GENERAL MODELS AND EXTENSIONALITY

by

Peter B. Andrews

Report 71-30

June 1971

/ps -- 6/18/71

GENERAL MODELS AND EXTENSIONALITY

by

Peter B. Andrews

Abstract

Construction of a general model falsifying the Axiom of Extensionality shows that the general models of Henkin[1] s article "Completeness in the Theory of Types" are not all sound interpretations of the system. A modification of the definition of general model remedies the situation.

GENERAL MODELS AND EXTENSIONALITY

by

Peter B. Andrews

§1. Introduction.

It is well known that equality is definable in type theory. Thus, in the language of [2], the equality relation between elements of type α is definable as $[\lambda x_\alpha \lambda y_\alpha \forall p_{o\alpha} . p_{o\alpha} x_\alpha \supset p_{o\alpha} y_\alpha]$, i.e., $x_\alpha = y_\alpha$ iff every set which contains x_α also contains y_α. However, in a non-standard model of type theory, the sets may be so sparse that the wff above does not denote the true equality relation. We shall use this observation to construct a general model in the sense of [2] in which the Axiom of Extensionality is not valid. Thus Theorem 2 of [2] is technically incorrect. However, it is easy to remedy the situation by slightly modifying the definition of general model.

Naturally, our construction provides an independence proof for the Axiom Schema of Extensionality.

We shall assume familiarity with, and use the notation of, [2] and §§2-3 of [1].

This research was partially supported by NSF Grant GJ-580.

§2. A non-extensional general model.

The language of [2] has primitive logical constants $N_{(oo)}$, $A_{((oo)o)}$, $\Pi_{(o(o\alpha))}$, $\iota_{(\alpha(o\alpha))}$, whereas the language $\mathcal{L}$ of [1] has primitive logical constants $Q_{((o\alpha)\alpha)}$. By modifying the proof of Theorem 1 of [1] in the obvious way, one obtains the following:

Proposition. A frame $\{\mathcal{A}_\alpha\}_\alpha$ is a general model in the sense of [2] iff it satisfies all of the following conditions (for all type symbols α, β, γ):

(a1) $\mathcal{A}_{oo}$ contains the negation function n such that $nt = f$ and $nf = t$.

(a2) $\mathcal{A}_{oo}$ contains $(\lambda x_o t)$ and $(\lambda x_o x_o)$. Also, $\mathcal{A}_{((oo)o)}$ contains the alternation (disjunction) function a such that $at = (\lambda x_o t)$ and $af = (\lambda x_o x_o)$.

(a3) $\mathcal{A}_{o(o\alpha)}$ contains a function $\pi_{o(o\alpha)}$ such that for all $g \in \mathcal{A}_{o\alpha}$, $\pi_{o(o\alpha)} g = t$ iff $g = (\lambda x_\alpha t)$.

(a4) $\mathcal{A}_{\alpha(o\alpha)}$ contains a function $i_{\alpha(o\alpha)}$ such that if g is any non-empty set in $\mathcal{A}_{o\alpha}$, $i_{\alpha(o\alpha)} g$ is in g.

(b) For all $x \in \mathcal{A}_\alpha$, $(\lambda y_\beta x) \in \mathcal{A}_{\alpha\beta}$.

(c) $(\lambda x_\alpha \lambda y_\beta x_\alpha) \in \mathcal{A}_{\alpha\beta\alpha}$.

(d) For all $x \in \mathcal{A}_{\alpha\beta\gamma}$ and $y \in \mathcal{A}_{\beta\gamma}$, $(\lambda z_\gamma . x z_\gamma . y z_\gamma) \in \mathcal{A}_{\alpha\gamma}$.

(e) For all $x \in \mathcal{A}_{\alpha\beta\gamma}$, $(\lambda y_{\beta\gamma} \lambda z_\gamma . x z_\gamma . y_{\beta\gamma} z_\gamma) \in \mathcal{A}_{\alpha\gamma(\beta\gamma)}$.

(f) $(\lambda x_{\alpha\beta\gamma}\lambda y_{\beta\gamma}\lambda z_{\gamma}\cdot x_{\alpha\beta\gamma}z_{\gamma}\cdot y_{\beta\gamma}z_{\gamma}) \in \mathcal{D}_{\alpha\gamma(\beta\gamma)(\alpha\beta\gamma)}$

Theorem. There is a general model in the sense of [2] in which the Axiom of Extensionality

$$\forall x_{\iota}\,[f_{\iota\iota}x_{\iota} = g_{\iota\iota}x_{\iota}] \supset .\, f_{\iota\iota} = g_{\iota\iota}$$

is not valid.

Proof. We construct a frame $\mathcal{D} = \{\mathcal{D}_{\alpha}\}_{\alpha}$ by induction on α. Simultaneously we define three equivalence relations $\stackrel{1}{=}$, $\stackrel{2}{=}$ and $\stackrel{3}{=}$ on each of the $\mathcal{D}_{\alpha}$. When it is more convenient to do so, we shall define $\stackrel{i}{=}$ in terms of the partition (set of equivalence classes) P_i^{α} of $\mathcal{D}_{\alpha}$ induced by $\stackrel{i}{=}$. A statement about $\stackrel{i}{=}$ is meant to apply to each of $\stackrel{1}{=}$, $\stackrel{2}{=}$ and $\stackrel{3}{=}$.

$\mathcal{D}_{o} = \{t,f\}$. $x_{o} \stackrel{i}{=} y_{o}$ iff $x_{o} = y_{o}$.

$\mathcal{D}_{\iota} = \{\ell,m,n\}$, where ℓ, m, n are distinct individuals.

$P_1^{\iota} = \{\{m,n\},\{\ell\}\}$. P_2^{ι} P_3^{ι}

Given $\mathcal{D}_{\alpha}$ and $\mathcal{D}_{\beta}$, let $\mathcal{D}_{\alpha\beta}$ be the set of all functions g from $\mathcal{D}_{\beta}$ into $\mathcal{D}_{\alpha}$ such that for all u and v in $\mathcal{D}_{\beta}$, if $u \stackrel{1}{=} v$ then $gu \stackrel{1}{=} gv$, and if $u \stackrel{2}{=} v$ then $gu \stackrel{2}{=} gv$, and if $u \stackrel{3}{=} v$ then $gu \stackrel{3}{=} gv$. If g and h are in $\mathcal{D}_{\alpha\beta}$, let $g \stackrel{i}{=} h$ iff for all $x \in \mathcal{D}_{\beta}$, $gx \stackrel{i}{=} hx$.

Having defined the frame $\mathcal{D}$, we use the Proposition above to show that it is a general model.

(a) Since $\stackrel{i}{=}$ is trivial on $\mathcal{D}_o$, $\mathcal{D}_{\alpha o}$ contains all functions from $\mathcal{D}_o$ into $\mathcal{D}_\alpha$. Hence (a_1) and (a_2) are satisfied. Also, if u and v are in $\mathcal{D}_{o\gamma}$, then $u \stackrel{i}{=} v$ iff $u = v$. Hence $\mathcal{D}_{\alpha(o\gamma)}$ contains all functions from $\mathcal{D}_{o\gamma}$ into $\mathcal{D}_\alpha$. Thus (a_3) and (a_4) are satisfied.

(b) Clearly $(\lambda y_\beta x_\alpha) \in \mathcal{D}_{\alpha\beta}$ since this is a constant function.

(c) If $u, v \in \mathcal{D}_\alpha$ and $u \stackrel{i}{=} v$, then $(\lambda y_\beta u) \stackrel{i}{=} (\lambda y_\beta v)$, so $(\lambda x_\alpha \lambda y_\beta x_\alpha) \in \mathcal{D}_{\alpha\beta\alpha}$.

(d) Suppose $x \in \mathcal{D}_{\alpha\beta\gamma}$, $y \in \mathcal{D}_{\beta\gamma}$, $z^1, z^2 \in \mathcal{D}_\gamma$, and $z^1 \stackrel{i}{=} z^2$. Then $xz^1 \stackrel{i}{=} xz^2$ and $yz^1 \stackrel{i}{=} yz^2$ so $xz^1(yz^1) \stackrel{i}{=} xz^1(yz^2) \stackrel{i}{=} xz^2(yz^2)$, so $(\lambda z_\gamma . x z_\gamma . y z_\gamma) \in \mathcal{D}_{\alpha\gamma}$.

(e) Suppose $x \in \mathcal{D}_{\alpha\beta\gamma}$, $y^1, y^2 \in \mathcal{D}_{\beta\gamma}$ and $y^1 \stackrel{i}{=} y^2$. Then for each $z \in \mathcal{D}_\gamma$, $y^1 z \stackrel{i}{=} y^2 z$ so $xz(y^1 z) \stackrel{i}{=} xz(y^2 z)$, so $(\lambda z_\gamma . x z_\gamma . y^1 z_\gamma) \stackrel{i}{=} (\lambda z_\gamma . x z_\gamma . y^2 z_\gamma)$. Hence $(\lambda y_{\beta\gamma} \lambda z_\gamma . x z_\gamma . y_{\beta\gamma} z_\gamma) \in \mathcal{D}_{\alpha\gamma(\beta\gamma)}$.

(f) Suppose $x^1, x^2 \in \mathcal{D}_{\alpha\beta\gamma}$ and $x^1 \stackrel{i}{=} x^2$. Then for each $z \in \mathcal{D}_\gamma$ and $y \in \mathcal{D}_{\beta\gamma}$, $x^1 z \stackrel{i}{=} x^2 z$ so $x^1 z(yz) \stackrel{i}{=} x^2 z(yz)$ so

$(\lambda z_\gamma . x^1 z_\gamma . y z_\gamma) \stackrel{i}{=} (\lambda z_\gamma . x^2 z_\gamma . y z_\gamma)$ so

$(\lambda y_{\beta\gamma} \lambda z_\gamma . x^1 z_\gamma . y_{\beta\gamma} z_\gamma) \stackrel{i}{=} (\lambda y_{\beta\gamma} \lambda z_\gamma . x^2 z_\gamma . y_{\beta\gamma} z_\gamma)$ so

$(\lambda x_{\alpha\beta\gamma} \lambda y_{\beta\gamma} \lambda z_\gamma . x_{\alpha\beta\gamma} z_\gamma . y_{\beta\gamma} z_\gamma) \in \mathcal{D}_{\alpha\gamma(\beta\gamma)(\alpha\beta\gamma)}$.

Thus $\mathcal{M}^i$ is a general model in the sense of [2].

We next examine some of the domains $\mathcal{D}_\alpha$. $\mathcal{D}_{oi}$ contains only the constant functions $(\lambda x_1 t)$ and $(\lambda x_I f)$. Hence for any wffs A and B and any assignment <p, V [A4 = B] = t, since [A = B] is equivalent to Vp [p A => p B], where p does not occur free in A or B . Consequently UVxff x = g x 1 = t for any assignment (p.

It can be seen that $\& _{it} = \{ (\lambda x_t x_t), (\lambda x_t I_1), (\lambda x_i m_2), (\lambda x_i n) \}$. To verify this, note that geiS$^{\iota\iota}$ iff gm - gn, gl = gn, and g-t$_3$ = gm. One can examine the twenty-seven functions from $\&\iota$ into $£\iota$ to see that only the identity and constant functions satisfy all three of these properties. Alternatively, one can reason as follows: Suppose gt_1 = m. Then gl_3 = gn so gn = m. Also gm = gn so gm e (m,n)^ and gl = gm so gm € {£,m}; hence gm = m. Thus if gl = m, then g = $(\lambda x_\iota m)$. Similarly, if g£ = n, then g = (Ax n). Thus if gl 7^ I, then g is a constant function. Similarly if g m ^ m or g n ^ n, then g must be a constant function. Thus the only members of & are the constant and identity functions.

Note that P$^{11}_{j}$ = {{(Ax m$_i$), (Ax n)}, (M), {(Ax x$_i$)}),

9$^{\lambda x}_{\iota}$ = {((^t),(kn)), {(Axm)}, {(Ax x)}}, and

P3^1 = {{(Ax^),(Ax$_t$m)}$_5$ {(Ax$_t$n)}, {(Ax^^}}.

$\&^{o(\iota\iota)}$ contains a function h such that $h(\lambda x^\iota x^\iota)$ = t but h(Ax I) = h(Ax m) = h(Ax n) = f. Hence if cp is an assignment such that <pf = (Ax x) and <pg = (Ax I), then

$1/_{\varphi}[f_{11} = g_{tt}] = \backslash s_{o}\ Vp_{o(\iota t)}[p_{o(\iota t)} f_{11}\ 3\ p_{o(\iota\iota)} g_{\iota\iota}] = f.$ Hence

$V_{\varphi}\ FVx_t\ Ff_t\ x_t = g_{11}\ x_i] \Rightarrow .\ f_{tt} = g_{11}] = f$, and the Axiom of Extensionality is not valid in the general model to. O

§3. General models.

We suggest that the definition of general model in [2] should be modified by adding the following requirement:

(a_o) For each a, $\mathcal{E}_{o\alpha\alpha}$ contains the identity relation $q_{o\alpha\alpha}$ on $\mathcal{D}_a$ (and hence $\mathcal{D}_{oa}$ contains the unit set $q_{oaa}\ x_a$ for each $x \in \mathcal{D}_{a}$).

Of course, if this is done, clauses (a_1), (a_2), and (a_3) of the Proposition above become redundant. Indeed, $n = q_{ooo}\ f$, $a = (\lambda x_o \lambda y_o . n . q_{o(ooo)(o(ooo))} (\lambda g_{ooo} . g_{ooo}\ x_o\ y_o) (\lambda g_{ooo} . g_{ooo}\ ff))$, and $\pi_{o(oa)} = q_{o(oa)(oa)} (\lambda x_a t)$. Thus the modified definition of general model is equivalent to the result of adding a requirement concerning $\backslash s\ t_{\varphi\ a(oa)}$ to the definition of general model in [1].

With this definition, the general models constitute sound interpretations of the system of [2]. Moreover, the model constructed in the proof of Theorem 1 of [2] actually satisfies (a_o), since it can be seen that $\mathcal{D}([Q_{o\alpha\alpha}]) = q_{\alpha\alpha}$ (in the notation of that proof). Thus Theorem 2 of [2] becomes correct under the new definition of general model.

One of the appealing properties of the definition of general model in [2] is that it is generated in a very natural way by the formation rules for the language. Our modified definition no longer has this property for the language of [2], although it has it for a language in which $Q_{o\alpha\alpha}$ is taken as a primitive constant. Thus it appears that in contexts where one wishes to assume extensionality and discuss general models, a language such as £ of [1], augmented by a description or selection operator, is more natural than the language of [2].

Bibliography

[1] Peter B. Andrews, "General Models, Descriptions, and Choice in Type Theory", C-MU Mathematics Department Report 71-29.

[2] Leon Henkin, "Completeness in the Theory of Types", Journal of Symbolic Logic, vol.15(1950), pp.81-91.

Carnegie-Mellon University