THE THEOREM OF RAYLEIGH-FABER-KRAHN FOR THE CHARACTERISTIC VALUES ASSOCIATED WITH A CLASS OF NONLINEAR BOUNDARY VALUE PROBLEMS

by

Catherine Bandle

Research Report 71-37

August, 1971

/elb

University libraries
Carnegie Mellon University
Pittsburg,

SEP 8 '71

HUNT LIBRARY
CARNEGIE-MELLON UNIVERSITY

THE THEOREM OF RAYLEIGH-FABER-KRAHN FOR THE CHARACTERISTIC VALUES ASSOCIATED WITH A CLASS OF NONLINEAR BOUNDARY VALUE PROBLEMS

Catherine Bandle*

Abstract

This paper is concerned with functionals which were introduced by Nehari [8,9] and also discussed by Coffman [2,3] in connection with the study of nonlinear boundary value problems. Their behavior under the Schwarz syitimetrization [12] is studied, and an isoperimetric inequality analogous to that of Rayleigh-Faber-Krahn [12] for the fundamental frequency of a vibrating membrane is derived.

* This work was supported by NSF Grant GU-2056.

THE THEOREM OF RAYLEIGH-FABER-KRAHN FOR THE CHARACTERISTIC VALUES ASSOCIATED WITH A CLASS OF NONLINEAR BOUNDARY VALUE PROBLEMS

by

Catherine Bandle

1. Let 0 be a bounded region in R^n for which the Green's function for the Laplace operator exists. We shall write P for an arbitrary point in R^n and R_+ for the positive real axis. Let F(s,P) be a positive function on "$\overline{R}_+$ y Cl with the following properties

(A) $F(^m, x)$ is continuous on $\overline{R}_+$ for almost all xefh

F(s,•) is measurable for all se$\overline{R}_+$.

(B) There exists a positive number £ such that for almost all Pen and for all $s_1 < s_2$

$$s_1^{-} * F(s_i ' p) \; 1 \; s_2^{-} * F(s_2,P) \cdot$$

We define the function G(t,P) by

$$6(t,P) = \int_0^t F(s,P)ds,$$

and consider the functional

$$H(v) = fi(v) - \int_{n} G(v^2,P)dx$$

[dx volume element in R^n, $j\&(v) = \int_{\Omega} \text{grad}^2 v dx$, $(x^1 \wedge^2, \ldots, x^n)$ Cartesian coordinates], within the class T of piecewise continuously differentiable functions which vanish on the boundary SO . This note will be concerned with isoperimetric inequalities for the functional

$$A(Q) = \underset{v}{\mathrm{Min}}\ H(v)$$

where v ranges over all functions in T satisfying the side condition

$$_{fi}(V) = \underset{0}{J}\ v^2F(v^2,P)dx \qquad (1)$$

Following Nehari we call $A(Q)$ the <u>characteristic value</u>, Nehari [8] proved that for every function veT satisfying (1) the inequality

$$H(v) \geq -\wedge \underset{0}{J}\ v^2F(v^2,P)dx$$

holds. A(fi) is therefore bounded from below. It was pointed out in [8] that for every function veT there exists a constant $a \wedge 0$ such that av satisfies the side condition (1). This is an immediate consequence of (B) and the fact that $\lim_{s - 0} F(s,P) = 0$ and $\lim_{S \sim *OD} F(s,P) = \infty$. Nehari [9] also showed that for fie $R^!$ there exists a function ueT subject to (1) which minimizes $H(v)$. This function is a solution of the boundary value problem $u'' + uF(u^2,p) = o$ in Cl, $u = 0$ on 3fi. This result has been generalized by Coffman [2] for the case where $\Omega \subset R^n$ $(n \geq 2)$. It can be stated as follows:

Let $F(s,P)$ be locally Hölder continuous on $\bar{R}_+ \times \bar{\Omega}$, and suppose that there are positive constants θ, c and $\gamma < \frac{2}{n-2}$ such that $F(s,P) \leq c s^{\gamma} + a$ for all $s \in R_+$. [In R^2 there is no restriction on γ]. If we assume further that (A) and (B) hold, then $A(\Omega)$ exists, and the minimizing function u is of class C^2 in $\bar{\Omega}$ and solves the Dirichlet problem $\Delta u + uF(u^2,P) = 0$ in Ω, $u = 0$ on $\partial\Omega$ $\left[\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \text{ Laplacian}\right]^{(1)}$. In order to estimate $A(\Omega)$ we shall use the following property of $H(v)$.

LEMMA 1: *If* v *satisfies* (1), *then the inequality*

$$H(av) \leq H(v)$$

holds for every real number a.

Proof, Since $F(s,P)$ is non-decreasing, $G(s,P)$ is concave and hence $G(s_0,P) - G(s_1,P) \geq (s_0 - s_1)F(s_0,P)$. Thus, observing (1) we have

$$H(av) - H(v) = (a^2-1)\beta(v) - \int_\Omega \{G(a^2v^2,P)-G(v^2,P)\}\, dx$$

$$\leq (a^2-1)\beta(v) - \int_\Omega (a^2v^2-v^2)F(v^2,P)dx = 0$$

---

(1) If $\Omega \subset R^2$, then more general results can be found in [6],[11],...

For the following considerations we shall need the Schwarz synme-trization [12]. By this synunetrization a somain $B \in R^n$ is transformed into a n-sphere B^* with the center at the origin and the same volume as B. A positive measurable function f on B with f = 0 on ∂B is transformed into a function f^* on B^* in the following way: Let B_t denote the region $B_t = \{P \in B; f(P) > t\}$. f^* is the radially symmetrical function with $f^* > t$ on B_t^* and $f^* = t$ on ∂B_t^*. The next result is based on the inequality of Rayleigh-Faber-Krahn [12] for vibrating membranes.

THEOREM I: Let F(s,P) = F(s) satisfy (A) and (B) and be independent of P. Then among all regions $\Omega \in R^n$ with a given volume the n-sphere yields the minimal value of A(0).

Proof. Let $\{u_n\}_{n=1}^{\infty}$ be a sequence of functions in T, subject to the side condition $(1)_3$ and with the property

$$A(\Omega) = \lim_{n \to \infty} H(u_n).$$

We denote by u_n^* the function obtained from u_n after the Schwarz symmetrization. For each u_n^* we determine a number a_n such that

$$V(a_n u_n^*) = \int_{\Omega^*} a_n^2 u_n^{*2} F(a_n^2 u_n^{*2}) dx.$$

It follows from the definition of u_n^* that $\int_\Omega G(a_n^2 u_n^2) dx = \int_{\Omega^*} G(a_n^2 u_n^{*2}) dx$. Since the symmetrization diminishes the Dirichlet integral, we have $\$Q({}^a{}_n{}^u{}_n)$ 2 $*n*({}^an^un*\wedge$ $t^{12\wedge}$ and thus by Lemma 1

$$A(Q) \wedge \lim_{n\text{-}ao} [^*)_{\hat{w}}(a_n u_n) - \int_{\Omega}^{*} G(a_n^2 u_n^2)dx]$$

$$\geq \underline{\text{lfm}} \quad t \wedge W * - J \quad G(a_n u_n{}^{*2})dx].$$

This inequality together with the minimum property of A(ft*) proves that A(ft) ^ A(ft*).

REMARKS.

(1) The same arguments show that A(ft) is diminished by the Steiner symmetrization [12].

(2) Suppose that ft is a sphere and that the minimizing function u of the variational problem exists. Then u is radially symmetric and non-increasing in r $[r^2 = \sum_{i=1}^{n} (x^i)^2]$.

From this fact it is not difficult to obtain an upper bound for the maximal value of the function u which solves the Dirichlet problem Au + uF(u) = 0 in ft = [x; |x| <^ R}, u = 0 on 50, and yields the minimum of H(v). As an example we consider the case value $F(s) = s^m$ and ft = { (x,y) ; $x^2+y^2 \leq 1$}. u can then be written as

$$u(x) = \wedge \iint_{n} \ln|z-z'|u^{2m+1}(z')dA_{z}, \tag{2}$$

$$[z = x + iy, \quad z' = x' + iy', \quad dA_{z'} = dx'dy']$$

Since $\max_{z \in Q} u(z) = u(0)$, and since u is decreasing,, we have

$$u(0) \geq 4^{\frac{1}{m}}$$

(3) Some growth conditions on $F(s)$ are necessary in order to obtain a minimizing function of class $C^2(Q)$. Indeed, consider the functional

$$A(\Omega) = \underset{v=0 \text{ on } \partial\Omega}{\text{Min}} \ \max_{a} \left(\alpha(av) - \frac{1}{m+1} \int_\Omega J(\alpha v)^{2m+2} dx\right), \tag{3}$$

and suppose that the minimizing function u is of class $C^2(Q)$. It is therefore a solution of the corresponding Euler equation $\Delta u + u^{m+1} = 0$ in Q, $u = 0$ on $\partial\Omega$. If $u(r)$ is the solution for $\Omega_1 = \{x;\ |x| \leq 1\}$, then $t^{-1/m} u(\frac{r}{t})$ is the solution for the sphere $\Omega_t = \{x;\ |x| \leq t\}$. An easy computation yields

$$A(Q_t) = t^{-\left(\frac{2m+2}{m}\right)+n} A(\Omega_1) \tag{4}$$

where n is the dimension of the space. Since $A(\Omega_t)$ is a monotonic functional of t, we must have $m \leq \frac{2}{n-2}$. This condition was obtained by Pohozaev [11] in a different way. If we compute the value for a, then (3) becomes

$$\Lambda(\Omega) = \operatorname*{Min}_{v=0 \text{ on } \partial\Omega} \frac{m}{m+1} \left[\frac{\mathcal{D}(v)^{m+1}}{\int_0^1 v^{2m+2}\, dx} \right]^{\frac{1}{m}} . \qquad (5)$$

We now consider the case $n = 3$, $m > 2$. If we take

$$v = \begin{cases} \cos \frac{\pi}{2} k r & \text{in } [0, \frac{1}{k}] \\ 0 & \text{in } [\frac{1}{k}, 1] \end{cases}$$

$k > 1$, then v is admissible for the variational characterization (5) of $\Lambda(\Omega_1)$. The computation shows that the right side of (5) tends to zero if $k \to \infty$. Hence $\Lambda(\Omega_1) = 0$, and by the same argument and the monotonicity of $\Lambda(\Omega)$ we can prove that $\Lambda(\Omega) = 0$ for an arbitrary domain Ω. If $n = 3$ and $m = 2$, there exists a constant $\beta > 0$ such that

$$\Lambda(\Omega) = \beta \quad \text{for all } \Omega .$$

Because of an inequality by Ladyzhenskaja [5]

$$\{\mathcal{D}(v)\}^3 \geq 48 \int_\Omega v^6 dx^1 dx^2 dx^3 ,$$

it follows that $\Lambda(\Omega) > 0$ for all Ω. If we can show that β is the same for all circles, then the assertion will be proved. Let $\Omega_1 \subset \Omega_2$, and $u_n(r)$ be a sequence of radially symmetrical functions subject to (1) such that

$$\Lambda(\Omega_2) = \lim_{n \to \infty} H(u_n)$$

The functions $v_n = t_o^{-\frac{1}{2}} u_n\left(\frac{r}{t_o}\right)$ with $t_o = \frac{t_2}{t_1}$ are admissible for the variational characterization of $A(n_{t_1})$. Hence

$$A(Q_{t_l}) \leq \lim_{v_n \to \infty} H(v_n) = A(0_{z_2}) .$$

On the other hand we have from the monotonicity

$$\Lambda(\Omega_{t_2}) \leq A(Q_{t_1}) .$$

2. Let fie R^2, and consider functions $F(s,P)$ of the form

$$F(s,P) = p(P)F_Q(s) + a(P)$$

where $p(P)$ and cx(P) a^e positive on 0 and $F_o(s)$ satisfies (A) and (B) of Section 1. If the least eigenvalue of the membrane problem Au + Agu = 0 in n, u = 0 on 3Q exceeds 1, then it is possible to find for each function weF a constant a such that (1) holds for v = aw [2,3,8,9]. Under all these assumptions A(fi) exists. If F(s,P) is locally Hölder continuous on $\overline{R}_+ \times 0$, then there is a minimizing function u which solves the boundary value problem $AU + u(p(P)F_o(u^2(P)) + a(P)) = 0$ in Q , u = 0 on SO [2] . It may be observed that $A_1 > 1$ is also necessary for the existence of the function u. Indeed, since u does not change sign[(1)], it can be interpreted as the first

---

(1) This follows immediately from the minimum property *of* A(Q) and from a simple reflection argument.

eigenf unction of the problem $h/ + /im(P)v = 0$ in Q, $v = 0$ on BO, where $m(P) = p(P)F_Q(U^2(P)) + c(p)$. We have $m(P) > a(P)$, and by the monotonicity of the eigenvalues $1 = ji_, < $ ^^.

We shall use the following notations:

$$M_\rho(B) = \iint_B \rho\, dxdy, \qquad M_\sigma(B) = \iint_B \sigma\, dxdy$$

where $B \subset fi$ is an arbitrary domain, and x, y are the Cartesian coordinates. Let $r = \sqrt{x^2 + y^2}$ and K be an arbitrary real number, then we define

$$g_K(r) = \begin{cases} \dfrac{4}{|K|(1 + r^2)^2} & \text{if } K > 0 \\ \dfrac{4}{(1 - r^2)^2} & \text{if } K < 0 \\ 1 & \text{if } K = 0 \end{cases}$$

and $M_{g_k}(B) = \iint_B g_k\, dxdy$. Let $F_o(s)$ be fixed, and consider $A(0) = A(0, e, p)$ as a function of 0, $<?$ and p. 0^*_σ denotes the circle with the property

$$\iint_{n^*} g_k\, dxdy = \iint_{n} 0 \cdot dxdy,$$

and Ω^*_ρ is defined in an analogous way. The next result is a generalization of Theorem 1 of Section 1. It is related to some extensions of the Rayleigh-Faber-Krahn inequality for inhomogeneous membranes [1,10].

In order to simplify the proof we shall assume that there exists a function $u \in C^2$ belonging to I and subject to (1) which yields the minimum of H(v). Otherwise we have to consider a minimizing sequence as we did in the proof of Theorem 1.

THEOREM II. Suppose that Ω is simply connected, $\lambda_1 > 1$ and that there exists a number K such that the following inequalities hold in Ω:

$$-\Delta \ln \rho / 2\rho \leq K, \quad -\Delta \ln \sigma / 2\sigma \leq K, \quad 4\pi - KM_\rho > 0, \text{ and } 4\pi - KM_\sigma > 0.$$

(a) If $\Omega^*_\rho \subset \Omega^*_\sigma$ and if the first eigenvalue of the problem $\Delta u + \mu g_K u = 0$ in Ω^*_σ, $u = 0$ on $\partial\Omega^*_\sigma$, exceeds 1, then we have, for fixed F(s),

$$\Lambda(\Omega,\sigma,\rho) \geq \Lambda(\Omega^*_\sigma, g_k, \frac{M_\rho}{M_\sigma} g_k)$$

$[M_\rho = M_\rho(\Omega),\ M_\sigma = M_\sigma(\Omega)]$.

(b) If $\Omega^*_\sigma \subset \Omega^*_\rho$, and if the first eigenvalue of the problem $\Delta u + \mu \frac{M_\sigma}{M_\rho} g_K u = 0$ in Ω^*_ρ, $u = 0$ on $\partial\Omega^*_\rho$, exceeds 1, then

$$\Lambda(\Omega,\sigma,\rho) \geq \Lambda(\Omega^*_\rho, \frac{M_\sigma}{M_\rho} g_k, g_k).$$

Proof of Part (a); If f is an arbitrary positive function, let B|f, denote the circle with center at the origin and the property that

$$\iint_B f\,dxdy = \iint_{B^*(f)} g_k\,dxdy.$$

Let n(t) = {PGO; U(P) > t) , and let u*(f) be the radially symmetrical function on ^(f\ such that u|f\ >. t i-n Ω*(f) (t) and U7f^ = t on ^^|f\ (t) • We shall write c = T~M σ / M ρ , p = c p and h(P) = max [uj^(ρ) (P) , u^ v (P) } • Since jx > 1 there exists a number a such that

$$\&_{n/\pi}(a)\,(ah) = f\int_{o^*} a^2h^2\{F\,(a^2h^2)c^{-1}g, + g.\}\,dxdy.$$

The proof is based on the following lemma [1].

LEMMA 2. Let v be em arbitrary positive function in 0 which vanishes on the boundary an. Let G(t) JDe the domain [PGQ; V(P) ^> t}. j^f a positive function f satisfies in Q the inequalities -A In f/2f ^ K and 4TT - K $\iint_n$ f dxdy > 0, then for every (t..,t9) (tn <^ t9)

$$\iint_{G(t_1)\setminus G(t_2)} \text{grad}^2 v\,dxdy \geq \iint_{G|f)(t_1)\setminus G|f)(t_2)} \text{grad}^2 v|_{f)}\,dxdy \qquad (6)$$

Because of the assumptions regarding σ, it follows therefore that

$$\iint_{\Omega(t_1)\setminus\Omega(t_2)} \operatorname{grad}^2 u \, dxdy \geq \iint_{\Omega^*_{(\sigma)}(t_1)\setminus\Omega^*_{(\sigma)}(t_2)} \operatorname{grad}^2 u^*_{(\sigma)} \, dxdy \qquad (7)$$

for all $t_1 \leq t_2$

Because of $c \geq 1$ we have $-\Delta \ln p/2p \leq \ldots \leq K$. Since

$4\pi - K \iint_{\Omega} p \, dxdy = 4\pi - K M_{\sigma} > 0$, we can apply Lemma 2 to

$u^*_{(p)}$, and we obtain

$$\iint_{\Omega(t_1)\setminus\Omega(t_2)} \operatorname{grad}^2 u \, dxdy \geq \iint_{\Omega^*_{(p)}(t_1)\setminus\Omega^*_{(p)}(t_2)} \operatorname{grad}^2 u^*_{(p)} \, dxdy \qquad (8)$$

From (7) and (8) we conclude that

$$B_{\Omega}(u) \geq B_{\Omega^*_{(\sigma)}}(u^*_{(\sigma)}) \qquad (10)$$

The following relations are immediate consequences of the definition of $u^*_{(\sigma)}$ and $u^*_{(p)}$:

$$\iint_{\Omega} \left\{ \int_0^{u} F(s)\,ds \right\} p \, dxdy = \iint_{\Omega^*_{(p)}} \left\{ \int_0^{u^*_{(p)}} F(s)\,ds \right\} c^{-1} g_k \, dxdy \qquad (11)$$

and

$$\iint a^2u^2 g \,dxdy = \iint_{\Omega^*(\sigma)} a^2u^2 g_k \,dxdy. \tag{12}$$

From (11) and (12) and the monotonicity of $F_0(s)$ we have

$$\iint_{\Omega} G(\alpha^2u^2)\,dxdy \leq \iint_{\Omega^*(\sigma)} \left[\left\{ c^{-1} \int_0^{\alpha^2 h^2} F_0\,ds \right\} + \alpha^2 h^2 \right] g_k \,dxdy,$$

and by (10) and the same arguments as in the proof of Theorem 1,

$$\Lambda(\Omega, \sigma, \rho) \geq \Lambda\left(\Omega^*(\sigma), g_k, \frac{M}{\sigma} g_k\right).$$

part (b) can be proved in a similar way.

EXAMPLE. Consider functions σ and ρ such that $\Delta\rho \leq 0$, $\Delta\sigma \leq 0$, $\frac{M}{\rho} \leq 2$ and $\frac{M}{\sigma} < \frac{\pi}{\sigma} j_0^2$ [$j_0 = 2,4048, \ldots$ first zero of the Bessel function of order zero]. It is easy to verify that $-\Delta \ln \rho/2\rho \leq 0$ and $-\Delta \ln \sigma/2\sigma \leq 0$. We have therefore $K = 0$ and $g_k = 1$. From the inequality of Nehari for inhomogeneous membranes [10] it follows that $\Lambda_1 \geq \frac{\pi j_0^2}{M_\sigma} \geq 1$, and from the Rayleigh-Faber-Krahn inequality $\mu_1 \geq \frac{\pi j_0^2}{M_\rho} \geq 1$. Hence, Theorem 2 yields

HUNT LIBRARY
CARNEGIE-MELLON UNIVERSITY

$$A(n,a,p) \geq \Lambda(\Omega^*_{(C)}, 1, \frac{M_\rho}{M})$$

where

$$\Omega^*_{(\sigma)} = \left\{(x,y) \in R^2;\ \sqrt{x^2+y^2} \leq \sqrt{\frac{M_\sigma}{\pi}}\right\} .$$

Acknowledgment

The author wishes to thank Professor Z. Nehari for having suggested the problem and for his help during the preparation of this paper. She is also indebted to Professor C. V. Coffman for many helpful discussions.

References

1. Bandle, C, "Konstruktion isoperimetrischer Ungleichungen der mathematischen Physik aus solchen der Geometrie", Comment. Math. Helv. 4(5, (1971) .

2. Coffman, C. V., "An existence theorem for a class of non-linear integral equations with applications to a nonlinear elliptic boundary value problem", J. of Math, and Mech. 18, (1968) 411-420.

3. ________________, "A minimum-maximum principle for a class of non-linear integral equations", J. Anal. Mathem 2J£, (1969) 391-419.

4. Courant, R. and D. Hilbert, Methods of Mathematical Physics, Vol. 1, New York, 1965.

5. Ladyzhenskaja, 0. A., The Mathematical Theory of Viscous Incompressible Flow, New York.

6. Levinson, N., "Positive Eigenfunctions for Au +Af(u) =0", Arch. Rat. Mech. Anal. JLL, (1962) 258-272.

7. Nehari, Z., "On a class of nonlinear second-order differential equations",, Trans. Am. Math. Soc. 9j^, (1960) 101-123.

8. ______________, "Characteristic values associated with a class of nonlinear second order differential equations", Acta. Math. 105, (1961) 141-175.

9. ______________, "On a class of nonlinear integral equations", Math. Z. 22, (1959), 175-183.

10. ______________, "On the principal frequency of a membrane", Pac. J. Math. J3, (1958) 285-293.

11. Pohozaev, S. T., "Eigenfunctions of the equation Au + Af(u) =0", Dokl. Akad. Nauk SSSR, 165, (1965) 36-39. Soviet Math. J5 (1965) 1408-1411.

12. Pólya, G. and G. Szegö, Isoperimetric Inequalities in Math. Physics, Princeton, 1951.