RADIAL AVERAGING TRANSFORMATIONS
WITH VARIOUS METRICS

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Introduction

In [3] one of the authors introduced the notion of a radial averaging transformation of domains in the plane, which was based on the metric $\frac{1}{r}\,dr\,d\theta$ where (r,θ) are polar coordinates. This transformation is useful in obtaining estimates for conformal capacity of condensers and conformal radius of domains. In this paper we discuss averaging transformations with various metrics of the form $g(r)\,dr\,d\theta$ where $g(r)$ is a positive, continuous function of r for $0 < r < \infty$. Using these transformations we are able to obtain estimates for energy integrals of the form

$$\iint_\Omega |\nabla F|^2 q^2\,dx\,dy$$

where $q^2 = rg(r)$. These estimates are then used in order to obtain further estimates for capacities and conformal radius. In this direction we have as yet only partial results, which concern the metrics $g\,dr\,d\theta$ with $g = r^\mu$, $\mu \geq -1$. But the method presented seems to be quite general and we plan to employ it also with other classes of metrics g in order to obtain further estimates of the type mentioned above.

§1. Estimates for Energy Integrals.

Let $g(r)$ be a positive continuous function for $0 < r < \infty$ and let $G(r)$ be a primitive of g. Let ρ be a fixed positive number and set:

$$(1.1)\qquad \begin{aligned} u &= G(r) - G(\rho) \qquad & (\rho \le r < \infty) \\ v &= \theta \qquad & (0 \le \theta < 2\pi) \end{aligned}$$

where (r,θ) are the polar coordinates in the (x,y) plane, (which will be referred to also as the z-plane).

If Ω is an open set which does not contain the circle $|z| \le \rho$, and if $F(x,y) \in C^1(\Omega)$ we obtain by a standard computation:

$$(1.2)\qquad |\text{grad } F|^2 dxdy = \left[\left(\frac{\partial F}{\partial u}\right)^2 rg(r) + \frac{1}{rg(r)}\left(\frac{\partial F}{\partial v}\right)^2\right] dudv.$$

Hence:

$$(1.3)\qquad \iint_{\Omega} |\nabla F|^2 rg(r)\,dxdy = \iint_{\tilde\Omega} [F_u^2 (rg(r))^2 + F_v^2]\,dudv$$

where $\tilde\Omega$ is the image of Ω by (1.1). We denote:

$$(1.4)\qquad q^2 = rg(r).$$

Definition 1.1.

Let D be an open set containing $|z| \le \rho$ and $D_\rho = D - \{|z| \le \rho\}$. Denote:

$$(1.5)\qquad \ell(\theta) = \int_{E_\theta} g(r)dr, \qquad E_\theta = D_\rho \cap \{\arg z = \theta\}$$

and:

(1.6) $R(6) = G\sim^{1}[\wedge(9) + G(p)]$.

Clearly R(9) does not depend on p. Set:

(1.7) $D^* = (z = re^{l6} : 0 \leq r < R(9),\ 0 \leq 6 < 2TT\}$.

The transformation D —> D will be called the radial concen-tration with metric g.

Remark.

If $\tilde{D}_\rho$ is the image of D_ρ by (1.1) and $\tilde{D}^*_\rho$ is the image of D^*_H by (1.1), then $\tilde{D}^*_{r*}$ is obtained from $\tilde{D}_{r\wedge}$ by:

(1.8) if $\rho = \{(u,v) \mid 0 < u < l(v),\ 0 \leq v < 2TT\}$,

where

(1.9) $\wedge(^v n\wedge$ = linear measure of $(D\wedge f\rho \{v = v\}9)$.

Definition 1.2.

Let D be a domain which contains |z [«< p and does not contain z = oo . Let F(x^y) be a continuous function in the extended (x,y) plane such that F = 0 outside D and F = 1 in a compact subset of D (denoted by E) such that { |z | $\leq$ *p)* c E. Suppose that in n = D - E ? O < F < 1 and that on every ray through the origin F obtains every value A, (0 < A < 1), only a finite number of times. Let

(1.10) $D_A(F) = \{(x_5 y) \mid F(x,y) > A\}$, $(0 \leq A < 1)$.

Let D_λ^* be the g-radial concentration of D_λ and let F^* be defined as follows:

$$(1.11)\quad F^* = \begin{cases} 1 & \text{in } E \\ \lambda & \text{on the boundary of } D_\lambda^*, \quad 0 < \lambda < 1 \\ 0 & \text{outside } D . \end{cases}$$

(Here E_λ^* is defined as in Definition 1 except that in (1.7) $0 \leq r \leq R(\theta)$.) Then F^* will be called the <u>radial concentration of</u> F, <u>with metric</u> g.

The following results are proved exactly in the same way as in [2]:

(i) D^* is a starlike domain.

(ii) E^* is a compact, connected, starlike set.

(iii) If F is continuous then F^* is continuous.

(iv) If F is continuous in the extended plane and Lip in every compact subset of D - E, then F^* has the same properties with respect to $D^* - E^*$.

Also the following basic result is obtained by essentially the same method as in [2]:

<u>Lemma</u> 1.1.

Let D, ft, F be as in Definition 1.2. Suppose also that $F \in C^1(\Omega)$, that $F \in C^0$ in the extend plane, and that on every ray arg z = θ, the set of points in Ω where $\frac{\partial F}{\partial r} = 0$ is at most a finite set. Finally suppose that:

$p(u) = [rg(r)]_{r=r(u)}$ is convex or monotone.

Then we have:

$$(1.12) \qquad \iint_{\Omega^*} |\nabla F^*|^2 q^2 \,dxdy \le \iint_{\Omega} |\nabla F|^2 q^2 \,dxdy$$

where $\Omega^* = D^* - E^*$, $\Omega = D - E$.

We now define the radial averaging transformation with metric g, in the same way as it was defined in [3] for the logarithmic measure.

Definition 1.3.

Let $\{D_1,\dots,D_n\} = \mathcal{D}$ be a family of open sets in the complex plane z, each containing the disk $|z| \le \rho$. Let $A = \{a_j\}_{j=1}^n$ where $a_j > 0$ and $\sum_{j=1}^n a_j = 1$. Let $\ell_j(\theta)$ be defined as in Definition 1.1 for D_j. Then set:

$$(1.13) \qquad \ell(\theta) = \sum_{j=1}^n a_j \ell_j(\theta),$$

$$(1.14) \qquad R(\theta) = G^{-1}[\ell(\theta) + G(\rho)],$$

and finally define D as in (1.7). We shall denote: $D = R_{g,A}(\mathcal{D})$ and the transformation $\mathcal{D} \to D$ will be called a radial averaging transformation with metric g.

Definition 1.4.

Let $\mathcal{D}$ and A be as above. Suppose that D_j does not contain $z = \infty$. Let E_j be a compact subset of D_j containing $\{|z| \le \rho\}$. Let $\mathcal{F} = \{F_j\}_{j=1}^n$ be a set of functions such that each F_j has the properties described in Definition 1.2 with respect to D_j and E_j. We define $D_j(F_j)$ as in (1.10) and $D_A^* = \mathfrak{R}_{g,A}(D_1(F_1), \ldots, D_n(F_n))$. Finally we define F^* as in (1.11). The transformation $\mathfrak{F} \to F^*$ will be called a radial averaging transformation on $\mathfrak{F}$ with metric g. We shall denote $F^* = \mathfrak{R}_{g,A}(\mathfrak{F})$.

The analogous properties to (i)-(iv) for the radial averaging transformation are verified exactly as in [3]. Also the following result is proved essentially in the same way as the parallel result in [3].

Theorem 1.1.

Let $\mathfrak{F}$, $\mathfrak{G}$ be as in Definition 1.4. Suppose also that each F_j has the properties described in Lemma 1.1 with respect to D_j, E_j. Finally suppose that p(u) (defined as in Lemma 1.1) is convex. Then we have:

$$(1.15)\qquad \iint_{\Omega^*} |\nabla F^*|^2 q^2 dxdy \le \sum_{j}^{n} a_j \iint_{\Omega_j} |\nabla F_j|^2 q^2 dxdy$$

where $\Omega^* = D^* - E^*$, $\Omega_j = D_j - E_j$, $D^* = \mathfrak{R}_{g,A}(\mathfrak{D})$, $E^* = \mathfrak{R}_{g,A}(E_1, \ldots, E_n)$.

We note that Lemma 1.1 is contained in Theorem 1.1, for the particular case n = 1. We remark also that one can obtain a more general inequality in Theorem 1.1, of the type discussed in Section 1 of [3].

The integrals in (1.15) may be interpreted as certain energy integrals. Hence Theorem 1.1 may be used in order to evaluate energy integrals of this type.

Finally, we remark that for $g(r) = \frac{1}{r}$ (i.e. the logarithmic metric) the results obtained here coincide with the results of

[3]. In this case the integrals in (1.15) may be interpreted as (conformal) capacities of condensers in the plane.

§2. Estimates for Conformal Capacity.

In this section we describe a method by which the result of Theorem 1.1. with various metrics g, may be used in order to derive inequalities for capacities of condensers.

Let D be a domain which does not contain z = oo, and E a compact subset of D which contains the disk |z | <£ p. We denote as usual 0 = D - E. Let 05 be a function which is continuous in the extended plane such that UteC ¹(&), CO = 0 outside D and to = 1 in E. We assume that the boundary of 0 is sufficiently smooth so that Green's theorem may be used. We shall denote by C the "inner boundary" of Q i.e. Q̄ ⁿ E.

Let h be defined, in 0, by h = to/q, where q is a positive function of r (0 < r < oo) such that qeC²(O,oo). Then the following formula is easily established:

(2.1) Jj |7co|²dxdy = Jj |7h|²q²dxdy - JjVqAqdxdy - £ |- |^ds .
n n Q

We now restrict our attention to the case where E = { |z | <C p} in which case C is the circly |z | = p. We also assume that CO is harmonic in Q and that q(r) is analytic for 0 < r < oo .

Let us apply the transformation of radial concentration with metric g, where q^2 = rg(r), to D and h. We denote the resulting domain and function by D , h respectively and we set n* = D* - E. (In this case E* = E.) It is easily verified that

(2.2) $\iint_{\Omega} h^2 \frac{q\Delta q}{r}\, dxdy = \iint_{\Omega^*} h^{*2} \frac{q\Delta q}{r}\, dxdy$.

Now suppose that q is chosen in such a manner that:

(2.3)
- (i) $q\Delta q = cg(r)/r$ where $q^2 = rg(r)$ and c is a constant;
- (ii) q is positive, non-decreasing.
- (iii) $p(u) = [q^2(r)]_{r=r(u)}$ is convex (where $r(u) = G^{-1}[u + G(\rho)]$, see (1.1)).

Since ω is harmonic in Ω we have $0 < \omega < 1$ in Ω and since q is non-decreasing $0 < h < \frac{1}{q(\rho)}$ in Ω with $h = \frac{1}{q(\rho)}$ on C and $h = 0$ on the boundary of D. Furthermore since h is an analytic function of r on the intersection of any ray $z = \theta$ with Ω, it is clear that h satisfies all the assumptions of Lemma 1.1 (if Ω has a smooth boundary). Hence we obtain:

(2.4) $$\iint_{\Omega^*} |\nabla h^*|^2 q^2\, dxdy \le \iint_{\Omega} |\nabla h|^2 q^2\, dxdy .$$

By (2.1), (2.2), (2.3), (2.4) we get:

(2.5) $$\iint_{\Omega} |\nabla\omega|^2 dxdy \ge \iint_{\Omega^*} |\nabla h^*|^2 q^2 dxdy - \iint_{\Omega^*} h^{*2} q\Delta q\, dxdy - \oint_C \frac{1}{q^2}\frac{\partial q^2}{\partial n}\, ds .$$

But, again by (2.1), the right-hand side of (2.5) is equal to:

$$\iint_{\Omega^*} |\nabla\omega^*|^2 dxdy$$

where $\omega^* = h^* q$; note that $\omega^* = 1$ on C and $\omega^* = 0$ on the boundary of D . Also, since h is Lip in every compact subset of Q , so is CO . Hence 60 is an admissible function for the variational definition of the capacity of the condenser fi* ; if ω^T is harmonic in fi* and $\omega^T = 1$ on C and $\omega^! = 0$ on the boundary of D , then:

$$(2.6) \qquad I(\Omega^*) = \iint_{\Omega^*} |\nabla\omega^*|^2 dxdy \le \iint_{\Omega^*} |\nabla af^*|^2 dxdy,$$

where $I(\Omega^*)$ is the capacity of Q . (As a reference for the facts quoted here see for instance Hayman [1]). From (2.5) and (2.6) we finally obtain:

$$(2.7) \qquad I(\Omega^*) \le I(\Omega),$$

where $I(\Omega)$ is the capacity of Ω.

To sum up this result we state:

Lemma 2.1.

Let D be a domain which does not contain $z = \infty$ and contains the disk $|z| \le p$. Let $\Omega = D - \{|z| \le p\}$. Let q be a positive analytic function of r for $0 < r < \infty$ satisfying (2.3). Let D denote the domain obtained by radial concentration with metric g from the domain D. We assume that D is not the entire plane (x,y) . Then:

$$(2.8) \qquad I(\Omega^*) \le I(\Omega)$$

where $\Omega^* = D^* - \{|z| \le p\}$.

Remark.

In the previous discussion we assumed that the boundary of D is smooth; but the result of Lemma 2.1 is obtained for general domains D by the standard method of approximating a given domain, by a sequence of domains with smooth boundary.

Using a result of Polya-Szego [4] on the connection between capacity and conformal radius, the following result is obtained as an immediate consequence of Lemma 2.1:

Lemma 2.2.

Let D be a domain containing the origin and let D^* be the domain obtained from D by radial concentration with metric g. Suppose that g is analytic for $0 < r < \infty$ and satisfies (2.3). Denote by r_0 (resp. r_0^*) the conformal radius of D (resp. D^*) at the origin. (We assume that D^* is not the entire plane.) Then:

$$r_0 \leq r_0^*. \tag{2.3}$$

By the same arguments used in the proof of Lemma 2.1, one obtains the following result (based on Theorem 1.1):

Theorem 2.1.

Let $\mathcal{D} = \{D_1,\dots,D_n\}$ be a family of domains each of which does not contain $z = \infty$ and contains the disk $|z| \leq \rho$. Let q be a positive analytic function of r for $0 < r < \infty$, satisfying (2.3). Let $D^* = \mathcal{R}_{g,A}(\mathcal{D})$ and suppose that D^* is not the entire

plane. Denote: fi . = D. - { |z | < p}, Q* = D* - { |z | < p}.
J J

Then:

(2.10) I(Q*) < S a. 1(0 .).
j=1 m

As a consequence of Theorem 2.1 we obtain, by using the result of Polya-Szego [4] mentioned above:

Theorem 2.2.

Let & = {D₁ ,...,Dn} be a family of domains containing the origin and let D = ft A($) where g(r) is positive and analytic for 0 < r < oo and satisfies (2.3). Denote by r . (resp. V9 J

rO) the conformal radius of D.J (resp. D) at the origin. (We assume that D is not the entire plane.) Then:

(2.11) n ai *
n r O j ^ V
i=1 '**

The family of functions q (or g) which satisfy (2.3) is easily established. If c is any positive constant the general solution of (2.3) (i) is given by the linear combinations of

q = r fc and q = r - fc . If c = 0 the general solution is given by the linear combinations of q = 1 and q = In r. If c < 0 the general solution is given by the linear combinations of q = sin(V-c In r) and q = cos(sf^c In r). Hence the functions:

(2.12) f q = r a , a ^ o
i
(g = r p L , j3 = 2a ^ 0

satisfy (2.3). (In this case, for j8 > 0, p(u) = j3u + o^{p}, which is certainly convex; and for /3 = 0, p(u) = 1. Also, condition (2.3) (ii) is satisfied.)

Examining Theorem 2.2 we observe that for g as in (2.12), this result is not more general than the corresponding result for the logarithmic measure ($g_0 = \frac{1}{r}$). Indeed, by the arithmetic-geometric mean inequality, it is clear that R .(&) 3 ft A(&) where $g = r^{B}$, j3 ≥ -1, and $g_0 = \frac{1}{r}$. This shows that in fact, Theorem 2.1 (and also Theorem 2.2), follow immediately from the corresponding theorems with logarithmic measure.

Furthermore, it can be shown that Theorem 2.2 cannot possibly hold for ~~every~~ metric $g = r^{-n}$, n = 1,2,... Assuming that this is true, it is not difficult to derive a contradiction.

On the other hand, an examination of the proof of Lemma 2.1 shows that the condition (2.3) is too restrictive for our arguments. The condition (2.3) (i) guarantees that the integral

rr2
jjh q/xqdxdy
n

is preserved under our transformation. But actually we only need that the integral does not decrease under this transformation. This might allow us to use some metrics, other than those discussed above.

Furthermore, although, for a given metric g, (2.11) might not hold for every family of domains &, it might hold for certain types of domains. Indeed, for a given family of domains, (2.3) may be replaced by a much weaker condition.

These observations, which we intend to investigate further, seem to us to justify the presentation of the method described above.

References

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