GENERAL MODELS, DESCRIPTIONS AND CHOICE IN TYPE THEORY

by

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§1. Introduction

In [4] Alonzo Church introduced an elegant and expressive formulation of type theory with A-conversion. In [8] Henkin introduced the concept of a general model for this system, such that a sentence $\mathbf{A}$ is a theorem if and only if it is true in all general models. The crucial clause in Henkin's definition of a general model to is that for each assignment cp of values in to to variables and for each wff $\mathbf{A}$, there must be an appropriate value If$_\varphi$ $\mathbf{A}$ of j\ in to. Hintikka points out in [10,p.3] that this constitutes a rather strong requirement concerning the structure of a general model. Henkin draws attention to the problem of constructing non-standard models for the theory of types in [9,p.324].

We shall use a simple idea of combinatory logic to find a characterization of general models which does not directly refer to wffs, and which is easier to work with in certain contexts. This characterization can be applied, with appropriate minor and obvious modifications, to a variety of formulations of type theory with A-conversion. We shall be concerned with a language £ with extensionality in which there is no description or selection operator, and in which (for convenience)

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the sole primitive logical constants are the equality symbols Q for each type a*

We shall give two applications of this characterization. First, we show that the Axiom of Descriptions (D) is independent of £. This axiom is very natural since a general model for £ with a finite domain of individuals is standard if and only if D is true in it. Secondly, we show how the Fraenkel-Mostowski method ([7],[11],[12]) can be adapted to £. We state our fundamental lemma concerning this method in fairly general form to facilitate possible future applications (analogous to those for axiomatic set theory mentioned in [11]), but confine ourselves here to simply showing that the Axiom of Choice is not derivable in £, even if the Axiom of Descriptions is assumed.

When a description operator l«/ol\ ^s included among the primitive symbols, the axiom of descriptions may be taken in the form

$$\forall p_{o\iota}\,.\,\exists_1 x_\iota\, p_{o\iota}\, x_\iota \supset p_{o\iota}[\iota_{\iota(o\iota)}\, p_{o\iota}],$$

so that i^Qtj[AX A Q] (which is abbreviated (12^ A Q)) denotes the unique x_ι such that J\o, when there is such an x_ι. Church showed in [4] that description operators for higher types can be introduced by definition, using the operators for lower types. Specifically, l aj3(o(ag)) ™ay be defined as

tAh o(a/3) AX θ l a(oa) Ay a 3f αβ •h o(αβ) f αβ Ay =α f Xαβ β •

R. 0. Gandy has pointed out (in a private communication) that $^{l}o/?(o(o/3))$ can be defined as

$$[A h_{o(o0)} A x/3^{3f}o0 * h_{o(o0)} f_{o0} A f_{o0} V ']$$

so description operators for certain higher types can be defined without using those for any other type. Also, Henkin noted in [9] that $^{l}o/_{oo})$ can be defined as

$$[A h_{oo} ' h_{oo} = [A x_o x_o]] \bullet$$

(A number of other definitions of $i_{o(oo)}^{*}$ are also possible, of which the shortest is perhaps the closely related

$$Q_{o(oo)(oo)} [A x_o x_o] \bullet>$$

Thus it is seen that description operators for all types can be introduced once one has $t_{\iota(o\iota)}x$. The argument in [2,pp.22-24] shows that the description operator $i_{\iota(o\iota)}v$ cannot be introduced by definition for the simple reason that there are no closed wffs of this type, and that the axiom of descriptions mentioned above is independent, since it is the sole axiom which describes the special characteristics of $t_{\iota(o\iota)}^{*}$.

If no description operator is included in the list of primitive symbols, the axiom of descriptions may be taken in the form

$$^{3i}t(ot)^{V}Pot'^{a}i^{x}tPoi^{x}t^{3}Poi^{[i}t(ot)\,Pot^{]}*$$

or equivalently

(D) $$^{ai}t(oi)^{V}V\ ^{i}t(ot)^{[Q}oti^{x}t^{1=x}t-$$

(The equivalence results from the theorem

$$^{a}i^{x}tPot^{X}t = {}^{ax}fPot == -{}^{Q}on^{X}t-)$$

Since in many logical systems descriptions can be eliminated, it is very natural to ask whether the wff D, which asserts the existence of a description operator, is in fact derivable. It will be seen that our independence proof below is conceptually very simple, and is compatible with any axioms concerning the cardinality of the domain of individuals which permit it to have at least two members.

Church mentions in [5] an unpublished proof by Lagerström of a closely related independence result using a complete non-atomic Boolean algebra for the domain of truth values. It seems unlikely that Lagerstrom*s proof applies to £, since in £, unlike the system treated by Lagerström, there is a strong axiom of extensionality for type o (Axiom 1 below) which permits one to derive $[p_o = q_o]$ z> $.p_o = q_o$.

§2. <u>The Language</u> £

The language £ is essentially the result of dropping the description operator from the language Q_o of [2], and is closely related to the system discussed in [9]. For the convenience of the reader we here provide a description of £.

We use a,j3,y, etc., as syntactical variables ranging over <u>type symbols</u>, which are defined inductively as follows:

(a) o is a type symbol (denoting the type of truth values).

(b) i is a type symbol (denoting the type of individuals).

(c) $(\alpha\beta)$ is a type symbol (denoting the type of functions from elements of type β to elements of type α).

The primitive symbols of $\mathcal{Q}$ are the following:

(a) Improper symbols: [ ] λ

(b) For each α, a denumerable list of variables of type α:

$f_\alpha\ g_\alpha\ h_\alpha\ \ldots\ x_\alpha\ y_\alpha\ z_\alpha\ f^1_\alpha\ g^1_\alpha\ \ldots\ z^1_\alpha\ f^2_\alpha\ \ldots$

We shall use $f_\alpha, g_\alpha, \ldots, x_\alpha, y_\alpha, z_\alpha$, etc., as syntactical variables for variables of type α.

(c) For each α, $Q_{((o\alpha)\alpha)}$ is a constant of type $((o\alpha)\alpha)$.

We write wff$_\alpha$ as an abbreviation for wff of type α, and use $A_\alpha, B_\alpha, C_\alpha$, etc., as syntactical variables ranging over wffs$_\alpha$, which are defined inductively as follows:

(a) A primitive variable or constant of type α is a wff$_\alpha$.

(b) $[A_{\alpha\beta} B_\beta]$ is a wff$_\alpha$.

(c) $[\lambda x_\beta A_\alpha]$ is a wff$_{(\alpha\beta)}$.

An occurrence of x_α is bound (free) in B_β iff it is (is not) in a wf part of B_β of the form $[\lambda x_\alpha C_\delta]$. A_α is free for x_α in B_β iff no free occurrence of x_α in B_β is in a wf part of B_β of the form $[\lambda y_\gamma C_\delta]$ such that y_γ is a free variable of A_α.

Brackets, parentheses in type symbols, and type symbols may be omitted when no ambiguity is thereby introduced. A dot stands for a left bracket whose mate is as far to the right as is consistent with the pairing of brackets already present and with the formula being well formed. Otherwise brackets and parentheses are to be restored using the convention of association to the left.

We introduce the following definitions and abbreviations:

$[\mathbf{A}_\alpha = \mathbf{B}_\alpha]$ stands for $[Q_{o\alpha\alpha}\mathbf{A}_\alpha\mathbf{B}_\alpha]$.

T_Q stands for $[Q_{Qoo} = Q_{ooQ}]$.

F_Q stands for $[Ap_o p_Q] = [Ap_o T_Q]$.

$[\forall x_a A_o]$ stands for $[Ax_a A_o] = [Ax_a T_o]$.

^{A}ooo stands for [APoA<V tAgooo'gooo Po q o] = lAgooo'goooTo T_o]].

t3 $_{oo}$AB$_o$] stands for $[\wedge_{ooo}\mathbf{A}_o\mathbf{B}_o]$

$\supset_{ooo}$ stands for $[\lambda p_o \lambda q_o . p_o \wedge q_o = p_o]$.

$[A_o \supset B_o]$ stands for $[\supset_{ooo} A_o B_o]$.

Other connectives and quantifiers are introduced in familiar ways.

$\mathbf{K}^{\alpha\beta}$ and $K_{a\,apa}$ stand for [Ax $_a$ Ay$_{Q\,\beta}$ x$_a$].

$\mathbf{S}^{\alpha\beta\gamma}$ and $\mathbf{S}_{\alpha\gamma(\beta\gamma)(\alpha\beta\gamma)}$ stand for

$[\lambda \mathbf{x}_{\alpha\beta\gamma}\ \lambda \mathbf{y}_{\beta\gamma}$

$\mathbf{B}^{\alpha\beta\gamma}$ and $B_{ay(ft0(op)}$ stand for [A f ^ Ag$_{py}$Ax$_y$. f^.g^$_y$ x$_y$] .

$C^{\alpha\beta\gamma}$ and $C_{\alpha\gamma\beta(\alpha\beta\gamma)}$ stand for

$$[\lambda f_{\alpha\beta\gamma} \lambda x_\beta \lambda y_\gamma . f_{\alpha\beta\gamma} y_\gamma x_\beta].$$

$W^{\alpha\beta}$ and $W_{\alpha\beta(\alpha\beta\beta)}$ stand for $[\lambda f_{\alpha\beta\beta} \lambda x_\beta . f_{\alpha\beta\beta} x_\beta x_\beta]$.

$S^{x_\alpha}_{A_\alpha} B_\beta$ stands for the result of substituting A_α for x_α at all free occurrences of x_α in B_β.

£ has a single rule of inference, which is the following:

Rule R: From C_o and $[A_\alpha = B_\alpha]$ to infer the result of replacing one occurrence of A_α (which is not an occurrence of a variable immediately preceded by λ) in C_o by an occurrence of B_α.

The axioms and axiom schemata for £ are the following:

1 $[g_{oo} T_o \wedge g_{oo} F_o] = \forall x_o \, g_{oo} x_o$

2 $x_\alpha = y_\alpha \supset h_{o\alpha} x_\alpha = h_{o\alpha} y_\alpha$

3 $f_{\alpha\beta} = g_{\alpha\beta} = \forall x_\beta . f_{\alpha\beta} x_\beta = g_{\alpha\beta} x_\beta$

4 $[\lambda x_\alpha B_\beta] A_\alpha = S^{x_\alpha}_{A_\alpha} B_\beta$ where A_α is free for x_α in B_β.

Let us denote by JT& the system obtained when the axioms of extensionality (6.LI of [3]) are added to the list of axioms of the system JT of [3]. This is essentially the system of [8] or [4] using axioms 1-6,10°,10^{a}^, and with the selection operators deleted. ff£ differs from £ in having primitive constants ~oo^ooo, and $II_{o(oa)}$ instead of Q ^ . There are natural translations A from £ into 2T& and 7 from !I£ into £ which involve replacing the primitive constants of one language by appropriate closed wffs of the other language. For example, if A is a wff of £, AA is the result of replacing each occurrence of $Q_n aa$ in iL by the wff [Ax. Ay« Vf^.f-j x. z> f g y^] of ff£. It is easy to establish that & and JT& are equivalent in the sense that for each wff A of £ and B of ffC, k, A iff .L.p AA. and $K_{ff£}B_o$ iff h£ VB_Q; moreover, $h_£ £_Q = V_A A_o$ and |^e B_Q = ΔVB$_o$. Hence our independence proofs below apply also to JT&.

Definition. $£_\gamma$ is contractible to D_γ (C_γ contr D_γ) iff p_γ can be obtained from $£_\gamma$ by a sequence of zero or more applications of the following two rules of A-conversion:

I. (Alphabetic change of bound variables). To replace any wf part [Ax_α B_{fl}] of a wff by [Ay_α $S^{x_a}_{y}$ Bg]$_5$ provided that y_α is not free in B_o and y is free for x in B_Q .

II. (A-contraction). To replace any wf part [[Ax_α Bg]A_α] of a wff by S^{x}_{α} B_o. provided that A is free for x in B_a .

Definition. E. is a KS-combinatorial wff iff every occurrence of A in E. is in a wf part of E of the form v.*t or $S^{\alpha\beta\gamma}$.

E_δ is a KBCW-combinatorial wff iff every occurrence of A in E. is in a wf part of E. of the form $K^{\alpha\beta}$, B , $C^{\alpha\beta\gamma}$, or $W^{\alpha\beta}$.

Clearly K^^S$^{01\beta\gamma}$, and all primitive constants and variables are KS-combinatorial wffs. Also, [$A_{\alpha\beta}$ QBQ] is such a wff iff $A_{\alpha g}$ and Bg are.

We next show that every wff of £ is convertible to a KS-combinatorial wff$_5$ and to a KBCW-combinatorial wff. This requires only a simple translation into the present context of familiar facts about combinatory logic (see [6], [13], for example).

<u>Lemma 1</u>. For any KS-combinatorial wff B_{fi} and variable x_γ there is a KS-combinatorial wff Pg_γ such that $JP_{fi\gamma}$ contr [Ax_γ B_{fl}].

Proof: By induction on the number of occurrences of [in B_β.

Case a: B_o is x .

~ j3 -y

Let $P_{\gamma\gamma}$ be s$^{y(yy)y}$ $_K$$^{y(yy)}$ K^{yy}. Thus $P_{\gamma\gamma}$ contr [Az_γ.K$^{y\wedge yy\wedge}$ z_γ.K^{yy}z_γ] contr [$Az_\gamma z_\gamma$] contr [$Ax_\gamma B_\gamma$] .

Case b: B_o does not contain x free. Let P,, be K^{By} gg. Then $P_{g\gamma}$ contr [Ax_γ Bg] .

<u>Case c</u>: B_Q has the form [D_{ofi} E^]. By inductive

~.p ^ po —o

hypothesis there are KS-combinatorial wffs G_o* and H_f

— p^y —5y

such that G^$_{6y}$ contr [Ax_y p^] and H_{fiy} contr [Ax_y E_{fi}] .

Let P_{3y} be [S$^{\wedge 6y}$G$^\wedge$$_yH_{6y}$]. Thus JP^$_y$ contr [S$^{\wedge 5y}$[Ax_y $D_{\beta\delta}$][$\lambda x_\gamma E_\delta$]] contr [$Ax_y$. [$Ax_y D$^] x_y.[Ax^E_6] x_{fi}] contr [λx_γ B_β] .

Since every KS-combinatorial wff B_{β} falls under at least one of these three cases, this completes the proof of the lemma. □

Proposition 1. For every wff A_{α} of £ there is a KS-combinatorial wff P_{α} such that P_{α} contr A_{α}.

Proof: by induction on the number of occurrences of [in A_{α}.

Case a: A_{α} is a primitive constant or variable. Let P_{α} be A_{α}.

Case b: A_{α} has the form $[D_{\alpha\beta} E_{\beta}]$.
By inductive hypothesis there are KS-combinatorial wffs $D_{\alpha\beta}$ and E_{β} such that $D_{\alpha\beta}$ contr $D_{\alpha\beta}$ and E_{β} contr E_{β}. Let P_{α} be $[D_{\alpha\beta} E_{\beta}]$.

Case c: A_{α} has the form $[\lambda x_{\beta} B_{\gamma}]$.
By inductive hypothesis there is a KS-combinatorial wff B_{γ} such that B_{γ} contr B_{γ}. Then by Lemma 1 there is a KS-combinatorial wff $P_{\gamma\beta}$ such that $P_{\gamma\beta}$ contr $[\lambda x_{\beta} B_{\gamma}]$.
Thus $P_{\gamma\beta}$ contr $A_{\gamma\beta}$. □

Proposition 2. For every wff A_{α} of £ there is a KBCW-combinatorial wff D_{α} such that D_{α} contr A_{α}.

Proof: it can be verified that

$B_{(\alpha\gamma(\beta\gamma))(\alpha\gamma(\beta\gamma)\gamma)(\alpha\beta\gamma)}[B_{(\alpha\gamma(\beta\gamma))(\alpha\gamma\gamma)(\alpha\gamma(\beta\gamma)\gamma)}$ ...

$[B_{(\alpha\gamma)(\alpha\gamma\gamma)(\alpha\beta\gamma)}W_{\alpha\gamma}]C_{(\alpha\gamma)(\beta\gamma)\gamma}][B_{(\alpha\gamma(\beta\gamma))(\alpha\beta)\gamma}B^{\alpha\beta\gamma}]$ contr $S^{\alpha\beta\gamma}$.

If one replaces $S^{\alpha}_{A_\gamma}$ by this wff everywhere in the wff $J P_\delta$ of Proposition 1, one obtains the desired wff $J D_\delta$. $\square$

§3. General Models for $\mathcal{L}$

We next define the general models for $\mathcal{L}$ by modifying appropriately the definition in [8].

Definition. A frame is a collection $\{\mathcal{D}_\alpha\}_\alpha$ of non-empty domains (sets), one for each type symbol α, such that $\mathcal{D}_o = \{t,f\}$ and $\mathcal{D}_{\alpha\beta}$ is a collection of functions mapping $\mathcal{D}_\beta$ into $\mathcal{D}_\alpha$. The members of $\mathcal{D}_o$ are called truth values and the members of $\mathcal{D}_\iota$ are called ~~individuals~~.

Definition. Given a frame $\{\mathcal{D}_\alpha\}_\alpha$, an assignment (of values in the frame to variables) is a function φ defined on the set of variables of $\mathcal{L}$ such that for each variable x_α, $\varphi x_\alpha \in \mathcal{D}_\alpha$. Given an assignment φ, a variable x_α, and an element $z \in \mathcal{D}_\alpha$, let $(\varphi : x_\alpha / z)$ be that assignment ψ such that $\psi x_\alpha = z$ and $\psi y_\beta = \varphi y_\beta$ if $y_\beta \neq x_\alpha$.

If h is a function of which x is an argument, we write the value of h at x as hx or (hx). If hx is itself a function of which y is an argument, we may write $(hx)y$ simply as hxy, using the convention of association to the left in our meta-language. We shall use dots to denote parentheses in our meta-language in the manner of our convention for brackets in $\mathcal{L}$. We shall also use λ-notation informally in our meta-language. Thus when A is an expression of our meta-language involving a variable x of our meta-language, then $(\lambda x\, A)$ shall serve as a name for the function whose domain is the range of the variable x and whose value at each argument x is A. In contexts where a frame has been specified, if α is a type symbol it will be understood that $x_\alpha, y_\alpha, z_\alpha$, etc., range over the domain $\mathcal{D}_\alpha$ of the frame. However, we reserve $q_{o\alpha\alpha}$ as a name for the identity relation over $\mathcal{D}_\alpha$; i.e., $q_{o\alpha\alpha} x_\alpha y_\alpha = T$ if $x_\alpha = y_\alpha$, and $q_{o\alpha\alpha} x_\alpha y_\alpha = F$ if $x_\alpha \neq y_\alpha$. We note for future reference that if $x_\alpha \in \mathcal{D}_\alpha$, then $q_{o\alpha\alpha} x_\alpha$ is $\{x_\alpha\}$, the unit set whose only member is x_α.

<u>Definition</u>. A frame $\{\mathcal{D}_\alpha\}_\alpha$ is a <u>general</u> model for $\mathcal{L}$ iff there is a binary function $\mathcal{V}$ such that for each assignment φ and wff A_α, $\mathcal{V}_\varphi A_\alpha \in \mathcal{D}_\alpha$ and the following conditions are satisfied for all assignments φ and all wffs:

(a) $\mathcal{V}_\varphi x_\alpha = \varphi x_\alpha$

(b) $\mathcal{V}_\varphi Q_{o\alpha\alpha} = q_{o\alpha\alpha}$

(c) $\mathcal{V}_\varphi[A_{\alpha\beta} B_\beta] = (\mathcal{V}_\varphi A_{\alpha\beta})(\mathcal{V}_\varphi B_\beta)$

(d) $\mathcal{V}_\varphi[\lambda x_\alpha B_\beta] = (\lambda y_\alpha \mathcal{V}_{(\varphi : x_\alpha / y_\alpha)} B_\beta)$

<u>Remark</u>. Clearly the crucial requirement above is that $\mathcal{V}_\varphi[\lambda x_\alpha B_\beta] \in \mathcal{D}_{\beta\alpha}$. Note that in a general model the function $\mathcal{V}$ is uniquely determined.

<u>Definition</u>. A frame $\{\mathcal{D}_\alpha\}_\alpha$ is a <u>standard model for</u> $\mathcal{L}$

iff for all α and β, $\mathcal{D}_{\alpha\beta}$ is the set of all functions from $\mathcal{D}_\beta$ into $\mathcal{D}_\alpha$.

Clearly a standard model is a general model, and is uniquely determined by $\mathcal{D}_\iota$.

A wff A_o is valid in a general model iff $\mathcal{V}_\varphi A_o = T$ for all assignments φ. It can be shown by an easy modification of the argument in [8] that a wff A_o is a theorem of $\mathcal{T}$ iff it is valid in every general model. Also, the rule of inference of $\mathcal{T}$ preserves validity in a general model.

Definition. A wff A_α is significant in a frame $\{\mathcal{D}_\alpha\}$ iff there is a function $\mathcal{V}$ such that for every assignment φ and for every wf part B_β of A_α (including A_α itself), $\mathcal{V}_\varphi B_\beta \in \mathcal{D}_\beta$, and $\mathcal{V}$ satisfies conditions (a)-(d) (in the definition of general model).

Thus a frame is a general model iff every wff is significant in it.

Before proving the next proposition we state the following lemmas, which can be proved by a straightforward induction on the construction of B_β.

Lemma 2. If B_β is significant in a frame and φ and ψ are assignments which agree on the free variables of B_β, then $\mathcal{V}_\varphi B_\beta = \mathcal{V}_\psi B_\beta$.

<u>Lemma 3</u>. If A_α and B_β are significant in a frame and A_α is free for x_α in B_β, then $S^{x_\alpha}_{A_\alpha} B_\beta$ is significant and for any assignment φ, $V_\varphi S^{x_\alpha}_{A_\alpha} B_\beta = V_{(\varphi:x_\alpha/V_\varphi A_\alpha)} B_\beta$.

<u>Proposition 3</u>. If C_γ is significant in a frame and C_γ contr D_γ, then D_γ is significant, and for any assignment φ, $V_\varphi C_\gamma = V_\varphi D_\gamma$.

Proof: Clearly it suffices to prove this proposition for the case where D_γ is obtained from C_γ by a single application of rule I or II of λ-conversion. In either case the proposition follows easily by induction on the construction of C_γ once one establishes it for the wf part of C_γ to which the rule is actually applied.

Thus in the case of rule I one may suppose C_γ is $[\lambda x_\alpha B_\beta]$ and D_γ is $[\lambda y_\alpha S^{x_\alpha}_{y_\alpha} B_\beta]$, where y_α is not free in B_β and y_α is free for x_α in B_β. We may assume that $y_\alpha \neq x_\alpha$. B_β is significant since C_γ is, so by Lemma 3 $S^{x_\alpha}_{y_\alpha} B_\beta$ is significant.

Note that for any $z_\alpha \in \mathcal{D}_\alpha$ we have $V_{(\varphi:y_\alpha/z_\alpha)} y_\alpha = z_\alpha$, so

$$V_{(\varphi:y_\alpha/z_\alpha)} S^{x_\alpha}_{y_\alpha} B_\beta = V_{((\varphi:y_\alpha/z_\alpha):x_\alpha/z_\alpha)} B_\beta \quad \text{(by Lemma 3)}$$

$$= V_{(\varphi:x_\alpha/z_\alpha)} B_\beta \quad \text{(by Lemma 2)}. \text{ Hence}$$

$$V_\varphi C_\gamma = (\lambda z_\alpha\, V_{(\varphi:x_\alpha/z_\alpha)} B_\beta) = (\lambda z_\alpha\, V_{(\varphi:y_\alpha/z_\alpha)} S^{x_\alpha}_{y_\alpha} B_\beta),$$

which is the desired value for $V_\varphi D_\gamma$, so D_γ is significant and $\mathcal{V}_{\varphi} C_{\gamma} = \mathcal{V}_{\varphi} D_{\gamma}$.

In the case of Rule II one may suppose that C_{γ} is $[[\lambda x_{\alpha} B_{\beta}] A_{\alpha}]$ and D_{γ} is $S^{x_{\alpha}}_{A_{\alpha}} B_{\beta}$, where A_{α} is free for x in B_{o}. Since C is significant, A and B_{β} are, so by Lemma 3 D_{γ} is significant. Also $\mathcal{V}_{\varphi} C_{\gamma} = (\mathcal{V}_{\varphi}[\lambda x_{\alpha} B_{\beta}])(\mathcal{V}_{\varphi} A_{\alpha}) = \mathcal{V}_{\varphi} D_{\gamma}$ by Lemma 3.

Remark: It is not true that if C_{γ} is significant in a frame and D_{γ} contr C_{γ}, then D_{γ} must be significant. For x_{ι} is always significant, but $[[\lambda x_{\iota} x_{\iota}] x_{\iota}]$ might not be.

Proposition 4. For any frame $\mathfrak{M}$, the following conditions are equivalent:

(a) $\mathfrak{M}$ is a general model for $\mathcal{L}$.

(b) Every KS-combinatorial wff of $\mathcal{L}$ is significant in $\mathfrak{M}$.

(c) Every KBCW-combinatorial wff of $\mathcal{L}$ is significant in $\mathfrak{M}$.

Proof: by Propositions 1,2, and 3. ☐

We now rephrase condition (b) to obtain a simple criterion for a frame to be a general model.

Theorem 1. A frame $\{\mathcal{D}_{\alpha}\}_{\alpha}$ is a general model for $\mathcal{L}$ iff it satisfies all of the following conditions (for all type symbols α, β, γ):

(a) $Q_{o\alpha\alpha} \in \mathcal{D}_{o\alpha\alpha}$.

(b) For all $x_\alpha \in \mathcal{D}_\alpha$, $(\lambda y_\beta x_\alpha) \in \mathcal{D}_{\alpha\beta}$.

(c) $(\lambda x_\alpha \lambda y_\beta x_\alpha) \in \mathcal{D}_{\alpha\beta\alpha}$.

(d) For all $x_{\alpha\beta\gamma} \in \mathcal{D}_{\alpha\beta\gamma}$ and $y_{\beta\gamma} \in \mathcal{D}_{\beta\gamma}$,

$(\lambda z_\gamma . x_{\alpha\beta\gamma} z_\gamma . y_{\beta\gamma} z_\gamma) \in \mathcal{D}_{\alpha\gamma}$.

(e) For all $x_{\alpha\beta\gamma} \in \mathcal{D}_{\alpha\beta\gamma}$,

$(\lambda y_{\beta\gamma} \lambda z_\gamma . x_{\alpha\beta\gamma} z_\gamma . y_{\beta\gamma} z_\gamma) \in \mathcal{D}_{\alpha\gamma(\beta\gamma)}$.

(f) $(\lambda x_{\alpha\beta\gamma} \lambda y_{\beta\gamma} \lambda z_\gamma . x_{\alpha\beta\gamma} z_\gamma . y_{\beta\gamma} z_\gamma) \in \mathcal{D}_{\alpha\gamma(\beta\gamma)(\alpha\beta\gamma)}$.

Proof: Clearly if the frame is a general model, the conditions (a)-(f) must be satisfied. To show they are sufficient, we show they imply condition (b) of Proposition 4. Since every variable is significant in every frame, and a wff $[A_{\alpha\beta} B_\beta]$ is significant in a frame iff $A_{\alpha\beta}$ and B_β are, it suffices to show that the wffs $Q_{o\alpha\alpha}$, $K^{\alpha\beta}$ and $S^{\alpha\beta\gamma}$ are significant in the frame. This is assured by conditions (a)-(f). (We note that condition (a) implies that for all $x_\alpha \in \mathcal{D}_\alpha$, $(Q_{o\alpha\alpha} x_\alpha) \in \mathcal{D}_{o\alpha}$.) ∎

Remark: We leave it to the reader to state the analogous theorem using $B^{\alpha\beta\gamma}$, $C^{\alpha\beta\gamma}$, and $W^{\alpha\beta}$ in place of $S^{\alpha\beta\gamma}$. Such a theorem may be useful since $B^{\alpha\beta\gamma}$, $C^{\alpha\beta\gamma}$, and $W^{\alpha\beta}$ are each conceptually simpler than $S^{\alpha\beta\gamma}$.

§4. The Axiom of Descriptions

We remind the reader that the Axiom of Descriptions is

(D) $$\exists i_{\iota(o\iota)} \forall x_\iota . i_{\iota(o\iota)} [Q_{o\iota\iota} x_\iota] = x_\iota$$

Theorem 2« D is not a theorem of £>

Proof: We partition the type symbols into two sets, $o and JT_t as follows: o e JT but _o o^ff ; I_t G J but_t i jL Z_o ; (ot|3) is in whichever set contains a* We then let C = {(ccP) |a€ ff_t and pe Z_Q).

We next define a frame IU = {& } by induction on a.
a a

&^o = {t,f}. &^ι = {m,n), where m and n are distinct individuals. (Actually &^ι may be taken to have any cardinality greater than one.) If (oc3)€C, & _Q is the set of all constant
ccp
functions (i.e., functions with the same value for all arguments) from &_Q into & • If (a)3)/C, & _a is the set of all functions
P a &p
from &_P into fi_a .

We next use Theorem 1 to verify that IU is a general model for £.

(a) Since (o_a)/C and (o_a a)/C, q € &_{oαα} •

(b) (Ay_Q x) is a constant function, and so is in & _a .
P a aP

(c) (aj3a)/C whether _a e JT_t or a € S^1^_o . Hence (Ax Ay_ft x)e& _Q •
oc p oc ocpoc

(d) We need consider only the case where (αγ)∈C.

We must show that if xefi _Q and ye&_o , then
apy * fiy

(Az^γ .xz^γ Y^z_y) is a constant function. So we let z^1 ,z^2 e$^γ
o
and show that (xz^1 _o yz^1) = (xz^2 .yz^2). Since a € 3^ and yeZ _x (aj3y)eC so xz^1 = xz^2_o

~~Case 1~~: fie Z^ι . Then {fiy)eC so yz^1 = yz^2 so (xz^yz^1) = (xz^2.yz^2).

Case 2; $\beta \in \mathcal{T}_o$. Then $(\alpha\beta) \in C$. Since $xz^1 = xz^2 \in \mathcal{D}_{\alpha\beta}$,
$xz^1(yz^1) = xz^2(yz^2)$.

(e) Suppose $(\alpha(\beta\gamma)) \in C$ and $x \in \mathcal{D}_{\alpha\beta\gamma}$.
We must show that $(\lambda y_{\beta\gamma}\, \lambda z_\gamma . xz_\gamma . y_{\beta\gamma} z_\gamma) \in \mathcal{D}_{\alpha\gamma(\beta\gamma)}$. So
suppose $y^1, y^2 \in \mathcal{D}_{\beta\gamma}$. We must show that
$(\lambda z_\gamma . xz_\gamma . y^1 z_\gamma) = (\lambda z_\gamma . xz_\gamma . y^2 z_\gamma)$. To do this we show that
for an arbitrary $z \in \mathcal{D}_\gamma$, $(xz.y^1 z) = (xz.y^2 z)$. But $\alpha \in \mathcal{T}_t$
and $\beta \in \mathcal{T}_o$ so $(\alpha\beta) \in C$ and $xz \in \mathcal{D}_{\alpha\beta}$, which contains only constant
functions. Hence $xz(y^1 z) = xz(y^2 z)$.

(f) $(\alpha\gamma(\beta\gamma)(\alpha\beta\gamma)) \notin C$ whether $\alpha \in \mathcal{T}_t$ or $\alpha \in \mathcal{T}_o$, so

$$(\lambda x_{\alpha\beta\gamma}\, \lambda y_{\beta\gamma}\, \lambda z_\gamma . x_{\alpha\beta\gamma}\, z_\gamma . y_{\beta\gamma}\, z_\gamma) \in \mathcal{D}_{\alpha\gamma(\beta\gamma)(\alpha\beta\gamma)} .$$

Now $q_{o\iota\iota} m$ and $q_{o\iota\iota} n$ are elements of $\mathcal{D}_{o\iota}$, so in
order that D be valid in $\mathfrak{M}$ there must be a function $h \in \mathcal{D}_{\iota(o\iota)}$
such that $h(q_{o\iota\iota} m) = m$ and $h(q_{o\iota\iota} n) = n$. However, $(\iota(o\iota)) \in C$,
so there is no such function in $\mathcal{D}_{\iota(o\iota)}$. Thus D is not valid
in the general model $\mathfrak{M}$, and so is not a theorem of $\mathcal{T}$. ∎

The idea behind the following theorem is contained in [9],
but the proof is short, so we give it here.

Theorem 3. Let $\mathfrak{M} = \{\mathcal{D}_\alpha\}_\alpha$ be a general model for $\mathcal{T}$
in which $\mathcal{D}_\iota$ is finite. Then $\mathfrak{M}$ is a standard model iff D
is valid in $\mathfrak{M}$.

Proof: The domains $\mathcal{D}_\alpha$ must, of course, all be finite.
If $\mathfrak{M}$ is standard one can enumerate the elements in $\mathcal{D}_{\iota(o\iota)}$,

to see that D is valid in to.

Suppose D is valid in to. We show that $A_{\alpha}g$ must contain all functions from $\$_p$ to & • So let g be any such function. Let «fl = {nu, ... ,1^}. By the methods mentioned in §1 one sees that there must be a description operator $h_{a(o_a)}$ K£& $_{a(oa)}$v such that for each n_a e&$_a$ $h_{a(oa)}$x[q^$_{oaa}$ n_a] = n_a.

Let cp be an assignment with values on the variables

^(oa)'^'---'^'--"^---'^ as follows: $cP^i a(o_a) = {}^h a(oa)$ '

φw^1_β = m^, ...,cpwg = m^, cpz = 9^{m}g* • • • 9 an(^ Vα = 9^m β •

Then 9 = V_{cp}[Ax_{J3}.$i_{a(oa)}$.Ay_a.[$x_\wedge$ = w^1_β ∧ y_α = z^1_α]

$v_{\#\#}$.V[x_D = w^k_D Ay = z]], so g must be in $\&_{Q}$ since to is a general model.

□

<u>Remark</u>, Theorem 3 provides a strong argument for always assuming the Axiom of Descriptions. If one does this by introducing a description operator $1_{\iota(o\iota)}$ and modifies the definition of general model in the natural way by introducing an appropriate requirement for Ix 1 , * (thus getting closer to the definition in [8]), one can again prove that the theorems are precisely the wffs valid in all general models. Thus it appears that the language Q^o of [2] is more natural than £.

§5. <u>The Axiom of Choice</u>

The Axiom of Choice (for individuals) is

(E) $\exists i_{\iota(o\iota)} \forall p_{o\iota} \exists x_\iota p_{o\iota} x_\iota \Rightarrow p_{o\iota} . i_{\iota(o\iota)} p_{o\iota}$.

Clearly [E ZD D] is a theorem of Z. We use the Fraenkel-Mostowski method to show that its converse is not. Thus E is not provable in Z, even if D is added to the list of axioms.

We first establish the following lemma, which is fundamental for applications of the Fraenkel-Mostowski method to Z. The lemma is true but trivial if $\&_\iota$ is finite, since in this case the conditions on 3 assure that to will be the standard model over $\&_\iota$. We use o to denote the composition of functions.

<u>Lemma 4</u>o Let $\$_\iota$ be an infinite set of individuals and P a set of permutations a of $\&_\iota$ such that or ©a = $(Ax_\iota\ x_\iota)$ o Let 3 be a family of subsets of P such that

(a) for each me &^ there is a set K€ 3 such that om = m for all σ e K, and

(b) for all H,Ke 3 there is a set Je 3 such that J c H fl K_o

Let the frame to = $[\&_\alpha\}_\alpha$ be defined, and each permutation a € P be extended to a permutation of $\$_\alpha$ (which we may denote by a^a) such that a^a o cr^a = $(Ax_{\alpha}\ x_a)$ for each a, as follows by induction on a:

$*_o$ = [t,f}? cr° = $(Ax_Q x_Q)$ for all a € P.

Given $\&_a$ and $\&_\beta$ and any function h from $\&_\beta$ into A_a, let σh = a^aohoa, and let $\&_{Q\alpha\beta}$ be the set of all functions h from $\mathcal{D}_{\beta}$ into $\mathcal{D}_{\alpha}$ such that there is some $K \in \mathfrak{J}$ such that $\sigma h = h$ for all $\sigma \in K$.

Then $\mathfrak{M}$ is a general model for $\mathcal{L}$ in which D is valid.

Proof: For notational convenience, if $h \in \mathcal{D}_{\gamma}$ we let K_h denote some $K \in \mathfrak{J}$ such that $\sigma h = h$ for all $\sigma \in K$. Clearly such a set K_h always exists. Note that if $h \in \mathcal{D}_{\alpha\beta}$ and $x \in \mathcal{D}_{\beta}$, then $(\sigma^{\alpha\beta} h)(\sigma^{\beta} x) = \sigma^{\alpha}(hx)$.

We use Theorem 1 to verify that $\mathfrak{M}$ is a general model.

(a) If $x, y \in \mathcal{D}_{\alpha}$ and $\sigma \in K_x$ then $(\sigma^{o\alpha\alpha} q_{o\alpha\alpha} x)y =$
$= \sigma^{o}(q_{o\alpha\alpha} x . \sigma y) = q_{o\alpha\alpha} x(\sigma y)$, which is t iff $\sigma y = x = \sigma x$ iff $x = y$, so $(\sigma^{o\alpha\alpha} q_{o\alpha\alpha} x)y = (q_{o\alpha\alpha} x)y$ for all $y \in \mathcal{D}_{\alpha}$ so $\sigma(q_{o\alpha\alpha} x) = q_{o\alpha\alpha} x$. Thus $(q_{o\alpha\alpha} x) \in \mathcal{D}_{o\alpha}$, and $q_{o\alpha\alpha}$ at least maps $\mathcal{D}_{\alpha}$ into $\mathcal{D}_{o\alpha}$. Also, for any $\sigma \in P_s$ $(\sigma q_{o\alpha\alpha}) xy =$
$(\sigma^{o\alpha\alpha} q_{o\alpha\alpha} \sigma^{\alpha} x)y = q_{o\alpha\alpha}(\sigma^{\alpha} x)(\sigma^{\alpha} y)$, which is t iff $\sigma x = \sigma y$ iff $x = y$, so $\sigma q_{o\alpha\alpha} = q_{o\alpha\alpha}$ and $q_{o\alpha\alpha} \in \mathcal{D}_{o\alpha\alpha}$.

(b) For any $x_{\alpha} \in \mathcal{D}_{\alpha}$ and $\sigma \in K_x$, $\sigma(\lambda y_{\beta} x_{\alpha}) = (\lambda y_{\beta} \sigma x_{\alpha}) = (\lambda y_{\beta} x_{\alpha})$, so $(\lambda y_{\beta} x_{\alpha}) \in \mathcal{D}_{\alpha\beta}$.

(c) For any $\sigma \in P$, $\sigma(\lambda x_{\alpha} \lambda y_{\beta} x_{\alpha}) = (\lambda x_{\alpha} . \lambda y_{\beta} \sigma x_{\alpha}) =$
$= (\lambda x_{\alpha} \lambda y_{\beta} \sigma \sigma x_{\alpha}) = (\lambda x_{\alpha} \lambda y_{\beta} x_{\alpha})$, so $(\lambda x_{\alpha} \lambda y_{\beta} x_{\alpha}) \in \mathcal{D}_{\alpha\beta\alpha}$.

Before checking (d)-(f) we observe that if $x \in \mathcal{D}_{\alpha\beta\gamma}$, $y \in \mathcal{D}_{\beta\gamma}$, and $z \in \mathcal{D}_{\gamma}$, then $\sigma^{\alpha}.(\sigma x)(\sigma z)_{o}(\sigma y).\sigma z =$
$= \sigma^{\alpha}.(\sigma^{\alpha\beta} xz).\sigma^{\beta}.yz = \sigma^{\alpha}.\sigma^{\alpha}.xz_{o}yz = xz.yz$.

(d) Suppose $x \in \mathcal{D}_{\alpha\beta\gamma}$ and $y \in \mathcal{D}_{\beta\gamma}$. Let J be a member of $\mathfrak{J}$

such that J c K_x n K_y. For any a e J, ax = x and ay = y so a(Az $_\gamma$.xz.yz) = a(Az $_\gamma$.(ax)z.(ay) z) = (Az $_\gamma$$a^a$. (ax) (az) . (ay) .az) = (Az$_\gamma$.xz.yz), which must therefore be in #$_{\alpha\gamma}$.

(e) If xe& g$_{\alpha,\gamma}$ and a e K_x, then cx(Ayg$_\gamma$ Az_y.xz.yz) = = σ(λy$_\beta$y Az$_\gamma$.(ax)z.yz) = (Ayg$_{\gamma\#}$ a. Az$_\gamma$. (ax) z_o (ay) z) = = (λyg_y Az$_\gamma$ .a^a. (ax) (az) .(ay) .az) = ^Y&y Az_y.xz.yz), which must therefore be in &$_{\alpha\gamma(\text{fi}_r\gamma)}$ x •

(f) For any ae P_5 a(Ax g$_{\alpha,\gamma}$ Ayg$_\gamma$ Az$_\gamma$.xz.yz) = = (Ax a^r AyPr ^V 0 ^* (ax)(az) • (ay) _ az) = (Ax apy AYPY Az Y- x z.yz),

which must therefore be in & $^\alpha$y(ffy)(·$^\#$ty)^m

Thus to is a general model for £.

We next verify that D is valid in to. Let n e &$_\iota$. We shall construct a description operator h mapping &$_{o\iota}$ to &$_\iota$ as follows. For each unit set q_{o1i*} x, we let h($q_{oIt\backslash}$ x) = x. If ge &$_{ox}$ is not a unit set, let hg = n. Now we verify that he fl$_N$. Let a e K$_{-*}$. For each unit set q_{ott} V ff(q_{ott} \} = (CTq_{o1}i)(aX t) = %ti(CTX t^

so (ah)($q_{oti}x_t$) = a(h.a.q_{Qil} x^ = ordi.q^.crx^ = = acrx = x- = h(q_{otI} x_t). If g_{ot} is not a unit set, then ag$_{OX}$ (i.e. g$_{OX}$o cr^l) is not either, so

(ah)g_{Qt} = a(h.ag_Q^) = an = n = hg_{Ql} • Thus ah = h, and hefi$_{(o\iota)^o}$

It is now easy to see that D is valid in to .

Theorem 4. [D 3 E] is not a theorem of £.

Proof: Let 9 be an infinite index set and for all j e 9 let \x? and x? be distinct individuals, so chosen that m^j ^ m^1 and n^j fi n^1 if j / i. Let A, = {m^j | j € 9} U {n^D | j € 2}. Let P be the set of all mappings a from $ to JD such that for all j € 9, oxx? = xc? and on?- - v?, or am^3 = n*^3 and an^3 = m^3. Thus for each a e P we have o*o = (Ax^ x). Let 3 be the family of all subsets K of P such that there is a finite subset j of 9 such that K = {aep| for all j € j, ow? = TO^3 and an^3 = n"^3}. It is easily checked that 3« satisfies the conditions of Lemma 4, so let to be the general model constructed as in Lemma 4.

We must see that E is false in to. Suppose it were true. Then there would be a choice function h e & , x i(oi) such that for every non-empty set g€ A Ot hg is in q, i.e., g(hg) = t. For each j€ 9$ iet 9^ = (Ax .x = m^] or x = n^), i.e., g-^1 = {m-',n-^J}. It is easy to see that erg-* = g-^5 for all a € P, so each g^]ei)Ot . Now for any K e 3 there is some j € 9 which is not in the finite subset of 9 which determines K, and hence some a e K such that am^3 = n^ and an-^5 = m-^1. Then (ah)g^D = a(h.ag^D) = cr(hg^D) / hg^D, so ah ^ h. Thus there can be no choice function h € & , x , i(ot) so E is false in **to.**

Thus [D z> E] is not valid in the general model to and so is not a theorem of £. rn

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