ON THE POSSIBILITY OF DETECTING INVARIANCE OF MATERIAL RESPONSE TO CHANGES IN TIME SCALE

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Report 71-31

June 1971

(Contract) Acknowledgement

The first named author was supported by United States Air Force Office of Scientific Research under Grant AFOSR-71-2057 during part of the period in which the research described here was carried out.

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Abstract

Our attention is focused here on a class of materials each of whose response to a particular motion is invariant under certain changes in time scale. We establish conditions on the given motion under which this invariance can in principle be detected by means of the following procedure: subject identical specimens to motions which differ from the given one only by the appropriate changes in time scale and measure the responses. In general, the class of motions for which this procedure is feasible includes only certain accelerationless or circulation-preserving motions. However, for the special case of simple, incompressible materials in homogeneous bodies, this restriction is not unduly severe.

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1. Introduction.

A constitutive relation represents an ideal description of material response in some usually unspecified set of motions. Such a relation is often formulated on the basis of measurements taken in a set of motions smaller than the original set. In some cases, if a motion yields a given measured response, then certain time rescalings of the motion (performed on identical specimens) will yield the original measured response modified only by the corresponding time rescaling. In such cases we say the response is invariant (under the appropriate set of rescalings). It is often the case that constitutive relations, are tacitly assumed to be invariant over the entire range of possible time rescalings. The theories of elasticity, placticity and hypoelasticity are based on such an assumption. Actually, invariance of material response to arbitrary rescalings, called rate-independence, represents an extreme case. Generally, one finds by measurement that materials are sensitive to at least some, if not all, changes in time scale.

It is natural to ask whether or not the response of a given material in a given motion is invariant under a given time rescaling, and whether or not such invariance or lack thereof may be detected by means of some conceptually simple procedure. In this paper, we attempt to evaluate the following, rather natural procedure: subject identical specimens to motions which differ only by the given change in time scale and measure the response at each point of the specimen in order to detect invariance or lack thereof. Our main interest will be the feasibility of this procedure in the case where some invariance exists. Regarding the procedure described above, we note that invariance of response under a rescaling necessitates a rescaling of applied surface tractions. This fact may be useful in carrying out such a procedure in a laboratory situation.

For simplicity, we fix the body force field once and for all; while we do take this field to be spatially steady, we make no other assumptions regarding it. Our main results, to be described presently, apply to non-simple materials occurring in non-homogeneous bodies, as well as in the case of simple, homogeneous bodies. Thus, we shall admit a broad class of materials and bodies in our discussion.

The following results allow us to evaluate the procedure outlined above. First, suppose in a given motion the response at every point of an <u>unconstrained</u> body is invariant under one linear and one non-linear change of time scale. It follows that (a) both the given motion and the two rescalings are realizable if and only if the given motion is realizable and is trivial (i.e. static), and (b) both the given motion and the one linear rescaling are realizable if and only if the given motion is realizable and is accelerationless. Next suppose that in a given motion the response at every point of an <u>incompressible</u> body is invariant under one linear and one non-linear change of time scale. It follows that (c) both the given motion and the two rescalings are realizable if and only if the given motion is realizable and is irrotational, and (d) both the given motion and the linear rescaling are realizable if and only if the given motion is realizable and is circulation preserving.

Here, the term "realizable" means that the condition of balance of linear momentum is satisfied at every point in the body at every time. In practical terms, we interpret the term "realizable" to mean that a motion can be produced in a laboratory provided suitable surface tractions are applied to the body. With this interpretation, we observe that our results (a) and (b) imply that invariance of response, in a given motion, to even one or two rescalings can be detected by the procedure described above if and only if the given motion is of a very trivial nature. Our results (c) and (d) again show that the procedure generally will not enable us to detect invariance, but the class of motions for which the procedure will be successful is expected to be larger for incompressible bodies than for unconstrained bodies.

In this paper, we also evaluate the procedure in the extreme case of rate-independent response. The results (a) and (b) then are modified as follows. Suppose that in a given motion the response at every point of an unconstrained body is invariant under arbitrary changes of time scale. It follows that (a') the given motion as well as all rescalings are realizable if and only if the motion is realizable and is trivial, and (b') the given motion as well as all linear rescalings are realizable if and only if the given motion is realizable and accelerationless. In particular, the motion and one linear rescaling are both realizable if and only if the motion and <u>all</u> linear rescalings are realizable. The analogous modifications of (c) and (d) in the case of rate-independent response are obtained by replacing the words "trivial" and "accelerationless", occurring in (a') and (b'), by the words "irrotational" and "circulation preserving". In particular, as is the case of (b'), one finds that the motion and one linear rescaling are both realizable if and only if the motion and all linear rescalings are realizable.

We remark that the additional rescalings which are introduced through the condition of rate-independence do not substantially affect the statements of the results (a)-(d). (Of course, the class of realizable, (say) irrotational motions in an incompressible body may be affected by varying the invariance properties of the response functions.) Hence, invariance of response to arbitrary rescalings of a motion can be detected in a rate-independent body, using our suggested procedure, only for a limited class of motions: trivial, realizable motions (if the body is unconstrained) and irrotational, realizable motions (if the body is incompressible).

It is worth noting that the class of realizable motions varies with changes in the body force field. Thus, a motion for which our procedure fails given one choice of body force field may be one for which our procedure succeeds given some other choice of body force field. Hence, the class of motions for which the procedure fails, given a body force field, is not devoid of physical significance.

The procedure we have described can be used to determine whether or not a simple, incompressible, homogeneous body,subject to a conservative (not necessarily steady) body force, has rate-independent response, i.e., response which is invariant under arbitrary rescalings of arbitrary motions. In fact, the Theorem on Homogeneous Motions ([l],p.74) implies that every homogeneous, isochoric, irrotational motion is realizable in such a body. This class of motions is invariant under rescalings, and the Theorem of

Determination ([l],p.75) implies that this same class of motions can be used to determine the response function of every simple, incompressible, homogeneous body. Thus/ given any motion in this class, every rescaling of the motion is realizable in every body of the above type. Hence, the procedure in question will enable one to detect invariance <u>or</u> lack thereof.

We conclude this paper with a brief discussion of invariance of mechanical and thermal response and establish restrictions on thermal fields under which such invariance can be detected. In particular, we show that a thermodynamical process which enables us to detect existing invariance of response must proceed so as to increase the entropy at each point of the body and to satisfy the classical heat conduction inequality.

2. Unconstrained Bodies.

In any realizable motion of an unconstrained body, the mass density field ρ, the body force field $\mathbf{b}$, the stress field $\mathbf{T}$, and the acceleration field $\ddot{\mathbf{x}}$ must satisfy the equation of balance of linear momentum for all times:

$$\ddot{\mathbf{x}} = \frac{1}{\rho}\,\operatorname{div}\mathbf{T} + \mathbf{b}.$$

If the stress response at every point in the body is invariant under some change of time scale, then the right hand side of this equation also is invariant, since the body force is steady and the density field is determined in an invariant way by the motion. If we denote the change in time scale by φ and the acceleration field for the rescaled motion by $\ddot{\mathbf{x}}_\varphi$, it follows from the chain rule that

$$\ddot{\mathbf{x}}_\varphi = \dot{\varphi}^2\,\ddot{\mathbf{x}} + \ddot{\varphi}\,\dot{\mathbf{x}},$$

where $\ddot{\mathbf{x}}$ and $\dot{\mathbf{x}}$ are evaluated at time $\varphi(t)$ whenever $\ddot{\mathbf{x}}_\varphi$, $\dot{\varphi}$ and $\ddot{\varphi}$ are evaluated at time t. If the motion and the rescaled motion both are realizable, then the left hand side of the equation of balance of momentum must be invariant since the right hand side is invariant, i.e.,

$$\ddot{\mathbf{x}}_\varphi = \ddot{\mathbf{x}}$$

(with $\ddot{\mathbf{x}}$ and $\ddot{\mathbf{x}}_\varphi$ evaluated as explained above). The last two equations imply that for all times:

$$\ddot{\mathbf{x}}(\dot{\varphi}^2 - 1) + \dot{\mathbf{x}}\,\ddot{\varphi} = \mathbf{0}.$$

Thus, the given invariance of the stress response under <p and the assumption that the motion and the rescaled motion are realizable yield the above restriction on the motion itself. It follows that if <p(t) = ct, with c ^ 1, i.e., if <*p* is a non-trivial linear rescaling, then $\dot{\underset{\sim}{x}} = \underset{\sim}{0}$ for all times. If, in addition, the above conditions hold when *o* is a non-linear rescaling, i.e. <*p* ^ 0 for all times, then we may also conclude that $\dot{x} = 0$ for all times.

It is easily seen that if the stress response to a realizable accelerationless motion is invariant under any set of linear rescalings, then each such rescaling of the motion is realizable. If the stress response to a realizable trivial motion is invariant under any set of rescalings (linear or non-linear),then each such rescaling of the motion is realizable.

These considerations establish the results (a), (b), (a*), and (b[1]) stated in the introduction.

3. Incompressible Bodies.

A motion of an incompressible body is realizable if the motion is volume preserving (isochoric) and if there is a scalar field p such that the following relation is satisfied for all times:

$$\ddot{\underset{\sim}{x}} + \frac{1}{\rho}\,\mathrm{grad}\ p = \frac{1}{\rho}\,\mathrm{div}\ \underset{\sim}{S} + \underset{\sim}{b}.$$

Here, $\underset{\sim}{S}$ is the deviatoric stress field. If the deviatoric stress response to a given realizable motion is invariant under some rescaling φ, then in order that the last relation be satisfied by the rescaled motion, it is not necessary that the acceleration field be invariant under the rescaling. Rather, it is necessary that for some second scalar field p', the relation

$$\ddot{\underset{\sim}{x}} + \frac{1}{\rho}\,\mathrm{grad}\ p = \ddot{\underset{\sim}{x}}_{\varphi} + \frac{1}{\rho_{\varphi}}\,\mathrm{grad}\ p'$$

be satisfied when the left hand side is evaluated at $\varphi(t)$ and the right hand side at time t. (The quantities ρ and ρ_{φ} which appear on the two sides of the equation are identical, since one gives the density at a point in space at time $\varphi(t)$ for the original motion and the other gives the density at the same point in space at time t for the rescaled motion.) The last relation may be rewritten in the form

$$\rho\ddot{\underset{\sim}{x}}(\dot{\varphi}^2-1) + \rho\dot{\underset{\sim}{x}}\ddot{\varphi} = \mathrm{grad}(p'-p)\ ,$$

using the relation between $\ddot{\underset{\sim}{x}}_{\varphi}$ and $\ddot{\underset{\sim}{x}}$ recorded in Section 2. Moreover, if ρ is spatially constant, we obtain:

$$\ddot{x}(\theta^2-1) + \dot{x}\dot{\theta} = \text{grad}\ £$$

where £ = (P!~p)/P* If this relation is valid for one non-trivial linear rescaling, then it follows that

$$\ddot{\tilde{x}} = \text{grad}\{£/(\theta^2-1)\}.$$

In other words, in order that both the original motion and one linear rescaling of the motion (which leaves the deviatoric stress invariant) be realizable, it is necessary that the original motion be circulation preserving. If a non-linear rescaling θ also satisfies these conditions, it follows that

$$\dot{\tilde{x}} = \text{grad}\ §$$

for some choice of scalar field ?. i.e. the original motion is irrotational. In fact, if

$$\ddot{x}(\phi^2-1) + \dot{x}\dot{\phi} = \text{grad}\ £^*$$

then the condition that the motion is circulation preserving and the non-linearity of θ imply that

$$\dot{\tilde{x}} = \text{grad}\{(C^*/\dot{\phi}) - (C[\theta^2-1]/\dot{\theta}[\theta^2-1])\}.$$

Hence, the original motion is irrotational.

It is easily seen that if the deviatoric stress response to a realizable, circulation preserving motion is invariant under any set of linear rescalings, then each such rescaling is also realizable. Moreover, if the deviatoric stress response to a realizable, irrotational motion is invariant under any set of rescalings (linear or non-linear), then each such rescaling is realizable. (In the verification of the second assertion, one uses the fact that an irrotational motion is necessarily circulation preserving.)

These considerations establish the results (c), (d), (c'), and (d') stated in the introduction.

4. Thermodynamic Considerations.

Let q be the heat flux vector, *8* the temperature, £ the internal energy, r the rate of supply of energy per unit mass of the body, and let

$$D = \frac{1}{2}[\nabla\dot{x} + (\nabla\dot{x})^T]$$

be the stretching tensor. The energy equation can then be written as follows:

$$\rho\dot{\imath} - tr(TD) = -div\ q + \rho r.$$

We consider materials for which £ , q and T are invariant under simultaneous rescalings of the motion and the temperature field and assume that the scalar field r (or ρr) is steady. The right hand side of the energy equation is invariant under such a rescaling, while the left hand side is never invariant under a non-trivial rescaling. If a motion and a temperature history satisfy the energy equation before and after a non-trivial linear rescaling, then both members of the energy equation must vanish:

$$-div\ q + \rho r = 0, \quad \rho\dot{£} - tr(TD) = 0,$$

so that the heat flux and the energy supply have no net effect on the internal energy.

We recall that the momentum equation imposes two conditions on realizable motions whose rescalings are also realizable. The first, in terms of the velocity or the acceleration field, is explicit.

The second requires that the motion be quasi-equibrated. This implicit condition is generally <u>not</u> independent of the above thermodynamic conditions since the stress may depend on both the motion and the history of the temperature field. But in the special case of incompressible isotropic linear elastic materials satisfying Fourier's law of heat conduction, every pair consisting of a spatially steady and uniform temperature field and an isochoric irrotational motion with infinitesimal displacement gradient is realizable and remains realizable after an arbitrary rescaling in a homogeneous body subject to a conservative body force and vanishing energy supply.

In the thermodynamics of materials with memory the entropy inequality is interpreted as a restriction on the constitutive functionals to be observed in all dynamically possible histories satisfying the energy equation. Accordingly, we assume that the response functions of the materials in question are such that the inequality

$$P\dot{V} \geq (q\text{-grad } d/d^2) + (-\text{div } g + pr)/9$$

is satisfied by every realizable motion-temperature pair. In particular, if a pair is realizable before <u>and</u> after a rescaling, it follows from this inequality and the last two equations that the specific entropy 77 can never decrease and the classical heat-conduction inequality

* This condition means that the momentum equation is satisfied when the inertia1 force term is ignored.

$$\mathbf{q} \cdot \operatorname{grad} \theta \leq 0$$

holds for that pair. It follows from our interpretation of the entropy inequality that these last two conditions on the entropy and the temperature field together with the condition that a pair be rescalable provide the same information concerning the pair as the condition that the pair be rescalable can alone provide.

Acknowledgement

The first named author was supported by United States Air Force Office of Scientific Research under Grant AFOSR-71-2057 during part of the period in which the research described here was carried out.

Reference

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