^2 - \gamma\epsilon'' = D. \tag{4.16}$$

Putting now

$$\zeta(\chi) = \frac{\beta}{\gamma}\epsilon(x), \qquad \chi = \frac{1}{\sqrt{-\gamma}}x, \qquad E = \frac{\beta}{\gamma}D, \tag{4.17}$$

one may write (4.16) in the form

$$\zeta'^2 + \zeta'' = E. \tag{4.18}$$

If E is positive, the solutions of this equation are#

#log denotes the natural logarithm (with base e).

$$\zeta(\chi) = c \pm\sqrt{E}\chi, \quad \text{if} \quad \zeta'(0+)^2 = E, \tag{4.19a}$$

$$\zeta(\chi) = c + \log\sinh(\pm\sqrt{E}\chi+\delta), \quad \text{if} \quad \zeta'(0+)^2 > E, \tag{4.19b}$$

$$\zeta(\chi) = c + \log\cosh(\pm\sqrt{E}\chi+\delta), \quad \text{if} \quad \zeta'(0+)^2 < E. \tag{4.19c}$$

When E is negative, for each choice of $\zeta'(0+)$, (4.18) has a solution of the form

$$\zeta(\chi) = c + \log \sin(\pm\sqrt{-E}\chi+\delta), \qquad \delta\in[0,\pi). \tag{4.20}$$

When E is zero, (4.18) has the solutions

$$\zeta(\chi) = c, \tag{4.21a}$$

$$\zeta(\chi) = c + \log(\chi+\delta). \tag{4.21b}$$

In (4.19)-(4.21), c and δ are constants.

The following proposition follows immediately from (4.17), and (4.19)-(4.21): <u>The linear excitation field (4.3) is in $\mathfrak{C}_D$, if and only if $\beta D \leq 0$ and $\epsilon'(0+)^2 = -D/\beta$, in which case,</u>

$$B = \sqrt{-D/\beta}, \qquad A = \epsilon(0+). \tag{4.22}$$

<u>If $\beta D < 0$, and $\epsilon'(0+)^2 > -D/\beta$, the field</u>

$$\left.\begin{aligned} \epsilon(x) &= A + \frac{\gamma}{\beta}\log\sinh\left(B \pm x\sqrt{-\beta D/\gamma^2}\right), \\ B &= \pm\mathrm{ctnh}^{-1}\left(\frac{\epsilon'(0+)}{\sqrt{-D/\beta}}\right), \qquad A = \epsilon(0+) - \frac{\gamma}{\beta}\log\sinh B \end{aligned}\right\} \tag{4.23}$$

<u>is in $\mathfrak{C}_D$. If $\beta D < 0$ and $\epsilon'(0+)^2 < -D/\beta$, then the field</u>

$$\left.\begin{aligned} \epsilon(x) &= A + \frac{\gamma}{\beta}\log\cosh\left(B \pm x\sqrt{-\beta D/\gamma^2}\right), \\ B &= \pm\tanh^{-1}\left(\frac{\epsilon'(0+)}{\sqrt{-D/\beta}}\right), \\ A &= \epsilon(0+) - \frac{\gamma}{\beta}\log\cosh B \end{aligned}\right\} \tag{4.24}$$

is in $\mathfrak{C}_D$. *If* $\beta D > 0$, *then an excitation field* ϵ *is in* $\mathfrak{C}_D$ *if and only if*

$$\left.\begin{aligned} \epsilon(x) &= A + \frac{\gamma}{\beta}\log\sin\left(B \pm x\sqrt{\beta D/\gamma^2}\right), \\ B &= \pm\operatorname{ctn}^{-1}\left(\frac{\epsilon'(0+)}{\sqrt{D/\beta}}\right), \\ A &= \epsilon(0+) - \frac{\gamma}{\beta}\log\sin B. \end{aligned}\right\} \tag{4.25}$$

If $D = 0$, *then* ϵ *is in* $\mathfrak{C}_D = \mathfrak{C}_0$ *if and only if either the equation* (4.3) *holds with* $B = \epsilon'(0+) = 0$, *or* $\epsilon'(0+) \neq 0$ *and* ϵ *has the form*

$$\left.\begin{aligned} \epsilon(x) &= A + \frac{\gamma}{\beta}\log(B \pm x/\sqrt{-\gamma}), \\ B = \frac{\mp\sqrt{-\gamma}}{\epsilon'(0+)\beta}, \qquad A &= \epsilon(0+) - \frac{\gamma}{\beta}\log B. \end{aligned}\right\} \tag{4.26}$$

The fields (4.3), (4.23)-(4.26) *are the only excitation fields for which* (4.1) *has a solution* ψ *obeying* (4.15) *with* D *constant*.

The proposition just given leads naturally to the following general problem: Given $\mathcal{R}$, a region in E, and given f, a real-valued function on $\mathcal{R}$, find $\mathfrak{B}(\mathcal{R};f)$, the set of all twice-differentiable response fields ψ on $\mathcal{R}$ which obey (3.15) with

$$\psi(\underset{\sim}{x}) - \epsilon(\underset{\sim}{x}) = f(\underset{\sim}{x}) \qquad \text{for all } \underset{\sim}{x}\in\mathcal{R}. \tag{4.27}$$

When β and γ are constants, independent of ψ, this problem of finding the response fields which differ from the corresponding excitation fields by a prescribed function can be reduced to the problem of solving a *linear* second-order differential equation. This is the content of the following

Remark. *Let* β *and* γ *be constants obeying* (4.2) ~~(4.12)~~, $\mathcal{R}$ *an open subset of* E, $\mathbf{x}_\circ$ *a fixed point in* E, *and* $\mathcal{N}$ *the image in* E *of* $\mathcal{R}$ *under the linear transformation* $\mathbf{x} \mapsto \mathbf{y}$ *with*

$$\mathbf{y} = \mathbf{x}_\circ + \frac{\mathbf{x} - \mathbf{x}_\circ}{\sqrt{-\gamma}} . \tag{4.28}$$

Suppose f *is a prescribed real-valued function on* $\mathcal{R}$.
A twice-differentiable real-valued function ψ *on* $\mathcal{R}$ *obeys the equation*

$$f(\mathbf{x}) = \beta\big(\nabla\psi(\mathbf{x})\big)^2 + \gamma\Delta\psi(\mathbf{x}) \tag{4.29}$$

[i.e. *is in* $\mathfrak{B}(\mathcal{R};f)$], *if and only if the positive function* u, *defined by*

$$u(\mathbf{y}) = \exp\left\{\frac{\beta}{\gamma}\,\psi\big(\sqrt{-\gamma}(\mathbf{y}-\mathbf{x}_\circ) + \mathbf{x}_\circ\big)\right\}, \tag{4.30}$$

obeys, at each $\mathbf{y}$ *in* $\mathcal{N}$, *the equation*

$$\Delta u(\mathbf{y}) + g(\mathbf{y})u(\mathbf{y}) = 0 \tag{4.31}$$

with

$$g(\mathbf{y}) = \frac{\beta}{\gamma}\, f\big(\sqrt{-\gamma}(\mathbf{y}-\mathbf{x}_\circ) + \mathbf{x}_\circ\big). \tag{4.32}$$

To verify the remark, one need merely note that if u is defined on $\mathcal{N}$ by (4.30), then u is twice-differentiable, and, whenever $\mathbf{x}$ and $\mathbf{y}$

are related by (4.28),

$$\Delta u(\underset{\sim}{y}) = -\left[\left(\beta^2 (\nabla\psi)^2 + \frac{\beta}{\gamma}\,\Delta\psi\right) e^{\beta\psi/\gamma}\right]_{\underset{\sim}{x}} = -\frac{\beta}{\gamma}\left[\beta(\nabla\psi)^2 + \gamma\Delta\psi\right]_{\underset{\sim}{x}} u(\underset{\sim}{y}). \qquad (4.33)$$

Let g be as in (4.32). Clearly (4.29) and (4.33) imply (4.31). Furthermore, as u is, by (4.30), never zero, the relations (4.31) and (4.33) imply (4.29).

5. On Infinitesimal Variations

In this section I assume that $\mathfrak{J}$ is a tame function of type $n \geq 2$ and that, for each $c > 0$, the domain $\mathfrak{D}_c$ of $\mathfrak{J}(\cdot;c)$ is a set of the form (2.9) with $1 \leq p < \infty$. Thus, for some p in $[1,\infty)$ and some positive continuous function h obeying (2.5) with $r > 2+\frac{2}{p}$, $\mathfrak{D}_c$ is, for each c, a subset of $\mathcal{L}_{h,p}$.

If λ_* is in $\mathfrak{D}_c$, then, by (2.11),

$$\begin{aligned} \mathfrak{J}(\lambda_*;c) &= \delta\mathfrak{J}_c[\lambda_* - c^\dagger] + \frac{1}{2}\,\delta^2\mathfrak{J}_c[\lambda_* - c^\dagger] + o\left(\|\overset{\lambda_*-c^\dagger}{\cancel{\phi_*}}\|^2_{h,p}\right) \\ &= \delta\mathfrak{J}_c[\lambda_* - c^\dagger] + o\left(\|\lambda_* - c^\dagger\|^2_{h,p}\right). \end{aligned} \tag{5.1}$$

It follows from the fundamental representation theorem for continuous linear functionals on function spaces of type L_p,[#] that for each c there

[#] F. Riesz [1910, 1, §11]. See also Riesz & Sz.-Nagy [1955, 2, p. 78].

exists a real-valued, measurable function K_c on V such that

$$\delta\mathfrak{J}_c[\phi_*] = \int_V K_c(\underset{\sim}{v})\phi_*(\underset{\sim}{v})d\underset{\sim}{v}, \quad \text{for every } \phi_* \in \mathcal{L}_{h,p}; \tag{5.2}$$

if $1 < p < \infty$, then

$$\int_V \left|\frac{K_c(\underset{\sim}{v})}{h(|\underset{\sim}{v}|)}\right|^q d\underset{\sim}{v} < \infty, \quad \text{with } q = \frac{p}{p-1}; \tag{5.3}$$

if $p = 1$, then

$$\operatorname*{ess\,sup}_V \left|\frac{K_c(\underset{\sim}{v})}{h(|\underset{\sim}{v}|)}\right| < \infty. \tag{5.4}$$

It is a consequence of ~~(2.3)~~ (2.13) and (5.1) that, for each $\mathbf{G}$ in $\mathcal{O}$,

$$\delta\mathfrak{J}_c[\phi_* \circ \mathbf{G}] \;=\; \delta\mathfrak{J}_c[\phi_*], \quad \text{for } \phi_* \in \mathcal{L}_{h,p}, \tag{5.5}$$

which, by (5.2), implies that K_c obeys, for each $\mathbf{G}$ in $\mathcal{O}$, the identity

$$K_c(\mathbf{v}) = K_c(\mathbf{G}\mathbf{v}), \quad \text{for all } \mathbf{v} \in V. \tag{5.6}$$

However, this can be the case only if there exists a function $k_c(\cdot)$ on $[0,\infty)$ such that

$$K_c(\mathbf{v}) = k_c(|\mathbf{v}|), \quad \text{for all } \mathbf{v} \in V. \tag{5.7}$$

Now, let ψ be a response field, let μ be a positive number, and suppose that

$$\iota \;\overset{\text{def}}{=\!=}\; \sup_E |\psi(\mathbf{x}) - \mu| \tag{5.8}$$

is finite. Then for each $\mathbf{x}$ in E,

$$\|\psi_{\mathbf{x}} - \mu^\dagger\|_{h,p} \;\leq\; B\iota, \tag{5.9}$$

where, by (2.6) and (2.~~4~~5) $\left(\text{with } r > 2 + \frac{2}{p}\right)$,

$$B \;=\; \sqrt[p]{\int_V h(|\mathbf{v}|)^p dv} \;<\; \infty. \tag{5.10}$$

Hence (2.4) and ~~(5.10)~~ (5.1) yield the following relation between ψ and the excitation field ϵ determined by ψ:

$$\psi(\mathbf{x}) \;=\; \epsilon(\mathbf{x}) + \delta\mathfrak{J}_\mu[\psi_{\mathbf{x}} - \mu^\dagger] + \partial\mathfrak{J}(\mu^\dagger;\mu)(\psi(\mathbf{x}) - \mu) + o(\iota^2), \tag{5.11}$$

where

$$\partial\mathfrak{J}(\mu^{\dagger};\mu) \;\stackrel{\text{def}}{=\!=}\; \frac{\partial}{\partial c}\,\mathfrak{J}(\mu^{\dagger};c)\Big|_{c=\mu}. \tag{5.12}$$

Now, if $\psi(\underset{\sim}{x}) = b$ for all $\underset{\sim}{x}$, then, by the definition of ϵ, we must have $\epsilon(\underset{\sim}{x}) \equiv b$, and, therefore, by (5.11), $\mathfrak{J}$ must be such that

$$\delta\mathfrak{J}_{\mu}[b^{\dagger}-\mu^{\dagger}] + \partial\mathfrak{J}(\mu^{\dagger};\mu)(b-\mu) \;=\; 0. \tag{5.13}$$

This equation, which also follows directly from (2.10), yields, by (5.2) and (5.7),

$$\int_V k_{\mu}(|\underset{\sim}{v}|)\,d\underset{\sim}{v} \;=\; -\partial\mathfrak{J}(\mu^{\dagger};\mu). \tag{5.14}$$

Employing (2.3), (5.2), (5.7), and ~~(5.4)~~ (5.14), one may write (5.11) in the form

$$\psi(\underset{\sim}{x}) \;=\; \epsilon(\underset{\sim}{x}) + \int_V k_{\mu}(|\underset{\sim}{v}|)\big(\psi(\underset{\sim}{x}+\underset{\sim}{v}) - \psi(\underset{\sim}{x})\big)\,d\underset{\sim}{v} + O(\iota^2). \tag{5.15}$$

The integral over V appearing here is $O(\iota)$, and hence if $\psi(\underset{\sim}{x})$ is close to the constant μ for all $\underset{\sim}{x}$, the equation obtained by striking out the term $O(\iota^2)$ in (5.15), i.e.

$$\psi(\underset{\sim}{x}) \;=\; \epsilon(\underset{\sim}{x}) + \int_E [\psi(\underset{\sim}{z}) - \psi(\underset{\sim}{x})]\,k_{\mu}(|\underset{\sim}{z}-\underset{\sim}{x}|)\,d\underset{\sim}{z}, \tag{5.16}$$

approximates the basic constitutive equation (2.4).

Linear equations related to (5.16) have been studied by Davidson [1968, 2] and Ratliff, Knight, & Graham [1969, 2]; of course, in the present theory (5.1~~6~~5) holds only for infinitesimal variations from a uniform response field μ.

When $\mathfrak{J}$ and μ are specified, the function $k_\mu(\cdot)$ in (5.1$\cancel{2}$5) is determined almost everywhere on $[0,\infty)$. The relation (3.29) is here equivalent to the assertion that

$$k_\mu(s) \leq 0 \quad \text{for almost all } s \geq 0, \tag{5.17}$$

while (3.30) is equivalent to

$$k_\mu(s) < 0 \quad \text{for almost all } s \geq 0. \tag{5.18}$$

For fields which vary only in the x-direction, (5.1$\cancel{2}$5) becomes

$$\psi(x) = \epsilon(x) + \int_{-\infty}^{\infty} \bar{k}_\mu(|s|)[\psi(x+s) - \psi(x)]ds + o(\iota^2), \tag{5.19}$$

with

$$\bar{k}_c(\sigma) \overset{\text{def}}{=\!=} \int_{-\infty}^{\infty} k_c\left([\sigma^2 + \eta^2]^{\frac{1}{2}}\right)d\eta, \qquad \sigma \geq 0, \quad c > 0. \tag{5.20}$$

The functions $\gamma(\cdot)$ and $\cancel{\kappa}\,\theta(\cdot)$ occurring in Theorem 2 are given by $(3.25)_1$ and $(3.11)_2$, which equations can here be written

$$\gamma(c) = \int_0^\infty \sigma^2 \bar{k}_c(\sigma)d\sigma, \tag{5.21}$$

$$\theta(c) = \int_0^\infty \sigma \bar{k}_c(\sigma)d\sigma. \tag{5.22}$$

As an application of (5.19), suppose that ψ has the form

$$\psi(x) = \mu + b \sin \omega x \tag{5.23}$$

with $\mu > 0$ and $0 < b < \mu$. Then (5.8) yields $\iota = b$, and (5.19) becomes

$$\mu + b \sin \omega x = \epsilon(x) + b\int_{-\infty}^{\infty} \bar{k}_{\mu}(|s|)[\sin(\omega x + \omega s) - \sin(\omega x)]ds + O(b^2); \tag{5.24}$$

that is,

$$\epsilon(x) = \mu + b[1 + \hat{k}_{\mu}(0) - \hat{k}_{\mu}(\omega)] \sin \omega x - O(b^2), \tag{5.25}$$

where

$$\hat{k}_{\mu}(\omega) = 2\int_{0}^{\infty} \bar{k}_{\mu}(\sigma) \cos \omega\sigma \, d\sigma, \tag{5.26}$$

and

$$\hat{k}_{\mu}(0) = 2\int_{0}^{\infty} \bar{k}_{\mu}(\sigma) d\sigma = \int_{V} k_{\mu}(|\underset{\sim}{v}|) dv = \lim_{\omega \to 0} \hat{k}(\omega). \tag{5.27}$$

Conversely, if it is known that ϵ has the form

$$\epsilon(x) = \mu + B \sin \omega x, \tag{5.28}$$

then, under neglect of terms $O(b^2)$, (5.19) has a solution of the form (5.23) with

$$\frac{b}{B} = F_{\mu}(\omega) = \frac{1}{1 + \hat{k}_{\mu}(0) - \hat{k}_{\mu}(\omega)} . \tag{5.29}$$

The function F_{μ} may be called the "spatial transfer function for infinitesimal variations about μ". If we use primes to denote the derivatives of $\hat{k}_{\mu}$, then, in view of (5.26) and (5.21), we have

$$\hat{k}'_{\mu}(0) = 0, \qquad \hat{k}''_{\mu}(0) = 2\gamma(\mu); \tag{5.30}$$

indeed,

$$\hat{k}_\mu(\omega) = \hat{k}_\mu(0) - \gamma\omega^2 + O(\omega^4), \tag{5.31}$$

and (5.~~30~~29)$_2$ yields

$$F_\mu(\omega) = 1 - \gamma(\mu)\omega^2 + O(\omega^4), \tag{5.32}$$

in agreement with (4.11).

Employing the number $\hat{k}_\mu(0)$ of (5.27), one may write the general relation (5.15) in the form

$$\epsilon(\underset{\sim}{x}) = \left(1 + \hat{k}_\mu(0)\right)\psi(\underset{\sim}{x}) + \int_V k_\mu(|\underset{\sim}{y}|)\psi_{\underset{\sim}{x}}(\underset{\sim}{y})d\underset{\sim}{y} + O(\iota^2). \tag{5.33}$$

The function k_μ occurring here, and in (5.15)-(5.20), is a Hankel transform of the function $\hat{k}_\mu$ in (5.26)-(5.31):

$$k_\mu(\sigma) = \frac{1}{2\pi}\int_0^\infty \hat{k}_\mu(\omega)J_0(\omega\sigma)\omega\, d\omega. \tag{5.34}$$

The linear theory derived in this section by neglecting terms $O(\iota^2)$ has a different range of applicability than the asymptotic theory obtained in Section 3 as a consequence of Theorems 1 and 2.

A theory based on (5.16) is valid only for response fields ψ
pt which are infinitesimally close, for all $\underset{\sim}{x}$, to a uniform field μ. In pt
deriving (5.16) one neglects the term in (5.1) arising from the second Fréchet-derivative $\underset{\sim}{\mathfrak{J}}$, of $\delta^2\mathfrak{J}_c$. This second derivative determines the quantities β and π occurring in the theory of Section 3, and knowledge of the function k_μ in (5.16) does not yield β and π.

The field equations (3.15) and (3.26), on the other hand, as they are derived from retardation theorems, are valid only for response fields which do not vary rapidly from point to point in E, albeit these equations do not, in principle, require that ψ remain close to a constant field for all $\underset{\sim}{x}$ in E.#

pt

#In Section 4 it is assumed, however, that ψ remains close to a constant on a subregion of the visual field; this assumption permits one to treat β and γ as constants.

pt

In summary: For fields that are four-times differentiable, the non-linear equation (3.15) is valid to within an error of order four in α, the "scale of distance", while (5.16) is, in general, valid to within an error of order two in ι, the "variation from a constant field".

Acknowledgement. This research was supported in part by the U. S. Air Force Office of Scientific Research under Grant AF-68-1538.

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